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Abstract. We examine the congruences and iterate the digit sums of integer sequences. We generate recursive number sequences from triple and quintuple product identities. And we use second order recursions to determine the primality of special number systems.

1. Iterated sum of digits

The Iterated Sum of Digits can be given as an algorithm where we represent a natural number n in base b as $n = \sum_{i\geq 0} c_i\, b^i$. We then add the b -ary coefficients to get their digit sum $s_b(n)=\sum_{i\geq 0} c_i$. We repeat the process by using this digit summation $s_b(n)$ as our new number n, and only stop when the digit sum is less than some predetermined value t.

We use a notation for the resulting Iterated Sum $\$_{b,t}(\chi)$ by the symbol I over S, for any set of natural numbers χ in base b with iteration constraint t. When the value t equals the base value b, we shorten the notation for the Iterated Sum to a single subscript as $\$_b(\chi)$, with potential values ranging from 1 to $b-1$.

A one–to–one correspondence exists between the arithmetic of Iterated Sum of Digits and the arithmetic of Congruences. In a clock example, the times in hours N for $1\leq N\leq 12$ have residues of N modulo 12 equal to $\{1, 2, \ldots, 10, 11, 0\}$. The same values for N written in base 13 have Iterated Sums $\$_{13}(N)$ of $\{1, 2, \ldots, 10, 11, 12\}$. For any natural number $N\geq 1$, it can be seen by induction that the residues and the Iterated Sums will repeat in their respective cycles. And by analogy, for the sequence of natural numbers $N\geq 1$ and given any integer $b\geq 1$, the residues of N modulo b will have an integer cycle $\{1, 2, \ldots, b-1, 0\}$, and the sequence of Iterated Sums for N in base $b+1$ will have the integer cycle $\{1, 2, \ldots, b-1, b\}$.

We can examine the fundamental cycles of special sequences of numbers by their Congruences and Iterated Digit Sums. For example, Pisano numbers $\pi(m)$

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give the periods of the Fibonacci Sequence $\{F_n\}$ modulo m, shown by the integer sequence in [7] as:

$$\{\pi(m)\} \;=\; \{1,3,8,6,20,24,16,12,24,60,10,\ldots\} \quad \text{for } \; m \geq 1. \tag{1}$$

The same period sequence (1) holds for the fundamental cycle of Iterated Sum of Digits $\$_b(\{F_n\})$ when $b{=}m{+}1$.

Likewise the periods for Lucas numbers have the same sequence (1) for the congruences $\{L_n\}\,(\mathrm{mod}\ m)$ and for the Iterated Sum of Digits $\$_{m+1}(\{L_n\})$, except that the period values for the Lucas numbers are $\frac{1}{5}$ that of the Fibonacci numbers when m is divisible by 5, according to

$$\text{Lucas cycle}(m) \;=\begin{cases} \;\;\text{Fibonacci cycle}(m) & \text{if} \quad 5 \not| \, m. \\ \frac{1}{5}\,\text{Fibonacci cycle}(m) & \text{if} \quad 5 \,|\, m. \end{cases}$$

Period values π may be reduced by accelerating the sequence index n in order to shorten the cycle length, as shown here for Lucas numbers L_n:

$$\pi(\$_{m+1}(\{L_{2n}\})) \rightarrow \tfrac{1}{2}\,\pi(\$_{m+1}(\{L_n\})).$$

2. Recursive sequences

We define recursive sequences like those for the Fibonacci and Lucas numbers $\{F_n, L_n\}$ by second order recursions for $a_i{=}a_i(r)$ and $b_i{=}b_i(r)$ with recursion variable r and certain initial conditions, given by

$$a_i \;=\; r\,a_{i-1} + a_{i-2}, \qquad a_0 = 0,\; a_1 = 1, \tag{2}$$

$$b_i \;=\; r\,b_{i-1} +\, b_{i-2}, \qquad b_0 = 2,\; b_1 = r. \tag{3}$$

Taking variable $r{=}1$, we get $a_n(1){=}F_n$ and $b_n(1){=}L_n$ for $n{\geq}0$. We get Pell numbers P_n and Pell Lucas numbers Q_n with $r{=}2$ for the respective values of $a_n(2)$ and $b_n(2)$ when $n{\geq}0$.

We can also define the sequence values $a_i(r)$ and $b_i(r)$ as the coefficients in the series expansion of their generating functions:

$$\sum_{i\geq 0} a_i(r)\,x^i = \frac{x}{1 - r\,x - x^2}. \qquad \sum_{i\geq 0} b_i(r)\,x^i = \frac{2 - x}{1 - r\,x - x^2}.$$

Separately we will define another variable $\gamma_j(r)$ by its generating function, noting that the even indexed variables $\gamma_{2j}(r)$ have a Chebyshev related generating function with $R{=}r^2{+}2$.

$$\sum_{j\geq 0} \gamma_j(r)\,x^j = \frac{2 - x}{1 - r\,x - x^2}, \qquad \sum_{j\geq 0} \gamma_{2j}(r)\,x^j = \frac{2 - r\,x}{1 - R\,x + x^2}. \tag{4}$$

We then generalize variables $a_i(r)$ and $b_i(r)$ by substituting any $\gamma_j(r)$ value for variable r at any integer $j{\geq}1$ in their generating functions to obtain

$$\sum_{i\geq 0} a_{ji}(r)\,x^i \;=\; \frac{a_j(r)\,x}{1 - \gamma_j(r)\,x + (-1)^j x^2}. \tag{5}$$

$$\sum_{i\geq 0} b_{ji}(r)\,x^i \;=\; \frac{2 - \gamma_j(r)\,x}{1 - \gamma_j(r)\,x + (-1)^j x^2}. \tag{6}$$

3. Multiple product identities

Now we show how these second order recursive sequences are derived from multiple product identities, such as the Jacobi triple product and the Watson quintuple product identities.

3.1 Jacobi Triple Product

The Jacobi triple product identity of [3] can be written as an infinite product with its power series expansion in q.

$$\prod_{n\geq 1}(1-z^{-1}\,q^{2n-1})(1-z\,q^{2n-1})(1-q^{2n}) = \sum_{k=-\infty}^{\infty}(-1)^k\,z^k q^{k^2}. \tag{7}$$

This formal power series in q is valid for complex variables z and q, where $z{\neq}0$ and $|q|{<}1$. We define another complex variable $r{=}z{+}z^{-1}$ and then get polynomials in q with coefficients in r. As in [2], we show the triple product TP as a q -series and in product form (8) after substituting $r{=}z{+}z^{-1}$.

$$\begin{aligned} TP &= \sum_{h\geq 0} c_h\,q^h = (zq,\, z^{-1}q,\, q^2\,;\, q^2)_\infty \\ &= \prod_{n\geq 1}(1-r\,q^{2n-1}+q^{4n-2})(1-q^{2n}), \end{aligned} \tag{8}$$

where $(a,b;q)_\infty = \prod_{n\geq 1}(1-aq^{n-1})(1-bq^{n-1})$ are q -Pochhammer symbols.

We create a companion triple product by substituting $r{\to}r/I$ and $q{\to}q/I$:

$$TP = \prod_{n\geq 1}(1-(-1)^n\,r\,q^{2n-1}-q^{4n-2})(1-(-1)^n q^{2n}). \tag{9}$$

Then we combine the results $(8,9)$ into a composite product identity $TP2(r)$ for even or odd j parity, with series expansion in q and coefficients $\Psi_k(r)$:

$$\begin{aligned} TP2(r) &= \prod_{n\geq 1}(1-(-1)^{nj}\,r\,q^{2n-1}+(-1)^j\,q^{4n-2})(1-(-1)^{nj}\,q^{2n}) \\ &= \sum_{k\geq 0} c_k\,q^{k^2} = \sum_{k\geq 0}(-1)^{s_k}\,\Psi_k(r)\,q^{k^2} \end{aligned} \tag{10}$$

with sign exponent given by $s_k = (j-1)\,k + j\,\lfloor \frac{k}{2} \rfloor$.

We extend the $\Psi_k(r)$ coefficients by substituting any j^{th} term from the sequence $\gamma_j(r)$ in (4) for the r variable. This generalizes the triple product $TP2(r)$ at any value $j{\geq}1$, giving a product identity with $\Psi_k(\gamma_j(r))$ coefficients:

$$\begin{aligned} TP2(\gamma_j(r)) &= \prod_{n\geq 1}(1-(-1)^{nj}\,\gamma_j(r)\,q^{2n-1}+(-1)^j\,q^{4n-2})(1-(-1)^{nj}\,q^{2n}) \\ &= \sum_{k\geq 0}(-1)^{s_k}\,\Psi_k(\gamma_j(r))\,q^{k^2} \quad \text{with } s_k = (j-1)\,k + j\,\lfloor \tfrac{k}{2} \rfloor. \end{aligned}$$

At each $\gamma_j(r)$ variable, this general power series has Ψ values given by the $b_{ji}(r)$ coefficients.

$$TP2(\gamma_j(r)) = \sum_{k\geq 0}(-1)^{s_k}\,\Psi_k(\gamma_j(r))\,q^{k^2} = 1 + \sum_{k\geq 1}(-1)^{s_k}\,b_{jk}(r)\,q^{k^2}. \tag{11}$$

The $b_{ji}(r)$ values can be obtained by recursion (3), generating function (6) or the binomial expansion with b_0=1 and

$$b_{ji}(r) \;=\; \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{k(j+1)} \;\frac{i}{i-k} \binom{i-k}{k} (\gamma_j(r))^{i-2k} \;\text{ for }\; i \geq 1.$$

For $r = \pm 1, \pm 2$, coefficients $b_{ji}(r)$ are Lucas L_n and Pell Lucas Q_n numbers.

$$\begin{aligned} \Psi_k(\gamma_j(1)) \;&=\; \Psi_k(L_j) \;=\; L_{jk}. \\ \Psi_k(\gamma_j(2)) \;&=\; \Psi_k(Q_j) \;=\; Q_{jk}. \end{aligned}$$

In the r =1 family, $TP2$ formulas yield j^{th} level Lucas number coefficients:

$$\begin{aligned} TP2(L_j) \;&=\; 1 + \sum_{k\geq 1} (-1)^{s_k} L_{jk}\, q^{k^2} \\ &= \begin{cases} 1 \;\; +1\; q \;\; - \;3\; q^4 \; - \;\; 4\; q^9 \;\cdots & \text{at } j=1 \\ 1 \;\; -3\; q \;\; + \;7\; q^4 \; - 18\; q^9 \;\cdots & \text{at } j=2 \\ 1 \;\; +4\; q \;\; -18\; q^4 \; - 76\; q^9 \;\cdots & \text{at } j=3 \\ 1 \;\; -7\; q \;\; +47\; q^4 \; -322\; q^9 \;\cdots & \text{at } j=4 \end{cases} \end{aligned}$$

with sign exponent $s_k = (j-1)\,k + j\,\lfloor \frac{k}{2} \rfloor$.

3.2 Watson Quintuple Product

The Watson quintuple product of [4] can be given by the form:

$$\begin{aligned} &\prod_{n\geq 1} (1 - z^{-1} q^{2n-1})(1 - z q^{2n-1})(1 - z^{-2}\, q^{4n-4})(1 - z^2\, q^{4n-4})(1 - q^{2n}) \\ &\qquad = \sum_{k=-\infty}^{\infty} q^{(3k-2)k}((z^{3k} + z^{-3k}) - (z^{3k-2} + z^{-(3k-2)})). \qquad (12) \end{aligned}$$

We obtain a balanced quintuple product QP as in [2], by dividing the Watson quintuple product (12) by $-((z^2-1)/z)^2$. We write this quintuple product QP in q -series notation and give its product form in (13) after variable substitutions r=z+z^{-1} and p=r^2−2.

$$\begin{aligned} QP \;&=\; \sum_{h\geq 0} c_h\, q^h = (zq,\, z^{-1}q,\, q^2\,;\, q^2)_\infty (z^2 q^4,\, z^{-2} q^4\,;\, q^4)_\infty \\ &=\; \prod_{n\geq 1} (1 - r q^{2n-1} + q^{4n-2})(1 - p q^{4n} + q^{8n})(1 - q^{2n}). \qquad (13) \end{aligned}$$

We get a companion quintuple product by substituting $r \to r/I$ and $q \to q/I$, with variable p^*=$-r^2$−2.

$$QP = \prod_{n\geq 1} (1 - (-1)^n\, r\, q^{2n-1} - q^{4n-2})(1 - p^* q^{4n} + q^{8n})(1 - (-1)^n q^{2n}). \qquad (14)$$

We use the results $(13,14)$ to create a composite quintuple product $QP2$ for even or odd j parity, yielding a series in q with coefficients $\Psi_{\omega_k}(r)$ with lacunary index $\omega_k = \lfloor \frac{3k+2}{2} \rfloor$:

$$\prod_{n\geq 1} (1-(-1)^{nj}\, r\, q^{2n-1} + (-1)^j q^{4n-2})(1-p^* q^{4n} + q^{8n})(1-(-1)^{nj} q^{2n})$$
$$= \; QP2 \; = \; \sum_{k\geq 0} c_k\, q^{\tau_k} \; = \; \sum_{k\geq 0} (-1)^{s_k}\; \Psi_{\omega_k}(r)\; q^{\tau_k} \qquad (15)$$

for variables $p^* = (-1)^j r^2 - 2$, $\tau_k = k^2 + k - (\lfloor \frac{k+1}{2} \rfloor)^2$ and $s_k = kj - k + j\lfloor \frac{k+1}{4} \rfloor$.

We extend the $\Psi_{\omega_k}(r)$ coefficients by substituting any j^{th} term from the sequence $\gamma_j(r)$ in (4) for the r variable in the composite product identity (15). This generalizes the quintuple product $QP2(r)$ at any value $j \geq 1$, with a product identity having $\Psi_{\omega_k}(\gamma_j(r))$ coefficients. At each $\gamma_j(r)$ variable, the generalized Ψ values are given by ratios of the $a_{ji}(r)$ coefficients.

$$QP2(\gamma_j(r)) \; = \sum_{k\geq 0} (-1)^{s_k}\; \Psi_{\omega_k}(\gamma_j(r))\; q^{\tau_k} = \sum_{k\geq 0} (-1)^{s_k}\; \frac{a_{j\omega_k}}{a_j}\; q^{\tau_k} \qquad (16)$$

for $\omega_k = \lfloor \frac{3k+2}{2} \rfloor$ and $\tau_k = k^2 + k - (\lfloor \frac{k+1}{2} \rfloor)^2$ as generalized octagonal numbers.

These $a_{ji}(r)$ values can be gotten by recursion (2), generating function (5) or the binomial expansion:

$$a_{ji}(r) \; = \; \alpha_j(r) \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^{\sigma_k} \binom{i-k-1}{k} (\gamma_j(r))^{i-2k-1}$$

with sign exponent $\sigma_k = k(j+1) + j\lfloor \frac{k+1}{6} \rfloor$.

At $r = \{1, 2\}$, the coefficients are Fibonacci F_n and Pell P_n number ratios.

$$\Psi_k(\gamma_j(1)) \; = \; \Psi_k(L_j) \; = \; F_{jk}/F_j.$$
$$\Psi_k(\gamma_j(2)) \; = \; \Psi_k(Q_j) \; = \; P_{jk}/P_j.$$

In the $r = 1$ family, the $QP2$ formulas yield j^{th} level Fibonacci number coefficient ratios:

$$QP2(L_j) \; = \sum_{k\geq 0} (-1)^{s_k}\; \frac{F_{j\omega_k}}{F_j}\; q^{\tau_k}$$
$$= \begin{cases} 1 \; +1\; q \; + \; 3\; q^5 \; - \; 5\; q^8 \; \cdots & \text{at } j=1 \\ 1 \; -3\; q \; + \; 21\; q^5 \; - \; 55\; q^8 \; \cdots & \text{at } j=2 \\ 1 \; +4\; q \; + \; 72\; q^5 \; - \; 305\; q^8 \; \cdots & \text{at } j=3 \\ 1 \; -7\; q \; +329\; q^5 \; -2255\; q^8 \; \cdots & \text{at } j=4 \end{cases}$$

with $\omega_k = \lfloor \frac{3k+2}{2} \rfloor$, $\tau_k = k^2 + k - (\lfloor \frac{k+1}{2} \rfloor)^2$ and $s_k = k(j-1) + j\lfloor \frac{k+1}{4} \rfloor$.

3.3 Other Variable Selections

We can reconstruct the polynomial $QP2$ of (16) in terms of variable z from $\gamma_j(r) = z + z^{-1}$ to get

$$QP2 = \begin{cases} \sum_{k\geq 0} q^{\tau_k} \dfrac{(-1)^{s_k}}{z^{\nu_k}} \sum_{h=0}^{\nu_k} D_{k,k-h}\, z^{2h} & \text{for odd } j \\ \sum_{k\geq 0} q^{\tau_k} \dfrac{(-1)^{k}}{z^{\nu_k - 1}} \sum_{h=0}^{\nu_k - 1} z^{2h} & \text{for even } j \end{cases}$$

$$\text{with } \tau_k = k^2 + k - (\lfloor \tfrac{k+1}{2} \rfloor)^2,\ \nu_k = \lfloor \tfrac{3k}{2} \rfloor, \text{ and } s_k = \lceil \tfrac{k-1+\operatorname{sgn}(r)}{4} \rceil$$

where the value for sgn(r) is 1 if $r>0$ and is -1 if $r<0$.

The $D_{n,k}$ coefficients are Delannoy numbers [1, 9], derived by the recursion

$$D_{n,k} = D_{n,k-1} + D_{n-1,k-1} + D_{n-1,k},$$

$$D_{0,0} = 1,\ D_{1,1} = 3,\ D_{n,0} = D_{0,n} = 1 \text{ for } n > 0.$$

We also can redefine polynomial $TP2$ of (11) in terms of variable z from $\gamma_j(r) = z + z^{-1}$ to obtain

$$TP2 = \begin{cases} \sum_{k\geq 0} q^{k^2} \dfrac{(-1)^{s_k}}{z^{k}} \sum_{h=0}^{k} D^*_{k,k-h}\, z^{2h} & \text{for odd } j \\ \sum_{k\geq 0} q^{k^2} (-1)^{s_k} \sum_{h=0}^{k} (z^h + z^{-h}) & \text{for even } j \end{cases}$$

$$\text{where } s_k = (j-1)\,(\,k\,\lfloor \tfrac{1+\operatorname{sgn}(r)}{2} \rfloor) + j\,\lfloor \tfrac{2k+1-\operatorname{sgn}(r)}{4} \rfloor.$$

The $D^*_{n,k}$ numbers are gotten from the same third order recursion, but with new initial values according to

$$D^*_{n,k} = D^*_{n,k-1} + D^*_{n-1,k-1} + D^*_{n-1,k},$$

$$D^*_{0,0} = 1,\ D^*_{1,1} = 4,\ D^*_{n,0} = D^*_{0,n} = 1 \text{ for } n > 0.$$

3.4 Delannoy Triangles

We will define Delannoy t triangles by the recursion:

$$d_{n,\,i} = d_{n-1,\,i} + d_{n-1,\,i-1} + d_{n-2,\,i-1}$$

with variable $d_{2,1} = t$ and $d_{n,0} = d_{n,n} = 1$ for $n \geq 0$.

Delannoy triangle row entries $d_{n,j}$ are the anti-diagonals of the Delannoy matrices $D_{n,k}$. And the anti-diagonals of the Delannoy triangle $d_{n,j}$ were shown in [11] to give the Tribonacci sequence $\{1, 1, 2, 4, 7, 13, \ldots\}$.

	Delannoy triangles $d_{n,i}$						
n	$i=0$	1	2	3	4	5	A_n
0	1						1
1	1	1					2
2	1	3	1				5
3	1	5	5	1			12
4	1	7	13	7	1		29
5	1	9	25	25	9	1	70

Delannoy triangle row entries $d^*_{n,j}$ are Delannoy matrices $D^*_{n,k}$ anti-diagonals.

	Delannoy triangles $d^*_{n,i}$						
n	$i=0$	1	2	3	4	5	A_n
0	1						1
1	1	1					2
2	1	4	1				6
3	1	6	6	1			14
4	1	8	16	8	1		34
5	1	10	30	30	10	1	82

The sum A_n of the entries in the n^{th} row of the Delannoy triangles $d_{n,j}$ is

$$A_{n+1} \;=\; Q_n + t\,P_n \quad \text{for } n \geq 0$$

with Pell numbers P_n and Pell Lucas numbers Q_n and taking $A_0 = 1$.

4. Primality testing

Second order recursions can be used to test primality of special number systems [5], like the Mersenne numbers and other number sequences.

4.1 The Lucas-Lehmer Test

In the Lucas-Lehmer test [10], Mersenne numbers $M_n{=}2^n-1$ are checked for primality by the recursion $s_n \equiv s^2_{n-1} - 2$ with initial condition $s_0 = 4$. The Mersenne number M_p is prime, if and only if $s_{p-2} \equiv 0 \pmod{M_p}$.

The $\{s_n\}_{n\geq 0}$ sequence in [6] begins $\{4,\ 14,\ 194,\ 37634,\ 1416317954, \ldots\}$. For example: Let $M_7 = 2^7-1 = 127$, the sequence residues for $\{s_n\} \pmod{M_7}$ are $\{4, 14, 67, 42, 111, 0, 125, 2, 2, 2, \ldots\}$. This primality result corresponds to the Iterated Sum case with base $b = M_7+1 = 2^7 = 128$ with Iterated Sum of Digits $\$_{2^7}(\{s_n\})$ of $\{4, 14, 67, 42, 111, 127, 125, 2, 2, 2, \ldots\}$.

We can thus state that comparable conditions exist for Mersenne primes $M_p = 2^p-1$ under either testing approach, whether by Congruences or by Iterated Sum of Digits, according to

$$\begin{aligned} s_{p-2} \pmod{M_p} &\equiv 0 & &\Leftrightarrow & \$_{2^p}(s_{p-2}) &= M_p. \\ s_{p-1} \pmod{M_p} &\equiv M_p-2 & &\Leftrightarrow & \$_{2^p}(s_{p-1}) &= M_p-2. \\ s_i \pmod{M_p} &\equiv 2 & &\Leftrightarrow & \$_{2^p}(s_i) &= 2 \quad \text{for } i \geq p. \end{aligned}$$

Although the above criteria do not accelerate the typical primality search ahead of the recursion index for s_{p-2}, the following conjecture does.

Conjecture: The Mersenne number $M_p = 2^p - 1$ is prime, if and only if

$$s_{p-3} \ (\text{mod } M_p) \equiv \pm 2^{\frac{p+1}{2}}. \quad \Leftrightarrow \quad \$_{2^p}(s_{p-3}) = \begin{cases} M_p - 2^{\frac{p+1}{2}} \text{ or} \\ 2^{\frac{p+1}{2}}. \end{cases}$$

The conjecture is based on empirical research, with both Congruence and Iterated Sum of Digits cases. For example: $111 \equiv s_4 (\text{mod } M_7) \equiv -16$.

4.2 Other Related Primality Tests

In Lucas-Lehmer primality testing, the fast Lucas type recursion comes from a full recursion $b_i(r)$ with $r=\sqrt{2}$, which has a corresponding $a_i(r)$ sequence of Fibonacci type. As in (2,3), these full recursions have the form

$$b_i = r\, b_{i-1} + b_{i-2}, \qquad b_0 = 2,\ b_1 = r, \tag{17}$$

$$a_i = r\, a_{i-1} + a_{i-2}, \qquad a_0 = 0,\ a_1 = 1. \tag{18}$$

	Sequences at $r=\sqrt{2}$								
	$i=0$	1	2	3	4	5	6	7	8
b_i	2	$\sqrt{2}$	4	$5\sqrt{2}$	14	$19\sqrt{2}$	52	$71\sqrt{2}$	194
a_i	0	1	$\sqrt{2}$	3	$4\sqrt{2}$	11	$15\sqrt{2}$	41	$56\sqrt{2}$

Using the $a_i(r)$ sequences, other fast primality tests for Mersenne numbers can be obtained by pairing the $a_i(r)$ and $b_i(r)$ recursions.

Fast Recursion Pair: Take recursions $\beta_{n+1}=\beta_n^2-2$ and $\alpha_{n+1}=\alpha_n\,\beta_n$, with initial values α_0=1, β_0=4, and define $M_p=2^p-1$, then

$$M_p \ \text{ is prime iff } \ \alpha_{p-1} \equiv 0 \ (\text{mod } M_p).$$

Proof: Identity $F_{2h}=F_hL_h$ in [8] is special case of $a_{2h}=a_hb_h$. Use a_h,b_h in recursions (17,18) at $r=\sqrt{2}$ for $h>0$. Repeating identity gives $a_{2^{n+1}}=a_{2^n}b_{2^n}$. Since $\beta_{n+1}=\beta_n^2-2$ with β_0=4 is the recursion for the Lucas-Lehmer test variable, we have $s_n=\beta_n=b_{2^{n+1}}$ with the associated variable $\alpha_n=a_{2^{n+1}}$, by dropping the common $\sqrt{2}$ factor. So for Mersenne primes $M_p=2^p-1$, the primality test $\alpha_{p-1} = \alpha_{p-2}\,\beta_{p-2} \equiv 0$ (mod M_p) corresponds to the Lucas-Lehmer test $s_{p-2}= \beta_{p-2} \equiv 0 (\text{mod } M_p)$ for primality.

Another Pair: Take recursions $\beta_{n+1}=\beta_n^2-2$ and $\alpha_{n+1}=\alpha_n\beta_n-1$, with initial values α_0=3, β_0=4, and define $M_p=2^p-1$, then

$$M_p \ \text{ is prime iff } \ \alpha_{p-1} \equiv -1 \ (\text{mod } M_p).$$

Proof: $F_{4h+1}=F_{2h+1}L_{2h}-1$ in [8] is a special case of $a_{4h+1}=a_{2h+1}b_{2h}-1$. Repeating the identity gives $a_{2^{n+1}+1}=a_{2^n+1}b_{2^n}-1$. Use a_h,b_h in recursions (17,18) at $r=\sqrt{2}$ for $h>0$. Since $\beta_{n+1}=\beta_n^2-2$ with β_0=4 is the recursion for the Lucas-Lehmer test variable, we get $s_n=\beta_n=b_{2^{n+1}}$ and paired variable $\alpha_n=a_{2^{n+1}+1}$, having dropped the common factor $\sqrt{2}$. For Mersenne primes $M_p = 2^p-1$, the primality test $\alpha_{p-1} = \alpha_{p-2}\,\beta_{p-2} - 1 \equiv -1$ (mod M_p) then corresponds to the Lucas-Lehmer primality test $s_{p-2}= \beta_{p-2} \equiv 0 (\text{mod } M_p)$.

Besides the primality tests for Mersenne primes discussed above, we conclude with conjectures for testing primality of two differently configured numbers, namely $E_t=4^t-5$ and $E_t^*=4^t-3$.

Conjecture for E_t: If we take the recursion $\alpha_n = 14\,\alpha_{n-1} - \alpha_{n-2}$ starting with initial values $\alpha_0{=}3$, $\alpha_1{=}1$, and define $E_t = 4^t - 5$ for $t \geq 2$, then

$$E_t \text{ is prime iff } \alpha_h \equiv 0 \pmod{E_t} \quad \text{at} \quad h = 2^{2t-3}.$$

We note: $\alpha_h{=}a_{4h-3}$ in recursion (18) at $r{=}\sqrt{2}$ for $h{>}0$. The α_h values are the coefficients in the series expansion of the generating function $\frac{3-41x}{1-14x+x^2}$.

Conjecture for E_t^*: If we take the same recursion $\beta_n = 14\,\beta_{n-1} - \beta_{n-2}$ with initial values $\beta_0{=}4$, $\beta_1{=}4$, and define $E_t^* = 4^t - 3$ for $t \geq 2$, then

$$E_t^* \text{ is prime iff } \beta_h \equiv 0 \pmod{E_t^*} \quad \text{at} \quad h = 2^{2t-3}.$$

We note: $\beta_h{=}b_{4h-2}$ in recursion (17) at $r{=}\sqrt{2}$ for $h{>}0$. The β_h values are the coefficients in the series expansion of the generating function $\frac{4-52x}{1-14x+x^2}$.

Of course, E_t and E_t^* are twin primes when their primality conditions are simultaneously satisfied at the same t value.

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