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Between Closed Sets and Generalized Closed Sets in Closure Spaces

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Abstract. The purpose of the present paper is to define and study ∂-closed sets in closure spaces obtained as generalization of the usual closed sets. We introduce the concepts of ∂-continuous and ∂-closed maps by using ∂-closed sets and investigate some of their properties.

1 Introduction

Generalized closed sets, briefly g-closed sets, in a topological space were introduced by N. Levine [10] in order to extend some important properties of closed sets to a larger family of sets. For instance, it was shown that compactness, normality and completeness in a uniform space are inherited by g-closed subsets. K. Balachandran, P. Sundaram and H. Maki [2] introduced the notion of generalized continuous maps, briefly g-continuous maps, by using g-closed sets and studied some of their properties.

Closure spaces were introduced by E. Čech in [4] and then studied by many mathematicians, see e.g. [5], [6], [14] and [15]. The concepts of generalized closed sets and generalized continuous maps of topological spaces were extended to closure spaces in [3]. In this paper, we introduce and study a new class of closed sets in closure spaces lying, as for generality, between the class of closed sets and the class of generalized closed sets. Using the concept of ∂-closed sets, we define two new kinds of spaces, namely $T'_{\frac{1}{2}}$-spaces and $T''_{\frac{1}{2}}$-spaces, and introduce ∂-continuous and ∂-closed maps. The two kinds of spaces and the two kinds of maps are investigated.

2 Preliminaries

A map $u\colon P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied:

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(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *additive* (respectively, *idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed. Let (X, u_1) and (X, u_2) be closure spaces. The closure u_1 is said to be *finer* than the closure u_2, or u_2 is said to be *coarser* than u_1, by symbols $u_1 \leq u_2$, if $u_2A \supseteq u_1A$ for every $A \subseteq X$. The relation $\leq$ is a partial order on the set of all closure operators on X.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too. A closure space (X, u) is said to be a *T_0-space* if, for any pair of points $x, y \in X$, from $x \in u\{y\}$ and $y \in u\{x\}$ it follows that $x = y$, and it is called a *$T_{\frac{1}{2}}$-space* if each singleton subset of X is closed or open.

Let (Y, v) be a closed subspace of (X, u). If F is a closed subset of (Y, v), then F is a closed subset of (X, u).

Let (X, u) and (Y, v) be closure spaces. A map $f\colon (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f\colon (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$. Clearly, if $f\colon (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f\colon (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$, is the closure space $\big(\prod\limits_{\alpha\in I} X_\alpha, u\big)$ where $\prod\limits_{\alpha\in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha\colon \prod\limits_{\alpha\in I}(X_\alpha, u) \to (X_\alpha, u)$, $\alpha \in I$, i.e., is defined by $uA = \prod\limits_{\alpha\in I} u_\alpha\pi_\alpha(A)$ for each $A \subseteq \prod\limits_{\alpha\in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta\colon \prod\limits_{\alpha\in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Proposition 1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $F \subseteq X_\beta$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod\limits_{\alpha\neq\beta\alpha\in I} X_\alpha$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$.*

Proof. Let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, hence $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ be a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β). □

Proposition 2. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $G \subseteq X_\beta$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.*

3 Generalized closed sets

Definition 1. Let (X, u) be a closure space. A subset $A \subseteq X$ is called a *generalized closed* set, briefly a *g-closed* set, if $uA \subseteq G$ whenever G is an open subset of (X, u) with $A \subseteq G$. A subset $A \subseteq X$ is called a *generalized open* set, briefly a *g-open* set, if its complement is g-closed.

The following statement is evident:

Proposition 3. *Let (X, u) be a closure space and let (Y, v) be a closed subspace of (X, u). If F is a g-closed subset of (Y, v), then F is a g-closed subset of (X, u).*

Theorem 1. *Let (X, u) be a closure space. Then (X, u) is a $T_{\frac{1}{2}}$-space if and only if every g-closed subset of (X, u) is closed.*

Proof. Let (X, u) be a $T_{\frac{1}{2}}$-space and let M be a g-closed subset of (X, u). Suppose that $x \notin M$. Then $\{x\} \subseteq X - M$ and hence $M \subseteq X - \{x\}$. Since M is g-closed and $X - \{x\}$ is open, $uM \subseteq X - \{x\}$ or, equivalently, $\{x\} \subseteq X - uM$. Therefore, $x \notin uM$ and thus $uM \subseteq M$. Hence, M is a closed subset of (X, u).

Conversely, suppose that $\{x\}$ is not closed. Then $X - \{x\}$ is not open. This implies that X is the only open set containing $X - \{x\}$. Therefore, $X - \{x\}$ is a g-closed subset of (X, u). Consequently, $X - \{x\}$ is closed. Hence, $\{x\}$ is an open subset of (X, u). Therefore, (X, u) is a $T_{\frac{1}{2}}$-space. □

Proposition 4. *Let (X, u) be a closure space and let (Y, v) be a closed subspace of (X, u). If (X, u) is a $T_{\frac{1}{2}}$-space, then (Y, v) is a $T_{\frac{1}{2}}$-space too.*

Proof. Let F be a g-closed subset of (Y, v). Then F is a g-closed subset of (X, u). Since (X, u) is a $T_{\frac{1}{2}}$-space, F is a closed subset of (X, u). This implies that F is a closed subset of (Y, v). Therefore, (Y, v) is a $T_{\frac{1}{2}}$-space. □

Proposition 5. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $F \subseteq X_\beta$. Then F is a g-closed subset of (X_β, u_β) if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g-closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Proposition 6. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $G \subseteq X_\beta$. Then G is a g-open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Proposition 7. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces. For each $\beta \in I$, let $\pi_\beta \colon \prod_{\alpha \in I} X_\alpha \to X_\beta$ be the projection map. Then*

(i) If F is a g-closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$, then $\pi_\beta(F)$ is a g-closed subset of (X_β, u_β).

(ii) If F is a g-closed subset of (X_β, u_β), then $\pi_\beta^{-1}(F)$ is a g-closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

Definition 2. Let (X, u) and (Y, v) be closure spaces. A map $f \colon (X, u) \to (Y, v)$ is called *generalized continuous*, briefly *g-continuous*, if $f^{-1}(F)$ is a g-closed subset of (X, u) for every closed subset F of (Y, v).

Clearly, a map $f \colon (X, u) \to (Y, v)$ is g-continuous if and only if $f^{-1}(G)$ is a g-open subset of (X, u) for every open subset G of (Y, v).

4 ∂-Closed Sets in Closure Spaces

In this section, we introduce and study a new class of closed sets lying, as for generality, between the class of closed sets and the class of generalized closed sets.

Definition 3. A subset A of closure space (X, u) is called a *∂-closed set* if $uA \subseteq G$ whenever G is a g-open subset of (X, u) with $A \subseteq G$. A subset A of X is called a *∂-open set* if its complement is a ∂-closed subset of (X, u).

Remark 1. For a subset A of a closure space (X, u), the following implications hold:

$$A \text{ is closed} \Rightarrow A \text{ is } \partial\text{-closed} \Rightarrow A \text{ is g-closed}.$$

None of these implications is reversible as shown by the following examples.

Example 1. Let $X = \{1, 2, 3, 4\}$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{1\} = \{1, 3\}$, $u\{2\} = \{2, 3\}$, $u\{3\} = u\{4\} = u\{3, 4\} = \{3, 4\}$ and $u\{1, 2\} = u\{1, 3\} = u\{1, 4\} = u\{2, 3\} = u\{2, 4\} = u\{1, 2, 3\} = u\{1, 2, 4\} = u\{2, 3, 4\} = u\{1, 3, 4\} = uX = X$. Then $\{1, 2, 3\}$ is ∂-closed set but it is not closed.

Example 2. Let $X = \{1, 2\}$ and define a closure operator u on X by $u\emptyset = \emptyset$ and $u\{1\} = u\{2\} = uX = X$. Then $\{1\}$ is g-closed but it is not ∂-closed.

The following statement is evident:

Proposition 8. *Let (X, u) be a closure space. If a subset A of (X, u) is both g-open and ∂-closed, then A is closed.*

Proposition 9. *Let (X, u) be a closure space and let u be idempotent. If A is a ∂-closed subset of (X, u) such that $A \subseteq B \subseteq uA$, then B is a ∂-closed subset of (X, u).*

Proof. Let G be a g-open subset of (X, u) such that $B \subseteq G$. Then $A \subseteq G$. Since A is ∂-closed, $uA \subseteq G$. As u is idempotent, $uB \subseteq uuA = uA \subseteq G$. Hence, B is ∂-closed. □

Proposition 10. *Let (X, u) be a closure space. If A is ∂-closed, then $uA - A$ has no nonempty g-closed subset.*

Proof. Suppose that A is ∂-closed. Let F be a g-closed subset of $uA - A$. Then $F \subseteq uA \cap (X - A)$ and so $A \subseteq X - F$. Consequently, $F \subseteq X - uA$. Since $F \subseteq uA$, $F \subseteq (X - uA) \cap uA = \emptyset$, thus $F = \emptyset$. Therefore, $uA - A$ contains no nonempty closed set. □

Theorem 2. *Let (X, u) be a closure space. A set $A \subseteq X$ is ∂-open if and only if $F \subseteq X - u(X - A)$ whenever F is a g-closed subset of (X, u) with $F \subseteq A$.*

Proof. Suppose that A is ∂-open and let $F \subseteq A$ be a g-closed subset of (X, u). Then $X - A \subseteq X - F$. But $X - A$ is ∂-closed and $X - F$ is g-open. It follows that $u(X - A) \subseteq X - F$ and hence $F \subseteq X - u(X - A)$.

Conversely, let $X - A \subseteq G$ where G is g-open. Then $X - G \subseteq A$. Since $X - G$ is g-closed, $X - G \subseteq X - u(X - A)$. Therefore, $u(X - A) \subseteq G$. Hence, $X - A$ is ∂-closed and so A is ∂-open. □

Proposition 11. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $G \subseteq X_\beta$. Then G is a ∂-open subset of (X_β, u_β) if and only if $G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-open subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Proof. Let F be a ∂-closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$ such that $F \subseteq G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$. Then $\pi_\beta(F) \subseteq G$. Since $\pi_\beta(F)$ is g-closed and G is ∂-open in (X_β, u_β), $\pi_\beta(F) \subseteq X_\beta - u_\beta(X_\beta - G)$. Therefore,

$$F \subseteq \pi_\beta^{-1}(X_\beta - u_\beta(X_\beta - G)) = \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} u_\alpha \pi_\alpha\Big(\prod_{\alpha \in I} X_\alpha - G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big).$$

By Theorem 2, $G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-open subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$.

Conversely, let F be a g-closed subset of (X_β, u_β) such that $F \subseteq G$. Then $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \subseteq G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$. Since $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is g-closed and $G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is ∂-open in $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$,

$$F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \subseteq \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} u_\alpha \pi_\alpha\Big(\prod_{\alpha \in I} X_\alpha - G \times \prod_{\alpha \neq \beta \alpha \in I} X_\alpha\Big)$$

by Theorem 2. Therefore,

$$\prod_{\alpha\in I} u_\alpha\pi_\alpha\Big((X_\beta - G)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) \subseteq \prod_{\alpha\in I} X_\alpha - F\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = (X_\beta - F)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Consequently, $u_\beta(X_\beta - G)\subseteq X_\beta - F$ implies $F\subseteq X_\beta - u_\beta(X_\beta - G)$. Hence, G is a ∂-open subset of (X_β, u_β). □

Proposition 12. *Let $\{(X_\alpha, u_\alpha) : \alpha\in I\}$ be a family of closure spaces, let $\beta\in I$ and let $F\subseteq X_\beta$. Then F is a ∂-closed subset of (X_β, u_β) if and only if $F\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.*

Proof. Let F be a ∂-closed subset of (X_β, u_β). Then $X_\beta - F$ is a ∂-open subset of (X_β, u_β). By Proposition 11,

$$(X_\beta - F)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = \prod_{\alpha\in I} X_\alpha - F\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$$

is a ∂-open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Hence, $F\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let G be a g-open subset of (X_β, u_β) such that $F\subseteq G$. Then $F\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $F\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is ∂-closed and $G\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-open in $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$,

$$\prod_{\alpha\in I} u_\alpha\pi_\alpha\Big(F\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\beta\Big) \subseteq G\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\,.$$

Consequently, $u_\beta F\subseteq G$. Therefore, F is a ∂-closed subset of (X_β, u_β). □

Proposition 13. *Let $\{(X_\alpha, u_\alpha) : \alpha\in I\}$ be a family of closure spaces. For each $\beta\in I$, let $\pi_\beta\colon \prod_{\alpha\in I} X_\alpha \to X_\beta$ be the projection map. Then*

(i) If F is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$, then $\pi_\beta(F)$ is a ∂-closed subset of (X_β, u_β).

(ii) If F is a ∂-closed subset of (X_β, u_β), then $\pi_\beta^{-1}(F)$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Proof. (i) Let F be a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ and let G be a g-open subset of (X_β, u_β) such that $\pi_\beta(F)\subseteq G$. Then $F\subseteq \pi_\beta^{-1}(G) = G\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since F is

∂-closed and $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-open, $\prod_{\alpha\in I} u_\alpha \pi_\alpha(F) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u_\beta\pi_\beta(F) \subseteq G$. Hence, $\pi_\beta(F)$ is a ∂-closed subset of (X_β, u_β).

(ii) Let F be a ∂-closed subset of (X_β, u_β). Then $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. By Proposition 12, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Therefore, $\pi_\beta^{-1}(F)$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. □

5 $T'_{\frac{1}{2}}$-spaces and $T''_{\frac{1}{2}}$-spaces

As applications of ∂-closed sets, two new kinds of spaces, namely $T'_{\frac{1}{2}}$-spaces and $T''_{\frac{1}{2}}$-spaces, are introduce.

Definition 4. A closure space (X, u) is said to be a $T'_{\frac{1}{2}}$-space if every ∂-closed subset of (X, u) is closed.

Definition 5. A closure space (X, u) is said to be a $T''_{\frac{1}{2}}$-space if every g-closed subset of (X, u) is ∂-closed.

We note that the concepts of a $T'_{\frac{1}{2}}$-space and a $T''_{\frac{1}{2}}$-space are independent as shown in the following examples.

Example 3. Let $X = \{a, b, c, d\}$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{a\} = \{a, c\}$, $u\{b\} = \{b, c\}$, $u\{c\} = u\{d\} = u\{c, d\} = \{c, d\}$ and $u\{a, b\} = u\{a, c\} = u\{a, d\} = u\{b, c\} = u\{b, d\} = u\{a, b, c\} = u\{a, b, d\} = u\{b, c, d\} = uX = X$. Then (X, u) is a $T''_{\frac{1}{2}}$-space. But (X, u) is not a $T'_{\frac{1}{2}}$-space since $\{a, c, d\}$ is ∂-closed but it is not a closed subset of (X, u).

Example 4. Let $X = \{a, b, c\}$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{a\} = \{a\}$, $u\{b\} = \{b\}$, $u\{c\} = \{a, c\}$ and $u\{a, b\} = u\{a, c\} = u\{b, c\} = uX = X$. Then (X, u) is not a $T''_{\frac{1}{2}}$-space since $\{c\}$ is g-closed but it is not a ∂-closed subset of (X, u). However, (X, u) is a $T'_{\frac{1}{2}}$-space.

Example 5. Let $X = \{p, q\}$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{p\} = u\{q\} = uX = X$. Then (X, u) is both a $T'_{\frac{1}{2}}$-space and a $T''_{\frac{1}{2}}$-space.

Proposition 14. *Let (X, u) be a closure space. Then*

(i) If (X, u) is a $T'_{\frac{1}{2}}$-space, then every singleton subset of X is either g-closed or open .

(ii) If every singleton subset of X is a g-closed subset of (X, u), then (X, u) is a $T'_{\frac{1}{2}}$- space.

Proof. (i) Suppose that (X, u) is a $T'_{\frac{1}{2}}$-space. Let $x \in X$ and assume that $\{x\}$ is not g-closed. Then $X - \{x\}$ is not g-open. This implies $X - \{x\}$ is ∂-closed since X is the only g-open set which contains $X - \{x\}$. Since (X, u) is a $T'_{\frac{1}{2}}$-space, $X - \{x\}$ is closed or equivalently, $\{x\}$ is open.

(ii) Let A be a ∂-closed subset of (X, u). Suppose that $x \notin A$. Then $\{x\} \subseteq X - A$ and we have $A \subseteq X - \{x\}$. Since A is ∂-closed and $X - \{x\}$ is g-open, $uA \subseteq X - \{x\}$, i.e., $\{x\} \subseteq X - uA$. Hence, $x \notin uA$ and thus $uA \subseteq A$. Therefore, A is a closed subset of (X, u). Hence, (X, u) is a $T'_{\frac{1}{2}}$-space. □

Proposition 15. *Let (X, u) be a closure space. If (X, u) is a $T''_{\frac{1}{2}}$-space, then every singleton subset of X is either ∂-open or closed.*

Proof. It follows from Proposition 14 (i). □

Clearly, if (X, u) is a $T_{\frac{1}{2}}$-space, then (X, u) is a $T''_{\frac{1}{2}}$-space. The converse need not be true as can be seen from the following example.

Example 6. In example 3, (X, u) is not a $T_{\frac{1}{2}}$-space since $\{a, c, d\}$ is g-closed but it is not closed in (X, u). However, (X, u) is a $T''_{\frac{1}{2}}$-space.

Clearly, if (X, u) is a $T_{\frac{1}{2}}$-space, then (X, u) is a $T'_{\frac{1}{2}}$-space. The converse need not be true as can be seen from the following example.

Example 7. Let $X = \{p, q\}$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{p\} = u\{q\} = uX = X$. Then (X, u) is not a $T_{\frac{1}{2}}$-space since $\{p\}$ is g-closed but it is not closed in (X, u). However, (X, u) is a $T'_{\frac{1}{2}}$-space.

The following statement is evident:

Proposition 16. *Let (X, u) be a closure space. Then (X, u) is a $T_{\frac{1}{2}}$-space if and only if (X, u) is both a $T'_{\frac{1}{2}}$-space and a $T''_{\frac{1}{2}}$-space.*

Proposition 17. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces. Then $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$ is a $T'_{\frac{1}{2}}$-space if and only if (X_α, u_α) is a $T'_{\frac{1}{2}}$-space for each $\alpha \in I$.*

Proof. Suppose that $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$ is a $T'_{\frac{1}{2}}$-space. Let $\beta \in I$ and let F be a ∂-closed subset of (X_β, u_β). Then $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$. Since $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$ is a $T'_{\frac{1}{2}}$-space, $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$. Consequently, F is a closed subset of (X_β, u_β). Hence, (X_β, u_β) is a $T'_{\frac{1}{2}}$-space.

Conversely, suppose that (X_α, u_α) is a $T'_{\frac{1}{2}}$-space for each $\alpha \in I$. Let F be a ∂-closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$ and let $(x_\alpha)_{\alpha \in I} \notin F$. Then there exists $\beta \in I$ such that $x_\beta \notin \pi_\beta(F)$. Since $\pi_\beta(F)$ is ∂-closed and (X_β, u_β) is a $T'_{\frac{1}{2}}$-space, $\pi_\beta(F)$ is a closed subset of (X_β, u_β). Thus, $x_\beta \notin u_\beta\pi_\beta(F)$ implies $(x_\alpha)_{\alpha\in I} \notin \prod_{\alpha\in I} u_\alpha\pi_\alpha(F)$. Therefore, F is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Hence, $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is a $T'_{\frac{1}{2}}$-space. □

Proposition 18. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces. Then $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is a $T_{\frac{1}{2}}$-space if and only if (X_α, u_α) is a $T_{\frac{1}{2}}$-space for each $\alpha \in I$.*

Proof. It follows from Proposition 17. □

Proposition 19. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces. If $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is a $T''_{\frac{1}{2}}$-space, then (X_α, u_α) is a $T''_{\frac{1}{2}}$-space for each $\alpha \in I$.*

Proof. Suppose that $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is a $T''_{\frac{1}{2}}$-space. Let $\beta \in I$ and let F be a g-closed subset of (X_β, u_β). Then $F \times \prod_{\alpha\neq\beta\alpha\in I} X_\alpha$ is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Since $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is a $T''_{\frac{1}{2}}$-space, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Then F is a ∂-closed subset of (X_β, u_β). Hence, (X_β, u_β) is a $T''_{\frac{1}{2}}$-space. □

6 ∂-Continuous Maps

In this section, we investigate a new class of maps called ∂-continuous maps. These maps are defined by the help of g-closed sets and they lie, as for generality, properly between the class of continuous maps and the class of generalized continuous maps. We also introduce the notion of ∂-closed maps and study some of its properties.

Definition 6. Let (X, u) and (Y, v) be closure spaces. A map $f\colon (X, u) \to (Y, v)$ is said to be *∂-continuous* if $f^{-1}(F)$ is a ∂-closed subset of (X, u) for every closed subset F of (Y, v).

Clearly, it is easy to prove that a map $f\colon (X, u) \to (Y, v)$ is ∂-continuous if and only if $f^{-1}(G)$ is a ∂-open subset of (X, u) for every open subset G of (Y, v).

Remark 2. The following implications hold for any map $f\colon (X, u) \to (Y, v)$:

$$f \text{ is continuous} \Rightarrow f \text{ is } \partial\text{-continuous} \Rightarrow f \text{ is g-continuous}.$$

None of these implications is reversible as shown by the following examples.

Example 8. Let $X = \{1, 2\} = Y$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{1\} = \{1\}$ and $u\{2\} = uX = X$. Define a closure operator v on Y by $v\emptyset = \emptyset$, $v\{1\} = \{1\}$, $v\{2\} = \{2\}$ and $vY = Y$. Let $\varphi\colon (X, u) \to (Y, v)$ be defined by $\varphi(1) = \varphi(2) = 1$. Then φ is ∂-continuous but φ is not continuous because $\varphi(u\{2\}) \nsubseteq v\varphi(\{2\})$.

Example 9. Let $X = \{1, 2\} = Y$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{1\} = u\{2\} = uX = X$. Define a closure operator v on Y by $v\emptyset = \emptyset$, $v\{1\} = \{1\}$, $v\{2\} = \{2\}$ and $vY = Y$. Let $\varphi\colon (X, u) \to (Y, v)$ be the identity map. Then φ is g-continuous but φ is not ∂-continuous because $\{1\}$ is a closed subset of (Y, v) but $\varphi^{-1}(\{1\}) = \{1\}$ is not a ∂-closed subset of (X, u).

Proposition 20. *Let (X, u) be a $T''_{\frac{1}{2}}$-space and let (Y, v) be a closure space. If $f\colon (X, u) \to (Y, v)$ is g-continuous, then f is ∂-continuous.*

Proof. Let F be a closed subset of (Y, v). Since f is g-continuous, $f^{-1}(F)$ is a g-closed subset of (X, u). Since (X, u) is a $T''_{\frac{1}{2}}$-space, $f^{-1}(F)$ is a ∂-closed subset of (X, u). Hence, f is ∂-continuous. □

The following statement is obvious:

Proposition 21. *Let (X, u), (Y, v) and (Z, w) be closure spaces. If $f\colon (X, u) \to (Y, v)$ is ∂-continuous and $g\colon (Y, v) \to (Z, w)$ is continuous, then $g \circ f\colon (X, u) \to (Z, w)$ is ∂-continuous.*

Proposition 22. *Let (X, u) and (Z, w) be closure spaces and let (Y, v) be a $T_{\frac{1}{2}}$-space. If $f\colon (X, u) \to (Y, v)$ is g-continuous and $g\colon (Y, v) \to (Z, w)$ is ∂-continuous, then $g \circ f\colon (X, u) \to (Z, w)$ is ∂-continuous.*

Proof. Let F be a closed subset of (Z, w). Since g is g-continuous, $g^{-1}(F)$ is a g-closed subset of (Y, v). Since (Y, v) is a $T_{\frac{1}{2}}$-space, $g^{-1}(F)$ is a closed subset of (Y, v). Since f is ∂-continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is a ∂-closed subset of (X, u). Therefore, $g \circ f$ is ∂-continuous. □

Proposition 23. *Let (X, u) and (Z, w) be closure spaces and let (Y, v) be a $T'_{\frac{1}{2}}$-space. If $f\colon (X, u) \to (Y, v)$ and $g\colon (Y, v) \to (Z, w)$ are ∂-continuous, then $g \circ f\colon (X, u) \to (Z, w)$ is ∂-continuous too.*

Proof. Let F be a closed subset of (Z, w). Since g is ∂-continuous, $g^{-1}(F)$ is a ∂-closed subset of (Y, v). Since (Y, v) is a $T'_{\frac{1}{2}}$-space, $g^{-1}(F)$ is a closed subset of (Y, v) which implies that $(g \circ f)^{-1}(F)$ is a ∂-closed subset of (X, u). Hence, $g \circ f$ is ∂-continuous. □

Proposition 24. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha) : \alpha \in I\}$ be families of closure spaces. For each $\alpha \in I$, let $f_\alpha\colon X_\alpha \to Y_\alpha$ be a map and $f\colon \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be the map defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. If $f\colon \prod_{\alpha \in I} (X_\alpha, u_\alpha) \to \prod_{\alpha \in I} (Y_\alpha, v_\alpha)$ is ∂-continuous, then $f_\alpha\colon (X_\alpha, u_\alpha) \to (Y_\alpha, v_\alpha)$ is ∂-continuous for each $\alpha \in I$.*

Proof. Let $\beta \in I$ and let F be a closed subset of (Y_β, v_β). Then $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} Y_\alpha$ is a closed subset of $\prod_{\alpha \in I} (Y_\alpha, v_\alpha)$. Since f is ∂-continuous,

$$f^{-1}\Big(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} Y_\alpha\Big) = f_\beta^{-1}(F) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$$

is a ∂-closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$. By Proposition 12, $f_\beta^{-1}(F)$ is a ∂-closed subset of (X_β, u_β). Hence, f_β is ∂-continuous. □

Definition 7. Let (X, u) and (Y, v) be closure spaces. A map $f\colon (X, u) \to (Y, v)$ is called *∂-closed* if $f(F)$ is a ∂-closed subset of (Y, v) for every closed subset F of (X, u).

Every closed map is ∂-closed but the converse is not true as may be seen from the following example.

Example 10. Let $X = \{1, 2, 3, 4\} = Y$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{1, 3, 4\} = \{1, 3, 4\}$ and $u\{1\} = u\{2\} = u\{3\} = u\{4\} = u\{1, 2\} = u\{1, 3\} = u\{1, 4\} = u\{2, 3\} = u\{2, 4\} = u\{3, 4\} = u\{1, 2, 3\} = u\{1, 2, 4\} = u\{1, 3, 4\} = u\{2, 3, 4\} = uX = X$. Define a closure operator v on Y by $v\emptyset = \emptyset$, $v\{1\} = \{1, 3\}$, $v\{2\} = \{2, 3\}$, $v\{3\} = v\{4\} = v\{3, 4\} = \{3, 4\}$ and $v\{1, 2\} = v\{1, 3\} = v\{1, 4\} = v\{2, 3\} = v\{2, 4\} = v\{1, 2, 3\} = v\{1, 2, 4\} = v\{1, 3, 4\} = v\{2, 3, 4\} = vY = Y$. Let $f\colon (X, u) \to (Y, v)$ be the identity map. Then f is ∂-closed but it is not closed because $\{1, 3, 4\}$ is a closed subset of (X, u) but $f(\{1, 3, 4\}) = \{1, 3, 4\}$ is not a closed subset of (Y, v).

The following statement is evident:

Proposition 25. *Let (X, u), (Y, v) and (Z, w) be closure spaces, let $f\colon (X, u) \to (Y, v)$ and $g\colon (Y, u) \to (Z, w)$ be maps. Then*

(i) If f is ∂-closed and g is closed, then $g \circ f$ is ∂-closed.

(ii) If $g \circ f$ is ∂-closed and f is continuous and surjective, then g is ∂-closed .

(iii) If $g \circ f$ is closed and g is ∂-continuous and injective, then f is ∂-closed.

Proposition 26. *Let (X, u) and (Y, v) be closure spaces. A map $f\colon (X, u) \to (Y, v)$ is ∂-closed if and only if, for each subset B of Y and each open subset G with $f^{-1}(B) \subseteq G$, there is a ∂-open subset V of (Y, v) such that $B \subseteq V$ and $f^{-1}(V) \subseteq G$.*

Proof. Suppose that f is ∂-closed. Let B be a subset of (Y, v) and G be an open subset of (X, u) such that $f^{-1}(B) \subseteq G$. Then $f(X - G)$ is a ∂-closed subset of (Y, v). Let $V = Y - f(X - G)$. Then V is ∂-open and

$$f^{-1}(V) = f^{-1}(Y - f(X - G)) = X - f^{-1}(f(X - G)) \subseteq X - (X - G) = G\,.$$

Therefore, V is ∂-open, $B \subseteq V$ and $f^{-1}(V) \subseteq G$.

Conversely, suppose that F is a closed subset of (X, u). Then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is open. By hypothesis, there is a ∂-open subset V of (Y, v) such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore, $F \subseteq X - f^{-1}(V)$. Hence,

$$Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$$

implies that $f(F) = Y - V$. Thus $f(F)$ is ∂-closed. Therefore, f is ∂-closed. □

Proposition 27. *Let (X, u) be a closure space and let $\{(Y_\alpha, v_\alpha) : \alpha \in I\}$ be a family of closure spaces. Let $f: X \to \prod_{\alpha \in I} Y_\alpha$ be a map. If $f: (X, u) \to \prod_{\alpha \in I} (Y_\alpha, v_\alpha)$ is ∂-closed, then $\pi_\alpha \circ f: (X, u) \to (Y_\alpha, v_\alpha)$ is ∂-closed for each $\alpha \in I$.*

Proof. Let f be ∂-closed. Since π_α is closed for each $\alpha \in I$, also $\pi_\alpha \circ f$ is ∂-closed for each $\alpha \in I$. □

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