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Reducibility of a special symmetric form

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Abstract. Irreducibility over $\mathbb{C}$ of a special symmetric form in a variables is proved for $n > 3$.

During the XVIIth Czech and Slovak International Conference on Number Theory A. Sładek has proposed the problem for which values $k \geq 2$, $n \geq 3$ the form

$$F_{k,n} = \prod_{i=1}^{n} x_i^k + \left(- \sum_{i=1}^{n} x_i \right)^k \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} x_j^k$$

is reducible over $\mathbb{C}$.

The following theorem gives a partial answer.

Theorem. *If $n > 3$, $F_{k,n}$ is irreducible over $\mathbb{C}$.*

In the proof, based on three lemmas we shall denote by $\tau_i(x_1, \dots, x_m)$ the i-th elementary symmetric polynomial of $x_1, \dots, x_m$ and set $\tau_i = \tau_i(x_1, \dots, x_n)$, $\tau_i' = \tau_i(x_1, \dots, x_{n-1})$.

Lemma 1. *For all $k \geq 1$ and all $n \geq 3$ the form*

$$A_{k,n} = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} x_j^k$$

is irreducible over $\mathbb{C}$.

Proof. We proceed by induction on n. For $n = 3$ we have

$$A_{k,3} = \left(x_1^k + x_2^k \right) x_3^k + x_1^k x_2^k.$$

Since $(x_1^k + x_2^k, x_1^k x_2^k) = 1$ reducibility of $A_{k,3}$ over $\mathbb{C}$ implies that $A_{k,3}$ viewed as a polynomial of x_3 is reducible over $\mathbb{C}(x_1, x_2)$, hence by Capelli's theorem (see [2], p. 662) $x_1^k + x_2^k$ is in $\mathbb{C}[x_1, x_2]$ a power with exponent $e > 1$ dividing k, a contradiction.

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Assume now that the lemma is true for $n-1$ variables ($n \geq 4$). We have

$$A_{k,n} = A_{k,n-1}x_n^k + \prod_{j=1}^{n-1} x_j^k.$$

By the inductive assumption $A_{k,n-1}$ is irreducible over $\mathbb{C}$, hence it is prime to $\prod_{j=1}^{n-1} x_j^k$ and is not a power with exponent greater than 1 in $\mathbb{C}[x_1,\dots,x_{n-1}]$. Hence, by Capelli's theorem $A_{k,n}$ is irreducible over $\mathbb{C}$.

Lemma 2. *For all positive integers k and n*

$$A_{k,n} = \sum (-1)^{k+\lambda_1+\ldots+\lambda_n} \frac{(\lambda_1+\ldots+\lambda_n-1)!k}{\lambda_1!\lambda_2!,\ldots,\lambda_n!} \tau_n^{k-\lambda_1-\ldots-\lambda_n}\tau_{n-1}^{\lambda_1}\cdot\ldots\cdot\tau_1^{\lambda_{n-1}},$$

where non-negative integers $\lambda_1,\ldots,\lambda_n$ satisfy $\lambda_1+2\lambda_2+\ldots+n\lambda_n = k$.

Proof. We have

$$A_{k,n} = \tau_n^k \sum_{i=1}^n x_i^{-k}$$

and it suffices to apply the formula (see [1], p. 155)

$$\sum_{i=1}^n x_i^{-k} = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=k} (-1)^{\lambda_1+\ldots+\lambda_n} \frac{(\lambda_1+\ldots+\lambda_n-1)!k}{\lambda_1!\cdot\ldots\cdot\lambda_n!} \frac{a_{n-1}^{\lambda_1}\cdot\ldots\cdot a_1^{\lambda_{n-1}}}{a_n^{\lambda_1+\ldots+\lambda_n}},$$

where $a_i = (-1)^i\tau_i$.

Lemma 3. *If $f \in \mathbb{C}[x_1,\ldots,x_n] \setminus \{0\}$ is a symmetric form of degree equal to the common degree d with respect to each variable, then*

$$f = a\tau_1^d + \sum\nolimits_1 c_{\delta_1,\ldots,\delta_n} \prod_{i=1}^n \tau_i^{\delta_i},$$

where $a \in \mathbb{C}^*$, $c_{\delta_1,\ldots,\delta_n} \in \mathbb{C}$ and the sum $\sum_1$ is taken over all non-negative integers $\delta_1,\ldots,\delta_n$ with $\delta_1+\delta_2+\ldots+\delta_n < d$, $\delta_1+2\delta_2+\ldots+n\delta_n = d$.

Proof. Since f is a symmetric form it equals $F(\tau_1,\ldots,\tau_n)$, where $F \in K[y_1,\ldots,y_n]\setminus 0$ is isobaric with respect to the common weight w of monomials of F and the common degree d of $F(\tau_1,\ldots,\tau_n)$ with respect to each variable x_i equals degree of F. Let M be a monomial of F of degree d,

$$M = a\prod_{i=1}^n y_i^{\alpha_i}.$$

We have

$$w = \sum_{i=1}^n i\alpha_i, \quad d = \sum_{i=1}^n \alpha_i$$

and the equality $w = d$ gives $\alpha_2 = \ldots = \alpha_n = 0$, $M = ay_1^d$. Hence

$$F = ay_1^d + \sum\nolimits_1 c_{\delta_1,\ldots,\delta_n} \prod_{i=1}^n y_i^{\delta_i},$$

which implies the lemma.

Proof of Theorem. By Lemma 1 at least one irreducible factor of $F_{k,n}$ viewed as a polynomial in x_n has the leading coefficient $A_{k,n-1}$. Let us call this factor f_1 and the complementary factor, assumed not constant, f_2. If for at least one transposition $\tau \in S_n$ we have $f_1^\tau / f_1 \notin \mathbb{C}$, then since $F_{k,n}^\tau = F_{k,n}$ we obtain

$$f_1 f_1^\tau \mid F_{k,n},$$

hence

$$2(n-2)k \leq 2 \deg f_1 \leq \deg F_{k,n} = kn;$$

$2(n-2) \leq n,\ n \leq 4,\ \deg f_1 = 2k,\ f_1 = A_{k,n-1}$ and

$$A_{k,n-1} \mid F_{k,n}\,(x_1, \ldots, x_{n-1}, 0) = {\tau'}_1^k {\tau'}_{n-1}^k,$$

which contradicts irreducibility of $A_{k,n-1}$. Therefore $f_1^\tau / f_1 \in \mathbb{C}$ for all transpositions $\tau \in S_n$. If for a transposition $\tau = (ij)$ we have $f_1^\tau = c f_1$, $c \neq 1$, then $\tau^2 = id$, $c^2 = 1$ implies $c = -1$ and since $f_1^\tau \equiv f_1 (\mathrm{mod}\ x_i - x_j)$, it follows that $x_i - x_j \mid f_1$, $f_1 = a(x_i - x_j)$, contrary to the choice of f_1. Therefore, $f_1^\tau = f_1$ for all transpositions $\tau \in S_n$ and since S_n is generated by transpositions, $f_1^\sigma = f_1$ for all $\sigma \in S_n$. Since $F_{k,n}^\sigma = F_{k,n}$ we have also $f_2^\sigma = f_2$, thus f_2 is a symmetric form,

$$f_\nu = F_\nu\,(\tau_1, \ldots, \tau_n) \qquad (\nu = 1, 2).$$

It follows now from Lemma 2 and the algebraic independence of $\tau_1, \ldots, \tau_n$ that

$$F_0 = F_1 F_2,$$

where

$$F_0 = y_n^k + y_1^k \sum\nolimits_2 (-1)^{\lambda_1 + \ldots + \lambda_n} \frac{(\lambda_1 + \ldots + \lambda_n - 1)!k}{\lambda_1! \ldots \lambda_n!} y_n^{k - \lambda_1 - \ldots - \lambda_n} y_{n-1}^{\lambda_1} \cdot \ldots \cdot y_1^{\lambda_{n-1}}$$

and the sum $\sum_2$ is taken over all nonnegative integers $\lambda_1, \ldots, \lambda_n$ with $\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n = k$.

On the other hand, f_2 as a factor of the form $F_{k,n}$ is itself a form and

$$\deg f_2 = \deg F_{k,n} - \deg A_{k,n-1} - \deg_{x_n} f_1 = 2k - \deg_{x_n} f_1 =$$

$$\deg_{x_n} F_{k,n} - \deg_{x_n} f_1 = \deg_{x_n} f_2,$$

hence, by Lemma 3

$$(*) \qquad F_2 = a y_1^d + \sum\nolimits_1 c_{\delta_1, \ldots, \delta_n} \prod_{i=1}^n y_i^{\delta_i}, \quad a \in C^*.$$

We have

$$F_2\,(y_1, \ldots, y_{n-1}, 0) \mid F_0\,(y_1, \ldots, y_{n-1}, 0) = y_1^k y_{n-1}^k,$$

thus

$$F_2\,(y_1, \ldots, y_{n-1}, 0) = b y_1^\alpha y_{n-1}^\beta, \quad b \in \mathbb{C}^*$$

and, by $(*)$

$$F_2\,(y_1, \ldots, y_{n-1}, 0) = a y_1^d.$$

If F_2 depends on y_{n-1} it follows that its leading coefficient with respect to y_{n-1} is divisible by y_n. However the leading coefficient of F_0 with respect to y_{n-1} is $(-1)^k y_1^k$, not divisible by y_n. Therefore, F_2 does not depend on y_{n-1} and it divides the leading coefficient of F_0 with respect to y_{n-1}, thus we obtain

$$F_2 \mid y_1^k, \quad y_1 \mid F_2,$$

$$y_1 \mid F_0, \quad y_1 \mid y_n^k.$$

The obtained contradiction completes the proof.

Remarks.

(1) In the theorem and the proof $\mathbb{C}$ can be replaced by any field K of characteristic not dividing k and the monomial $\prod_{i=1}^{n} x_i^k$ by any polynomial $F(\tau_1, \dots, \tau_n)$, where $F \in K[y_1, \dots, y_n]$ and
  1) F is isobaric of weight kn,
  2) degree $F < 2k$,
  3) $\deg_{y_{n-1}} F < k$,
  4) $F \not\equiv 0 \bmod y_1$, $F \equiv 0 \bmod y_n$.

(2) The condition $n > 3$ cannot be omitted in the theorem, since for k odd $F_{k,3}$ is reducible, divisible by $x_1 + x_2$ (this remark has also been made by A. Sładek).

References

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