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# Transcendence of Certain Infinite Sums Involving Rational Functions

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**Abstract.** The main result of this paper is the method which shows how to find by a simple technique the sum of the infinite series of the form

$$\sum_{n=0}^{\infty} \frac{\alpha^n P(n)}{Q(n)}$$

where $P(n)$ is a polynomial with algebraic coefficients, $Q(n)$ is reduced polynomial with integer coefficients and $\alpha \in \overline{\mathbb{Q}}^*$. The connection between algebraic and transcendental infinite series and linear forms of logarithms via Baker's theorem is included.

## 1. Introduction

Recently Tijdeman proved several interesting criteria concerning the irrationality of the infinite series of the type $\sum_{n=1}^{\infty} \frac{P(n)}{\prod_{j=1}^{n} Q(j)}$ where $P(x)$ and $Q(x)$ are polynomials with integer coefficients. His results can be found in [2], [3], [4], [5] or [14]. In 1966 Baker proved the following theorem about the transcendence of linear form in logarithms over $\overline{\mathbb{Q}}$.

**Theorem 1.1 (Baker** [7]**).** *Let $\alpha_1, \dots, \alpha_m \in \overline{\mathbb{Q}}$ and $\prod_{j=1}^{m} \alpha_j \neq 0$, $\beta_1, \dots, \beta_m \in \overline{\mathbb{Q}}$. Then the number*

$$\beta_1 \ln \alpha_1 + \cdots + \beta_m \ln \alpha_m$$

*is transcendental or zero.*

This theorem was used several times by many authors, e.g. in [1] or [9]. The resultst in the mentioned papers [1] and [9] are further based on the concept of the numbers $\gamma(r, k)$ and a theorem presented by D. H. Lehmer in [6]. Lehmer analysed the infite series of the form $\sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$ for a special choice of the polynomial $Q(n) \in \mathbb{Q}[n]$ with help of the numbers $\gamma(r, k)$ which are a kind of the generalization of the Euler-Mascheroni constant $\gamma$ as follows.

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**Definition 1.1 (Lehmer [6]).** *The Euler constants $\gamma(r,k)$ are defined by*

$$\gamma(r,k) := \lim_{x\to\infty}\left(H_x(r,k) - \frac{1}{k}\log x\right)$$

*where*

$$H_x(r,k) := \sum_{\substack{0<i\le x\\ i\equiv r \pmod k}} \frac{1}{i}.$$

The following theorem can be treated as the main tool for the investigation of the infinite series $\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}$ in [1].

**Theorem 1.2 (Lehmer [6]).** *Let $m\ge 2$ and let*

$$(r_1,k_1),\ (r_2,k_2),\dots,\ (r_m,k_m)$$

*be pairs of positive integers for which $0<r_j\le k_j$ for $j=1,\dots,m$ and the $m$ rational numbers $r_j/k_j$ are distinct. Finally let $p(x)$ be any polynomial of degree $\le m-2$, a necessary condition for convergence. Then*

$$S=\sum_{n=0}^{\infty}\frac{p(n)}{(k_1n+r_1)(k_2n+r_2)\cdots(k_mn+r_m)}=\sum_{j=1}^{m}c_j\left(\gamma(r_j,k_j)+\frac{\log k_j}{k_j}\right),$$

*where the coefficients $c_j$ are defined by the partial fraction decomposition*

$$\frac{p(x)}{(k_1x+r_1)(k_2x+r_2)\cdots(k_mx+r_m)}=\sum_{j=1}^{m}\frac{c_j}{k_jx+r_j}. \tag{1}$$

The results in [1] and [13] involve the following convenient concept.

**Definition 1.2.** *We call the polynomial $Q(x)$ reduced if $Q(x)\in\mathbb{Q}[x]$ and it has only simple rational zeros which are all in the interval $[-1,0)$.*

Adhikari, Saradha, Shorey and Tijdeman proved in [1] the following theorem.

**Theorem 1.3 (Adhikari, Saradha, Shorey, Tijdeman [1]).** *Let $P(x)\in\overline{\mathbb{Q}}[x]$. Let $Q(x)\in\mathbb{Q}[x]$ be reduced. If*

$$S=\sum_{n=0}^{\infty}\frac{P(n)}{Q(n)}$$

*converges, then $S=0$ or $S\notin\overline{\mathbb{Q}}$.*

More general are the following theorems.

**Theorem 1.4 (Adhikari, Saradha, Shorey, Tijdeman [1]).** *Let $f:\mathbb{Z}\mapsto\overline{\mathbb{Q}}$ be periodic* mod $q$. *Let $Q(x)\in\mathbb{Q}[x]$ be reduced. If*

$$S=\sum_{n=0}^{\infty}\frac{f(n)}{Q(n)}$$

*converges, then $S=0$ or $S\notin\overline{\mathbb{Q}}$.*

**Theorem 1.5 (Adhikari, Saradha, Shorey, Tijdeman [1]).** *Let $P_1(x), \ldots, P_\ell(x) \in \overline{\mathbb{Q}}[x]$ and $\alpha_1, \ldots, \alpha_\ell \in \overline{\mathbb{Q}}$. Put $g(x) = \sum_{\lambda=1}^{\ell} P_\lambda(x)\alpha_\lambda^x$. Let $Q(x) \in \mathbb{Q}[x]$ be reduced. If*

$$S = \sum_{n=0}^{\infty} \frac{g(n)}{Q(n)}$$

*converges, then $S = 0$ or $S \notin \overline{\mathbb{Q}}$.*

But Theorem 1.5 presented in [1] involves an error. Namely, for the choice $g(n) = \alpha_1^n Q(n)$, $Q(n)$ be a reduced polynomial, and $\alpha_1 \in \mathbb{Q} \setminus \{0\}$, $|\alpha_1| < 1$, we deduce the fact that

$$S = \sum_{n=0}^{\infty} \frac{g(n)}{Q(n)} = \sum_{n=0}^{\infty} \frac{\alpha_1^n Q(n)}{Q(n)} = \frac{1}{1-\alpha_1^{-1}} \in \mathbb{Q} \setminus \{0\}$$

which contradicts the statement in Theorem 1.5.

## 2. Main Results

This section presents very similar results as in [1] and [6]. But the method explored in the section Proofs is more elementar. Actually, we use only the integration and other simple techniques.

In the next theorem there is defined a general term $\Psi_w(x)$. A reccursion formula for $\Psi_w(x)$ is found.

**Theorem 2.1.** *Let $(a, w) \in \mathbb{N}^2$, $a \geq 2$ and $\{b_v\}_{v=1}^{w}$ be an increasing sequence of dicstinct positive integers. Assume that $b_v \leq a$ for all $v = 1, \ldots, w$ and $|x| < 1$. Denote*

$$\Psi_w(x) := \sum_{n=0}^{\infty} \frac{x^{an+b_w}}{\prod_{v=1}^{w}(an+b_v)}.$$

*If $\Psi_w(x)$ converges, then*

$$\Psi_w(x) = \sum_{j=1}^{w-1} \frac{(-1)^{j+1} x^{b_w - b_{w-j}}}{\prod_{s=1}^{j}(b_w - b_{w-s})} \Psi_{w-j}(x) + \frac{(-1)^{w+1}}{\prod_{s=1}^{w-1}(b_w - b_{w-s})} \cdot \int_0^x \frac{y^{b_w-1}}{1-y^a}\, \mathrm{d}y \tag{2}$$

*and $\Psi_w(x)$ can be expressed as a linear form in logarithms.*

**Remark 2.1.** *From Definition 1.2 we see that the polynomial $\prod_{v=1}^{w}(a_v n + b_v)$, $0 < \frac{a_v}{b_v} \leq 1$, is reduced.*

A more genaral case is treated in the proof of the next theorem.

**Theorem 2.2.** *Let $(a, w) \in \mathbb{N}^2$, $a \geq 2$, $\{b_v\}_{v=1}^{w}$ be an increasing sequence of distinct positive integers and let $b_v \leq a$ for all $v = 1, \ldots, w$. Suppose that $P(X) \in \overline{\mathbb{Q}}[X]$ be a polynomial and $\alpha \in \overline{\mathbb{Q}}^*$. If*

(1) *$|\alpha| > 1$, then the sum of the series*

$$S_{P(X)} := \sum_{n=0}^{\infty} \frac{P(n)}{\alpha^n \prod_{v=1}^{w}(an+b_v)} \tag{3}$$

*is a transcendental or a computable algebraic number;*

(2) $|\alpha| = 1, \deg P(n) < w+1$, *then the sum of the series* (3) *is a transcendental number or zero.*

**Example 2.1.**

$$\sum_{n=0}^{+\infty} \frac{n^2}{2^n(2n+1)} = \frac{\sqrt{2}}{4} \cdot \ln\left(1+\sqrt{2}\right) + \frac{1}{2}.$$

**Example 2.2.** *Let $a \in \mathbb{N}$, $a \geq 2$, $(b_1, b_2) \in \mathbb{N}^2$ and $a > b_2 > b_1$. Then*

$$\sum_{n=0}^{\infty} \frac{1}{(an+b_1)(an+b_2)} = \frac{1}{b_2 - b_1} \int_0^1 \frac{y^{b_1-1} - y^{b_2-1}}{1-y^a}\, \mathrm{d}y.$$

**Example 2.3.**

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(4n+1)} = 2\int_0^1 \frac{1-y}{1-y^4}\, \mathrm{d}y = \frac{1}{2}\ln 2 + \frac{1}{4}\pi.$$

**Example 2.4.**

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(3n+1)(4n+1)} = 72\int_0^1 \frac{\frac{y^2-y^3}{2} - \frac{y^2-y^5}{6}}{1-y^{12}}\, \mathrm{d}y =$$

$$= \frac{\sqrt{3}}{2}\pi - \frac{9}{2}\ln 3 + 8\ln 2 + \pi.$$

**Example 2.5.** *Let $p_j$, $j \in \mathbb{N}$, denotes the $j^{\text{th}}$ prime. Then*

$$\sum_{n=0}^{\infty} \prod_{j=1}^{4} \frac{1}{p_j n + 1} = \int_0^1 \frac{224y^{104} - 945y^{69} + 1750y^{41} - 1029y^{29}}{4(y^{210}-1)}\, \mathrm{d}y.$$

**Corollary 2.1.** *Let $(a, w) \in \mathbb{N}^2$, $a \geq 2$, $\{b_v\}_{v=1}^{w}$ be an increasing sequence distinct positive integers with $b_v/a \leq 1$ for all $v = 1, \ldots, w$. Assume that $P(X) \in \overline{\mathbb{Q}}[X]$ be a polynomial such that $\deg P(X) < w+1$. Then the sum of the series*

$$\sum_{n=0}^{\infty} \frac{P(n)}{\prod_{v=1}^{w}(an+b_v)}$$

*is a transcendental number or zero.*

**Corollary 2.2.** *Let $(a, w) \in \mathbb{N}^2$, $a \geq 2$, $\{b_v\}_{v=1}^{w}$ be an increasing sequence of distinct positive integers with $b_v/a \leq 1$ for all $v = 1, \ldots, w$. Assume that $P(X) \in \overline{\mathbb{Q}}[X]$ be a polynomial and $\kappa(n) : \mathbb{N}_0 \mapsto \overline{\mathbb{Q}}$ be a periodic function. If the series*

$$\sum_{n=0}^{\infty} \frac{\kappa(n) \cdot P(n)}{\alpha^n \prod_{v=1}^{w}(an+b_v)} \tag{4}$$

*converges, then the series* (4) *is a transcendental number or a computable algebraic number.*

## 3. Proofs

**Lemma 3.1.** *Let* $(a_v, w) \in \mathbb{N}^2$ *for* $\forall v = 1, \ldots, w$, $w \geq 2$ *and*

$$0 < \frac{a_v}{b_v} \leq 1.$$

*Put*

$$A := \operatorname{lcm}(a_1, \ldots, a_w), \quad B'_v := b_v \cdot \frac{\operatorname{lcm}(a_1, \ldots, a_w)}{a_v} \quad \text{and} \quad K := \frac{A^w}{\prod_{v=1}^{w} a_v}.$$

*Then*

$$\sum_{n=0}^{\infty} \frac{1}{\prod_{v=1}^{w}(a_v n + b_v)} = K \cdot \sum_{n=0}^{\infty} \frac{1}{\prod_{v=1}^{w}(An + B'_v)}$$

*with*

$$(A, B'_v)_{v=1,\ldots,w} \in \mathbb{N}^2 \qquad \text{and} \qquad 0 < \frac{A}{B'_v} \leq 1.$$

*Proof.*

$$\sum_{n=0}^{\infty} \frac{1}{\prod_{v=1}^{w}(a_v n + b_v)} = \frac{\operatorname{lcm}^w(a_1, \ldots, a_w)}{\prod_{v=1}^{w} a_v} \times$$

$$\times \sum_{n=0}^{\infty} \frac{1}{\prod_{v=1}^{w}\left(a_v \cdot \frac{\operatorname{lcm}(a_1,\ldots,a_w)}{a_v} n + b_v \cdot \frac{\operatorname{lcm}(a_1,\ldots,a_w)}{a_v}\right)} =$$

$$= K \cdot \sum_{n=0}^{\infty} \frac{1}{\prod_{v=1}^{w}(An + B'_v)}.$$

From this we obtain immediately the fact that

$$(A, B'_v)_{v=1,\ldots,w} \in \mathbb{N}^2 \qquad \text{and} \qquad 0 < \frac{A}{B'_v} \leq 1.$$

□

**Remark 3.1.** *Denote*

$$\mathcal{B} := \{B'_j | j = 1, \ldots, w \geq 2, B'_j\}$$

*and let* $\phi$ *be a bijection*

$$\phi : \mathcal{B} \mapsto \mathcal{B}$$

*such that the sequence* $\{\phi(B'_j)\}_{j=1}^{w}$ *is increasing. Then we define*

$$\{B_j\}_{j=1}^{w} := \{\phi(B'_j)\}_{j=1}^{w}.$$

*Proof of Theorem 2.1.* (1) For $w = 1$ we have

$$\Psi'_1(x) = \sum_{n=0}^{\infty} x^{an+b_1-1} = \frac{x^{b_1-1}}{1-x^a} = \frac{(-1)^{a+1} x^{b_1-1}}{\prod_{s=1}^{a}(\zeta_s - x)} = (-1)^{a+1} \sum_{s=1}^{a} \frac{\gamma_s}{\zeta_s - x}, \tag{5}$$

where the coefficients $\gamma_s \in \overline{\mathbb{Q}}$ and $\zeta_s \in \overline{\mathbb{Q}}$ are defined by the decomposition (5) into partial fractions. Using (5) we deduce

$$\Psi_1(x) = \int_0^x f'(y)\,\mathrm{d}y = \int_0^x (-1)^{a+1} \sum_{s=1}^{a} \frac{\gamma_s}{\zeta_s - y} \mathrm{d}y = \\ = (-1)^a \sum_{s=1}^{a} \gamma_s \ln(\zeta_s - x) + (-1)^{a+1} \sum_{s=1}^{a} \gamma_s \ln \zeta_s .$$

The proof for $w = 1$ is complete.

(2) Now, assume that $w \geq 2$. Applying $(w-1)$-times integration by parts, we obtain (2). By view of the fact that

$$\Psi'_w(x) = x^{b_w - b_{w-1} - 1} \Psi_{w-1}(x)$$

we have

$$\begin{aligned} \Psi_w(x) &= \int_0^x y^{b_w - b_{w-1} - 1} \Psi_{w-1}(y)\,\mathrm{d}y = \\ &= \frac{x^{b_w - b_{w-1}}}{b_w - b_{w-1}} \Psi_{w-1}(x) - \frac{1}{b_w - b_{w-1}} \int_0^x y^{b_w - b_{w-2} - 1} \Psi_{w-2}(y)\,\mathrm{d}y = \\ &= \cdots = \sum_{j=1}^{w-1} (-1)^{j+1} \cdot \frac{x^{b_w - b_{w-j}}}{\prod_{s=1}^{j} (b_w - b_{w-s})} \Psi_{w-j}(x) + \\ &+ \frac{(-1)^{w+1}}{\prod_{s=1}^{w-1} (b_w - b_{w-s})} \cdot \int_0^x \frac{y^{b_w - 1}}{1 - y^a}\,\mathrm{d}y \end{aligned}$$

and (2) follows. The fact that $\Psi_1(x)$ can be expressed as the linear form in logarithms and (2) imply that $\Psi_w(x)$ for $w \geq 2$ can be also expressed as the linear form in logarithms.

□

The following Lemma 3.2 is very known. We omit the proof.

**Lemma 3.2.** *Let $k \in \mathbb{N}_0$, $|x| < 1$. Then*

$$\sum_{n=0}^{\infty} n^k x^n = \sum_{r=1}^{k+1} \frac{a_r}{(1-x)^r}$$

*with $a_r \in \mathbb{Z}$ for all $r = 1, \ldots, k+1$ and $\sum_{r=1}^{k+1} a_r = 0$ for $k \geq 1$.*

*Proof of Theorem 2.2.* (1) There exist the polynomials $\Gamma_1(n)$, $\Gamma_2(n) \in \overline{\mathbb{Q}}[n]$ such that

$$\frac{P(n)}{\prod_{v=1}^{w} (an + b_v)} = \Gamma_1(n) + \frac{\Gamma_2(n)}{\prod_{v=1}^{w} (an + b_v)}$$

with $t := \deg \Gamma_2(n) < w + 1$. This and Lemma 3.2 yield

$$\begin{aligned}
S_{P(X)} :&= \sum_{n=0}^{\infty} \frac{P(n)}{\alpha^n \prod_{v=1}^{w}(an+b_v)} = \\
&= \sum_{n=0}^{\infty} \frac{\Gamma_1(n)}{\alpha^n} + \sum_{n=0}^{\infty} \frac{\Gamma_2(n)}{\alpha^n \prod_{v=1}^{w}(an+b_v)} = \\
&= \sum_{j=0}^{J} \sum_{n=0}^{\infty} \frac{\gamma_j n^j}{\alpha^n} + \sum_{n=0}^{\infty} \sum_{i=0}^{t} \sum_{p=0}^{i} \beta_{i,p} \frac{\prod_{v=w-i+p+1}^{w}(an+b_v)}{\alpha^n \prod_{v=1}^{w}(an+b_v)} = \\
&= \sum_{j=1}^{J+1} \frac{\delta_j}{(1-\alpha^{-1})^j} + \sum_{i=0}^{t} \sum_{p=0}^{i} \beta_{i,p} \sum_{n=0}^{\infty} \frac{1}{\alpha^n \prod_{v=1}^{w-i+p}(an+b_v)} = \\
&= \sum_{j=1}^{J+1} \frac{\delta_j}{(1-\alpha^{-1})^j} + \sum_{i=0}^{t} \sum_{p=0}^{i} \beta_{i,p} \alpha^{-b_{w-i+p}/a} \Psi_{w-i+p}\left(\alpha^{-1/a}\right)
\end{aligned}$$

where $\beta_{i,p}$, $\gamma_j$ and $\delta_j \in \overline{\mathbb{Q}}$ for all possible idexes $i, j$ and $p$. Theorem 2.1 and Theorem 1.1 complete the proof of the first part of Theorem 2.2.

(2) Assume that $t := \deg P(n) < w + 1$. We can write

$$\sum_{n=0}^{\infty} \frac{P(n)}{\alpha^n \prod_{v=1}^{w}(an+b_v)} = \sum_{i=0}^{t} \sum_{p=0}^{i} \sum_{n=0}^{\infty} \frac{\beta_{i,p}}{\alpha^n \prod_{v=1}^{w-i+p}(an+b_v)}$$

where $\beta_{i,p} \in \overline{\mathbb{Q}}$ for all possible indexes $i$ and $p$.

(a) If $\alpha \in \{x |\, x^a = 1\}$, then the fact that

$$\lim_{\substack{x \to y \\ (x,y) \in \mathbb{C}^2}} (x-y)\ln(x-y) = 0, \tag{6}$$

Theorem 2.1 and Theorem 1.1 complete the proof.

(b) If $\alpha \notin \{x |\, x^a = 1\}$, then Theorem 2.1 and Theorem 1.1 complete the proof.

□

**Remark 3.2.** *If the infinite series* $\Psi^\star(\alpha_0) := \sum_{n=0}^{\infty} \frac{\alpha_0^n P(n)}{Q(n)}$ *converges for a reduced polynomial* $Q(n)$ *and* $\alpha_0 \in \mathbb{C} \setminus \{0\}$, $|\alpha| = 1$ *then* $\Psi^\star(\alpha_0) = \lim_{\substack{x \to \alpha_0 \\ |x|<1}} \Psi(x)$ *where* $\Psi(x)$ *is defined in Theorem 2.1.*

*Proof of Corollary 2.1.* Use Remark 3.2 and take $\alpha \to 1^-$ in Theorem 2.2. □

*Proof of Corollary 2.2.* Suppose that $\kappa(n)$ is a periodic function $\bmod M$, $M \in \mathbb{N}$. Then for those $\alpha \in \overline{\mathbb{Q}}$ for which $|\alpha| < 1$, we can write

$$\sum_{n=0}^{\infty} \frac{\kappa(n) \cdot P(n)}{\alpha^n \prod_{v=1}^{w}(an+b_v)} = \sum_{\mu=0}^{M-1} \sum_{\substack{n=0 \\ n \equiv \mu \bmod M}}^{\infty} \frac{z_\mu P(n)}{\alpha^n \prod_{v=1}^{w}(an+b_v)}$$

where $z_\mu \in \overline{\mathbb{Q}}$. Now apply Theorem 2.2 $M$-times to every infinite series

$$S_\mu := \sum_{\substack{n=0 \\ n \equiv \mu \bmod M}}^{\infty} \frac{z_\mu P(n)}{\alpha^n \prod_{v=1}^{w}(an+b_v)}, \qquad \mu = 0, 1, \cdots, M-1.$$

The finite summation $\sum_{\mu=0}^{M-1} S_\mu$ gives the statement in Corollary 2.2 for any algebraic $\alpha$, $|\alpha| < 1$.

If the infinite series in Corollary 2.2 converges and $|\alpha| = 1$ then we can use the definition of $\Psi^\star(\alpha)$, the formula (6) and Theorem 2.1. □

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