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Simultaneous inhomogeneous Diophantine approximation of the values of integral polynomials with respect to Archimedean and non-Archimedean valuations

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Abstract. We prove an analogue of the convergence part of Khintchine's theorem for the simultaneous inhomogeneous Diophantine approximation on the Veronese curve $(x, x^2, \dots, x^n)$ with respect to the different valuations. It is an extension of the author's earlier results.

1. Introduction

The problem under the consideration belongs to the metric theory of Diophantine approximation on manifolds. This theory was formed in papers of V. Sprindžuk [18], [19], W.M. Schmidt [17]. Nowadays it is intensively developed ([2]-[16], [20]-[22]). We prove an analogue of the result [16] for the simultaneous inhomogeneous Diophantine approximation on the Veronese curve $(x, x^2, \dots, x^n)$ with respect to the different valuations.

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonically decreasing function and $\sum_{n=1}^{\infty} \psi(n) < \infty$. Let $P_n = P_n(y) = a_n y^n + \cdots + a_1 y + a_0 \in \mathbb{Z}[y]$, $\deg P_n = n$ and $H = H(P_n) = \max_{0 \le i \le n} |a_i|$. Let $p \ge 2$ be a prime number, $\mathbb{Q}_p$ be the field of p-adic numbers, $|\cdot|_p$ be the p-adic valuation. Suppose that $\mathcal{O} = \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$. We define a measure

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μ in $\mathcal{O}$ as a product of the Lebesque measures μ_1 and μ_2 in $\mathbb{R}$ and $\mathbb{C}$, and the Haar measure μ_3 in $\mathbb{Q}_p$, that is, $\mu = \mu_1\mu_2\mu_3$. We consider the system of inequalities

$$\begin{aligned} |P_n(x) + d_1| &< H^{\lambda_1}\psi^{\nu_1}(H), \\ |P_n(z) + d_2| &< H^{\lambda_2}\psi^{\nu_2}(H), \\ |P_n(\omega) + d_3|_p &< H^{\lambda_3}\psi^{\nu_3}(H) \end{aligned} \tag{1}$$

where $(d_1, d_2, d_3) \in \mathcal{O}$, $(x, z, \omega) \in \mathcal{O}$, $\lambda_i \leq 1$ $(i = 1, 2)$, $\lambda_3 \leq 0$, $\lambda_1 + 2\lambda_2 + \lambda_3 = 3 - n$, $\nu_i \geq 0$ $(i = 1, 2, 3)$, $\nu_1 + 2\nu_2 + \nu_3 = 1$, $\lambda_i - \nu_i < 1$ $(i = 1, 2)$, $\lambda_3 - \nu_3 < 0$. We prove the following

Theorem. *For every vector* $(d_1, d_2, d_3) \in \mathcal{O}$ *the system of inequalities* (1) *has only a finite number of solutions in polynomials* $P_n \in \mathbb{Z}[y]$ *for almost all* $(x, z, \omega) \in \mathcal{O}$.

E. Lutz (1955) was the first who considered an inhomogeneous problem in $\mathbb{Q}_p$ for polynomials with $n = 1$ and a function $H^{-2-\varepsilon}$ in the right hand part of the third inequality of (1) without two others inequalities. Inhomogeneous questions are rather different in character to the homogeneous ones in that they concern questions of how points are distributed rather than how close it is possible to get to the integers. Also it is well known that if $P_n(y)$ is irreducible then for any two its different roots ξ_1 and ξ_2 the following inequality holds

$$|\xi_1 - \xi_2| > c(P_n)H^{-n+1}.$$

The situation is different when $P_n(y) + d$ is considered instead of $P_n(y)$, where $d \in \mathbb{R}$. With a point of view of a continuity it is readily proved that for any $w > 0$ we can select such a number d that $|\rho_1 - \rho_2| < H^{-w}$, where ρ_1 and ρ_2 are the roots of the polynomial $P_n(y) + d$. Besides, the roots of the $P_n(y) + d$ are the transcendental number if d is one.

Note that the inhomogeneous Diophantine approximation for the Veronese curve were investigated earlier by V. Bernik, H. Dickinson and M. Dodson [5] in $\mathbb{R}$ when $\lambda_1 = 1 - n$, $\nu_1 = 1$, V. Bernik, H. Dickinson and J. Yuan [6] in $\mathbb{Q}_p$ when $\lambda_3 = -n$, $\psi^{\nu_3}(H) = H^{-1-\varepsilon}$, by A. Ustinov [21] in $\mathbb{Q}_p$ when $\lambda_3 = -n$, $\nu_3 = 1$, and [22] in $\mathbb{C}$ when $\lambda_2 = -(n-2)/2$, $\nu_2 = 1/2$, separately. Pan Giong from the X'ian University (China) acquainted us with her result [11]. It is the same as [21].

In order to prove the Theorem we develop the Sprindžuk's method of essential and inessential domains, use a proof scheme of the Bernik—Borbat result [4] and one lemma of Bernik—Kalosha [8]. Proving the Theorem we investigate 7 cases dependent on the values of the derivative $|P_n'(y)|$, that is, we consider the domains where the value of $|P_n'(y)|$ is large and the domains where the value of $|P_n'(y)|$ is small. Then, we combine these domains with respect to above-mention valuations.

2. Notation and Lemmas

According to the metric ideas [18] we may put $x \ll 1$, $z \ll 1$, $|\omega|_p \ll 1$, where $\ll$ is Vinogradov's symbol ($x \ll y$ means that $x = O(y)$). Let $\alpha_1^{(n)}, \ldots, \alpha_n^{(n)}$ be the roots of the polynomial P_n in $\mathbb{C}$ and $\beta_1^{(n)}, \ldots, \beta_n^{(n)}$ be the roots of the polynomial P_n in $\mathbb{Q}_p^*$, where $\mathbb{Q}_p^*$ is the least field containing $\mathbb{Q}$ and all algebraic numbers.

Lemma 1. *Let $P_n(y) \in \mathbb{Z}[y]$, P_n as above. Then*

$$\max_{0 \le m \le n} |P_n(m)| \ll \max_{0 \le i \le n} |a_i|. \tag{2}$$

This is Lemma 7 of [18, p. 19]. We denote the smallest m for which (2) is true by m_0.

Lemma 2. *Let $P_n(y) \in \mathbb{Z}[y]$, P_n as above with $|a_n| > cH(P_n)$, where c is a constant depending only on n, $0 < c \le 1$. Then $|\alpha_i^{(n)}| < \max(n/c, 1)$ for every root $\alpha_i^{(n)}$ $(i = 1, \dots, n)$ of P_n.*

This is Lemma 1 of [18, p. 13].

Lemma 3. *Let $P_n(y) \in \mathbb{Z}[y]$, P_n as above with $|a_n|_p > c_1$, where c_1 is a constant depending only on n. Then $|\beta_i^{(n)}|_p < \max(c_1^{-1}, 1)$ for every root $\beta_i^{(n)}$ $(i = 1, \dots, n)$ of P_n.*

This is Lemma 3 of [6].

Lemma 4. *Let*

$$|P_n(x)| < H^{\lambda_1}\psi^{\nu_1}(H), \quad |P_n(z)| < H^{\lambda_2}\psi^{\nu_2}(H), \quad |P_n(\omega)|_p < H^{\lambda_3}\psi^{\nu_3}(H), \tag{3}$$

be a system of inequalities with $(x, z, \omega) \in \mathcal{O}$ where H, ψ, $\mathcal{O}$ and parameters λ_i, ν_i $(i = 1, 2, 3)$ are defined as in Theorem. Then the system (3) *is satisfied by at most finitely many polynomials $P_n \in \mathbb{Z}[y]$ for almost all $(x, z, \omega) \in \mathcal{O}$.*

This is a main theorem in [16].

3. Reduction to a polynomial $P_n^{(D)}(y)$.

Let $Q(y) = y^n(P_n(y^{-1} + m_0) + d_3)$, where m_0 is the fixed integer from Lemma 1. Consider the third inequality of (1). If it holds infinitely often for a set of positive measure one can be readily verify that the set of solutions of the inequality

$$|Q(y)|_p < H(P_n)^{\lambda_3}\psi^{\nu_3}(H(P_n))$$

also has positive measure (see [1, Lemma 5] for details). It is easy to show that Q takes the form

$$Q(y) = (P_n(m_0) + d_3)y^n + b_{n-1}y^{n-1} + \dots + b_1 y + b_0 \tag{4}$$

where $b_i \in \mathbb{Z}$, $|b_i| \ll H(P_n)$ for $i = 0, 1, \dots, n-1$. If the value of $|P_n(m_0) + d_3|_p$ is very small then we shall consider $|P_n(m_0) + d_3 + 1|_p$ (or $|P_n(m_0) + d_3 - 1|_p$) instead of it. Then the new value equals 1. Let S_{d_3+1} be a set of ω for which the third inequality in (1) holds with $d_3 + 1$ for infinitely many P_n. Let S_{d_3} is the same set with d_3. Let $\mu_3(S_{d_3+1}) = 0$. It follows that $\mu_3(S_{d_3}) = 0$ as otherwise we obtain a contradiction (replacing $P_n(m_0) + d_3$ by $(P_n(m_0) - 1) + d_3 + 1$). Hence, we may assume without loss of generality that $|P_n(m_0) + d_3|_p > c_2$, where c_2 is a constant depending only on n and d_3. Therefore the roots of Q are bounded according to Lemma 3. Thus, instead of the third inequality of (1), we can consider the inequality

$$|P_n^{(d_3)}(\omega)|_p < H(P_n^{(d_3)})^{\lambda_3}\psi^{\nu_3}(H(P_n^{(d_3)})),$$

where

$$P_n^{(d_3)}(y) = (a_n + d_3)y^n + b_{n-1}y^{n-1} + \dots + b_1 y + b_0 = N_1 y^n + b_{n-1}y^{n-1} + \dots + b_1 y + b_0$$

and the roots of $P_n^{(d_3)}$ lie in the disk $|y|_p \ll 1$. Note that N_1 is not necessarily an integer. Let

$$\mathcal{P}_n(N, d_3) = \{P_n^{(d_3)} :\ a_n + d_3 = N,\ |b_i| \ll N,\ i = 2, \ldots, n,$$
$$|\beta_j^{(n)}|_p \ll 1,\ j = 1, 2 \ldots, n\},$$

where N depends on the height of the polynomial P_n associated with $P_n^{(d_3)}$, e.g. $N \ll H(P_n)$.

Further, according to (4) we can write the left hand parts of the first and the second inequalities in (1) as

$$(P_n(m_0) + D + (d_j - D))y^n + b_{n-1}y^{n-1} + \cdots + b_0 \quad (j = 1, 2)$$

where D denotes the integer part of d_3 and $|d_j - d_3| \ll 1$ $(j = 1, 2)$. Hence, we have $|P_n(m_0) + D + (d_j - D)| \asymp |a_n + (P_n(m_0) - a_n) + D|$ $(j = 1, 2)$ when the height $H = H(P_n) = |a_n|$ is sufficient large. If $H(P_n^{(d_3)}) = N$ where N is sufficient large then $N \asymp |a_n| = H(P_n)$. Therefore according to Lemma 1 we get $|a_n + (P_n(m_0) - a_n) + D| \asymp |a_n + D| \asymp H(P_n)$. Thus, $H(P_n^{(D)}) \asymp H(P_n)$ and according to Lemmas 1, 2 and 3, instead of (1) we can consider without loss of generality the following system of inequalities

$$|P_n^{(D)}(x| \ll H(P_n^{(D)})^{\lambda_1}\psi^{\nu_1}(H(P_n^{(D)})),\ \ |P_n^{(D)}(z| \ll H(P_n^{(D)})^{\lambda_2}\psi^{\nu_2}(H(P_n^{(D)})),$$

and

$$|P_n^{(D)}(\omega)|_p \ll H(P_n^{(D)})^{\lambda_3}\psi^{\nu_3}(H(P_n^{(D)})), \tag{5}$$

where $P_n^{(D)} \in \mathcal{P}_n^{'}(N, D)$ with

$$\mathcal{P}_n^{'}(N, D) = \{P_n^{(D)} :\ a_n + D = N,\ |b_i| \ll N,\ i = 2, \ldots, n,$$
$$|\alpha_j^{(n)}| \ll 1,\ |\beta_j^{(n)}|_p \ll 1,\ j = 1, 2, \ldots, n\}. \tag{6}$$

and $N \in \mathbb{N}$, N is a sufficient large number, $|D| \ll 1$.

4. Proof of Theorem.

As in [18] (see also [3, p. 40-42]), the investigation of the system (5) can be reduced to the case of primitive irreducible polynomials $P_n^{(D)} \in \mathcal{P}_n^{'}(N, D)$ when $|a_n + D|_p = |N|_p > p^{-n}$. Let $\mathcal{P}_n^{*}(N, D)$ be the set of those polynomials. Further, the proof of the Theorem is the same as the proof of the Lemma 4 (for more details see [16, Theorem], or [3, p. 40-52], where we consider only the third inequality in (5)). As in [16], we investigate 7 cases depending on the behavior of the values of the derivative $|P_n^{'(D)}|$ and $|P_n^{'(D)}|_p$. The distinctions in the proofs appear in the cases 2, 4, 7 which are related to dividing all considered polynomials into the following classes. Two polynomials

$$P_{n1}^{(D)} = Ny^n + b_{n-1}^{(1)}y^{n-1} + \cdots + b_1^{(1)} + b_0^{(1)},$$
$$P_{n2}^{(D)} = Ny^n + b_{n-1}^{(2)}y^{n-1} + \cdots + b_1^{(2)} + b_0^{(2)}$$

belong to one class if $b_{n-1}^{(1)} = b_{n-1}^{(2)}, \ldots, b_{n-r}^{(1)} = b_{n-r}^{(2)}$, where $0 < r < n$ is the fixed integer defined in [16, p. 483], $r = [\theta] - 1$, where $[\theta]$ is the integer part of θ and

$\theta = n+1-(q_1+2r_1+m_1+(k_2+2l_2+m_2)/T)$ with $k_1, k_2, m_1, m_2, r_1, l_2, T$ belonging to $\mathbb{N}\bigcup\{0\}$ and characterizing the differences between the roots $|\alpha_i^{(n)} - \alpha_j^{(n)}|$ and $|\beta_i^{(n)} - \beta_j^{(n)}|_p$ $(1 \le i, j \le n)$, and the number $T = T(\varepsilon)$ is sufficiently large, the number $\varepsilon > 0$ is sufficiently small.

There exists a class which has at least $\ll N^{0,9\varepsilon}$ polynomials according to Dirichlet's principle. Denote the polynomials of this class by $P_{n1}^{(d)}, \ldots, P_{nt}^{(d)}$ and construct $(t-1)$ new polynomials $R_{nj}^{(d)}(y) = P_{n(j+1)}^{(d)} - P_{nj}^{(d)}$ $(1 \le j \le t-1)$. Thus, starting from polynomials of degree n we reduce the problem to polynomials of degree k not greater than $(q_1+2r_1+m_1+(k_2+2l_2+m_2)/T)-1 \le n-1$. Then we make the new reduction to a polynomial $P_k^{(D)}(y)$ as in the section 3 of the paper. Further we use the arguments as case 2 [16, p. 483-484] or [3, p. 46-49]. Thus, the proof is complete.

References

[1] BAKER A.: *On a theorem of Sprindžuk* , Proc. Roy. Soc. London Ser. A **242** (1966), 92-104.

[2] BERESNEVICH V. — BERNIK V. — KLEINBOCK D. — MARGULIES G.A. *Metric Diophantine approximation: the Khintchine-Groshev theorem for non-degenerate ma nifolds*, Moscow Math. J. **2** (2002), 203-225.

[3] BERESNEVICH V.V. — BERNIK V.I. — KOVALEVSKAYA E.I. *On approximation of p-adic numbers by p-adic algebraic numbers* , J. Number Theory **111** (2005), 33-56.

[4] BERNIK V.I. — BORBAT V.N.: *Simultaneous approximation of zero values of integral polynomials in $\mathbb{Q}_p$*, Proc. of the Steklov Institute of Math. **218** (1997),
53-68 (translated by S.Varhromeev/MAHK Hayka /Interperiodica Publishing (Russia)).

[5] BERNIK V.I. — DICKINSON H. — DODSON M.: *Approximation of real numbers by integral polynomials*, Dokl. Nats. Akad. Nayk Belarusi **42** (1997), 51-54.

[6] BERNIK V. — DICKINSON H. — YUAN J.: *Inhomogeneous diohantine approximation on polynomials*, Acta Arith. **90** (1999), 37-48.

[7] BERNIK V.I. — DODSON M.M.: *Metric Diophantine approximation on manifolds.* Cambridge Tracts in Math. Vol. **137**. Cambridge Univ. Press, 1999.

[8] BERNIK V.I.—KALOSHA N.I.: *Simultaneous approximation of zero values of integral polynomials in $\mathbb{R} \times \mathbb{Q}_p$*, Dokl. Nats. Akad. Nauk Belarusi. **47** (2003), 19-22.

[9] BERNIK V. — KLEINBOCK D. — MARGULIES G.A. *Khintchine type theorems on manifolds: the convergence case for standard and multiplicative versions*, Int. Math. Research Notices. **9** (2001), 453-486.

[10] BUGEAUD Y.: *Approximation by algebraic numbers.* Cambridge Tracts in Math. **169**, Cambridge Univ. Press, Cambridge, 2004.

[11] GIONG PAN. *Metric inhomogeneous Diophantine approximation on polynomial curves in* $\mathbb{Q}_p$ (to appear).

[12] HARMAN G.: *Metric Number Theory.* LMS Monograph. New Series. Vol. **18**. Claredon Press, 1998.

[13] KLEINBOCK D. — MARGULIES G.A. *Flows on homogeneous space and Diophantine approximation on manifolds*, Ann. Math. **148** (1998), 339-360.

[14] KLEINBOCK D. — TOMANOV G. *Flows on S-arithmetic homogeneous space and application to metric Diophantine approximation*, Preprint Max Plank Institute. Bonn. **2003-65** (2003).

[15] KOVALEVSKAYA E.I.: *A metric theorem on the exact order of approximation of zero by values of integral polynomials in $\mathbb{Q}_p$*, Dokl. Nats. Akad. Nauk Belarusi. **43** (1999), 34-36.

[16] KOVALEVSKAYA E. *Simultaneous approximation of zero by values of integral polynomials with respect to different valuations*, Math. Slovaca. **54** (2004), 479-486.

[17] SCHMIDT W.M.: *Diophantine Approximation.* Lecture Notes in Math. Vol. **785**. Springer-Verlag, 1980.

[18] SPRINDŽUK V.G.:*Mahler's problem in metric Number Theory*, Nauka i Tehnika, Minsk, 1967 [Transl. Math. Monogr. **25**, Amer. Math. Soc., Providence, R.I., 1969]. (Russian)

[19] SPRINDŽUK V.G.:*Metric Theory of Diophantine Approximation*, Izdatel'stvo Nauka, Moskva, 1977 [Jihn Wiley, New York-Toronto-London, 1979 (translation by R. A. Silverman)]. (Russian)

[20] ŽELUDEVICH F.: *Simultine diophantishe Approximationen abhắngiger Grőssen in mehreren Metriken*, Acta Arithm. **46** (1986), 285-296.

[21] USTINOV A. E.: *Inhomogeneous approximation on manifolds in* $\mathbb{Q}_p$, Vestsi Nats. Akad. Nauk Belarusi. Ser. fiz.-mat. nauk. **No 3** (2005), 30-34

[22] USTINOV A. E.: *Inhomogeneous approximation on manifolds in* $\mathbb{C}$‘, Vestsi Nats. Akad. Nauk Belarusi. Ser. fiz.-mat. nauk. **No 1** (2006), 9-14.

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