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# Lucas balancing numbers

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**Abstract.** A positive $n$ is called a balancing number if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r).$$

We prove that there is no balancing number which is a term of the Lucas sequence.

## 1. Introduction

The sequence $\{R_n\}_{n=0}^{\infty}=R(A,B,R_0,R_1)$ is called a second order linear recurrence if the recurrence relation

$$R_n=AR_{n-1}+BR_{n-2}\qquad(n>1)$$

holds for its terms, where $A$, $B\neq 0$, $R_0$ and $R_1$ are fixed rational integers and $|R_0|+|R_1|>0$. The polynomial $x^2-Ax-B$ is called the companion polynomial of the second order linear recurrence sequence $R=R(A,B,R_0,R_1)$. The zeros of the companion polynomial will be denoted by $\alpha$ and $\beta$. Using this notation, as it is well known, we get

$$R_n=\frac{a\alpha^n-b\beta^n}{\alpha-\beta},$$

where $a=R_1-R_0\beta$ and $b=R_1-R_0\alpha$ (see [6]).

A positive integer $n$ is called a *balancing number* [3] if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

for some $r\in\mathbb{N}$. Here $r$ is called the *balancer* corresponding to the balancing number $n$. For example 6 and 35 are balancing numbers with balancers 2 and 14. In a joint paper A. Behera and G. K. Panda [3] proved that that the balancing numbers fulfil the following recurrence relation

$$B_{n+1}=6B_n-B_{n-1}\quad(n>1)$$

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where $B_0 = 1$ and $B_1 = 6$. In [5] we proved that there is no Fibonacci balancing number. We call a balancing number *Lucas balancing number* if it is a Lucas number, too. In the next section we prove that there are no Lucas balancing numbers. In the prove we use the same method as in [5]. Using an other method Szalay in [7] got the same result.

## 2. Lucas balancing numbers

In [5] we proved that the balancing numbers are solutions of a Pell's equation. We proved the following theorem.

**Theorem 1.** *The terms of the second order linear recurrence $B(6, -1, 1, 6)$ are the solutions of the equation*

$$z^2 - 8y^2 = 1 \tag{1}$$

*for some integer $z$.*

In the proof of our main result we need the following theorem of P. E. Ferguson [4].

**Theorem 2.** *The only solutions of the equation*

$$y^2 - 5x^2 = \pm 4 \tag{2}$$

*are $y = \pm L_n, x = \pm F_n$ $(n = 0, 1, 2, \dots)$, where $L_n$ and $F_n$ are the $n^{th}$ terms of the Lucas and Fibonacci sequences, respectively.*

Next we use the method of A. Baker and H. Davenport and we prove that there are finitely many common solutions of the Pell's equations (1) and (2). In the proof we use the following theorem of A. Baker and H. Wüstholz [2].

**Theorem 3.** *Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers not 0 or 1, and let*

$$\Lambda = b_1 \log \alpha_1 + \dots + b_n \log \alpha_n,$$

*where $b_1, \dots, b_n$ are rational integers not all zeroes.*

*We suppose that $B = max(|b_1|, \dots, |b_k|, e)$ and $A_i = max\{(H(\alpha_i), e\}$ $(i = 1, 2, \dots n)$. Assume that the field $K$ generated by $\alpha_1, \alpha_2, \dots \alpha_n$ over the rationals has degree at most $d$. If $\Lambda \neq 0$ then*

$$\log |\Lambda| > -(16nd)^{2(n+2)} \log A_1 \log A_2 \dots \log A_n \log B.$$

*($H(\alpha)$ is equal to the maximum of absolute values of the coefficients of the minimal defining polynomial of $\alpha$.)*

The following theorem is the main result of this paper.

**Theorem 4.** *There is no Lucas balancing number.*

*Proof.* First we show that there are finitely many common solutions of the equations (1) and (2). The general solution of equation (1) is given by

$$z + \sqrt{8}y = (3 + \sqrt{8})^n \quad n = 0, 1, 2, \dots \tag{3}$$

The equations (2) can be written as

$$(y + x\sqrt{5})(y - x\sqrt{5}) = 4 \tag{4}$$

and

$$(y + x\sqrt{5})(y - x\sqrt{5}) = -4. \tag{5}$$

If we put

$$y + x\sqrt{5} = (y_0 + x_0\sqrt{5})(9 + 4\sqrt{5})^m$$

where $m \geq 0$, it is easily verified (by combining this equation with its conjugate) that $y_0$ is always positive but $x_0$ is negative if $m$ is large. Hence we can choose $m$ so that $x_0 > 0$; but if $x_1$ is defined by

$$y_0 + x_0\sqrt{5} = (y_1 + x_1\sqrt{5})(9 + 4\sqrt{5})$$

then $x_1 \leq 0$. Since

$$y_0 + x_0\sqrt{5} = (9y_1 + 20x_1) + (9x_1 + 4y_1)\sqrt{5}$$

we have $y_0 = 9y_1 + 20x_1$ and $x_0 = 9x_1 + 4y_1$. From the previous equations we have $x_1 = 9x_0 - 4y_0$ and $x_0 \leq \frac{4y_0}{9}$. Using the positive part of equation (2) we have

$$y_0^2 - 4 = 5x_0^2 \leq \frac{80}{81}y_0^2.$$

Hence $y_0 = 3, 7, 18$ and $x_0 = 1, 3, 8$, respectively. Thus the general solution of the positive part of equation (2) is given by

$$y + x\sqrt{5} = (3 + \sqrt{5})(9 + 4\sqrt{5})^m \tag{6}$$

$$y + x\sqrt{5} = (7 + 3\sqrt{5})(9 + 4\sqrt{5})^m \tag{7}$$

$$y + x\sqrt{5} = (18 + 8\sqrt{5})(9 + 4\sqrt{5})^m \tag{8}$$

where $m = 0, 1, 2 \ldots$

Using the same method as before with the negative part of equation (2), we find that $y_1 = 9y_0 - 20x_0 \leq 0$ (in this case $x_0$ is always positive), whence $y_0^2 = 5x_0^2 - 4 \leq \frac{400}{81}x_0^2$, so that $x_0 = 1, 2, 5$ and $y_0 = 1, 4, 11$, respectively. Thus the general solution of the negative part of equation (2) is given by

$$y + x\sqrt{5} = (1 + \sqrt{5})(9 + 4\sqrt{5})^m \tag{9}$$

$$y + x\sqrt{5} = (4 + 2\sqrt{5})(9 + 4\sqrt{5})^m \tag{10}$$

$$y + x\sqrt{5} = (11 + 5\sqrt{5})(9 + 4\sqrt{5})^m \tag{11}$$

where $m = 0, 1, 2, \ldots$ We are looking for the common solutions of the equation (6), (7), (8), (9), (10), (11) with the equations (3). Using equation (3), (6) and their conjugates we have

$$2y = \frac{(3 + \sqrt{8})^n}{\sqrt{8}} - \frac{(3 - \sqrt{8})^n}{\sqrt{8}} = (3 + \sqrt{5})(9 + 4\sqrt{5})^m + (3 - \sqrt{5})(9 - 4\sqrt{5})^m$$

and so

$$\begin{gathered}\frac{(3 + \sqrt{8})^n}{\sqrt{8}} - \frac{(3 - \sqrt{8})^n}{\sqrt{8}} = \\ (3 + \sqrt{5})(9 + 4\sqrt{5})^m + (3 - \sqrt{5})(9 - 4\sqrt{5})^m\end{gathered} \tag{12}$$

Putting

$$Q = \frac{(3+\sqrt{8})^n}{\sqrt{8}}, \quad P = (3+\sqrt{5})(9+4\sqrt{5})^m,$$

in equation (12) we obtain

$$Q - \frac{1}{8}Q^{-1} = P - 4P^{-1}.$$

Since

$$Q - P = \frac{1}{8}Q^{-1} - 4P^{-1} < 4(Q^{-1} - P^{-1}) = 4\frac{P-Q}{QP}$$

we have that $Q < P$. Obviously $P > 90$ and

$$P - Q = 4P^{-1} - \frac{1}{8}Q{-}1 < 4P^{-1}.$$

It follows that

$$\begin{aligned} 0 < \log\frac{P}{Q} &= -\log\left(1 - \frac{P-Q}{P}\right) < 4P^{-2} + 16P^{-4} \\ &< 4.16P^{-2} < \frac{0.00052}{(9+4\sqrt{5})^{2m}}. \end{aligned} \tag{13}$$

Morover

$$0 < \log\frac{P}{Q} = m\log(9+4\sqrt{5}) - n\log(3+\sqrt{8}) + \log(3+\sqrt{5})\sqrt{8} = \Lambda.$$

Since $P > Q$ we have $4m > n$.

We apply Theorem 3 with $n = 3$, $\alpha_1 = 9+4\sqrt{5}$, $\alpha_2 = 3+\sqrt{8}$ and $\alpha_3 = (3+\sqrt{5})\sqrt{8}$. The equations satisfied by $\alpha_1, \alpha_2, \alpha_3$ are $\alpha_1^2 - 18\alpha_1 + 1 = 0$, $\alpha_2^2 - 6\alpha_2 + 1 = 0$ and $\alpha_3 - 224\alpha_3^2 + 1024 = 0$. In the previous equations we have $A_1 = 18, A_2 = 6, A_3 = 1024$. Using Theorem 3 we have

$$\log|\Lambda| > -(16 \times 3 \times 4)^{10} \log 18 \log 6 \log 1024 \log 4m.$$

We use that

$$0.00052\left((9+4\sqrt{5})^2\right)^{-m} < \exp(-5.77m)$$

and we have

$$m < \frac{1}{5.77}(16 \times 3 \times 4)^{10} \log 18 \log 6 \log 1024 \log 4m < 10^{24} \log 4m.$$

Thus we have $m < 10^{26}$. Using this method we get the same result for $m$ in the other cases, too. In those cases $\alpha_1 = 9+4\sqrt{5}$, $\alpha_2 = 3+\sqrt{8}$ as before but $\alpha_3^{(2)} = (7+3\sqrt{5})\sqrt{8}$, $\alpha_3^{(3)} = (18+8\sqrt{5})\sqrt{8}$, $\alpha_3^{(4)} = (1+\sqrt{5})\sqrt{8}$, $\alpha_3^{(5)} = (4+2\sqrt{5})\sqrt{8}$, and $\alpha_3^{(6)} = (11+5\sqrt{5})\sqrt{8}$. In these cases $A_3^{(2)} = 1504$, $A_3^{(3)} = 10304$, $A_3^{(4)} = 1024$, $A_3^{(5)} = 1024$ and $A_3^{(6)} = 3936$.

This upper bound is treatable. Using the computer algebraic program package Maple we can show easily that there is no Lucas balancing number. $\square$

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