Acta Mathematica Universitatis Ostraviensis

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Acta Mathematica Universitatis Ostraviensis, Vol. 14 (2006), No. 1, 21--25

Persistent URL: `http://dml.cz/dmlcz/137479`

Truncatable primes and unavoidable sets of divisors

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Abstract. We are interested whether there is a nonnegative integer u_0 and an infinite sequence of digits $u_1, u_2, u_3, \dots$ in base b such that the numbers $u_0 b^n + u_1 b^{n-1} + \dots + u_{n-1} b + u_n$, where $n = 0, 1, 2, \dots$, are all prime or at least do not have prime divisors in a finite set of prime numbers S. If any such sequence contains infinitely many elements divisible by at least one prime number $p \in S$, then we call the set S *unavoidable with respect to b*. It was proved earlier that unavoidable sets in base b exist if $b \in \{2, 3, 4, 6\}$, and that no unavoidable set exists in base $b = 5$. Now, we prove that there are no unavoidable sets in base $b \geqslant 3$ if $b - 1$ is not square-free. In particular, for $b = 10$, this implies that, for any finite set of prime numbers $\{p_1, \dots, p_k\}$, there is a nonnegative integer u_0 and $u_1, u_2, \dots \in \{0, 1, \dots, 9\}$ such that the number $u_0 10^n + u_1 10^{n-1} + \dots + u_n$ is not divisible by $p_1, \dots, p_k$ for each integer $n \geqslant 0$.

1. Truncatable primes

Following [1], [8] and [10], we call a positive integer N *right truncatable prime* if N itself and all numbers obtained by successively removing the rightmost digits are prime. For instance, the number 31193 is right truncatable prime, because the numbers 31193, 3119, 311, 31 and 3 are all prime. There are exactly 83 right truncatable primes primes in base 10, the largest being 73939133 (see [1] and [10]).

Similarly, a positive integer N without zero digits is called *left truncatable prime* if N itself and all numbers obtained by successively removing the leftmost digits are prime. It is known that there are exactly 4260 left truncatable primes (in base 10), the largest being

$$357686312646216567629137$$

(see, e.g., the sequence $A024785$ in N.J.A. Sloane's "On-Line Encyclopedia of Integer Sequences" `http://www.research.att.com/~njas/` or [1]).

Received: November 28, 2006.

2000 *Mathematics Subject Classification:* 11A41, 11A63, 11B50.

Key words and phrases: Prime numbers, truncatable primes, integer expansions, square-free numbers.

This research was supported in part by the Lithuanian State Studies and Science Foundation.

Naturally, one can extend these definitions to any integer base $b \geqslant 2$. However, since all above truncatable primes were found by a routine computation and since there are no theoretical results, even some simply looking questions, as one expects, are very difficult to answer. For example, it is not known whether, for any base b, there only finitely many or infinitely many right truncatable primes and left truncatable primes (without zero digits). See, however, the sequence $A033664$ in the above mentioned Sloane's Encyclopedia if zero digits are permitted in left truncatable primes in base 10.

Note that instead of removing digits from the right we can add them step-by-step to the right and ask whether, in base b, there is an infinite word $u_0u_1u_2u_3\dots$, where $u_0, u_1, u_2, \dots \in \{0, 1, \dots, b-1\}$, such that its every *prefix* (beginning) in base b, namely, the number $u_0b^n + u_1b^{n-1} + \dots + u_n$ is prime. Moreover, let us allow u_0 to be an arbitrary nonnegative integer. We call an infinite word $u_0u_1u_2u_3\dots$, where $u_0 \in \mathbb{N}$ and $u_1, u_2, u_3, \dots \in \{0, 1, \dots, b-1\}$, *infinite right truncatable prime in base* b if the numbers $u_0b^n + u_1b^{n-1} + \dots + u_n$, $n = 0, 1, 2, \dots$, are all prime.

We begin with the following conjecture:

Conjecture 1. *If $b \geqslant 2$ is an integer then there are no infinite right truncatable primes in base b.*

2. Integer expansions

Note that writing a positive number ξ by the sum of its integral and fractional parts $[\xi] + \{\xi\}$ and expanding the fractional part $\{\xi\}$ in its b-adic expansion we obtain that

$$\xi = [\xi] + \{\xi\} = u_0 + u_1b^{-1} + u_2b^{-2} + u_3b^{-3} + \dots .$$

Here, $u_0 = [\xi] \geqslant 0$ is an integer, $u_1, u_2, u_3, \dots \in \{0, 1, \dots, b-1\}$, and no $n \in \mathbb{N}$ exists for which $u_n = u_{n+1} = \dots = b - 1$.

Conjecture 2. *If $b \geqslant 2$ is an integer then, for any real $\xi > 0$, the sequence $[\xi b^n]$, $n = 0, 1, 2, \dots$, contains infinitely many composite numbers.*

We will show below that Conjecture 1 is equivalent to Conjecture 2. Apparently, Conjecture 2 is true for any real number $b > 1$. By a result of Koksma [9], Conjecture 2 holds for almost all $b > 1$. However, there are very few specific b for which this result was established. See, e.g., [3], [5], [6], [11] and Problem E19 in [7], where some special cases with some $b > 1$ have been treated. In [2] and [4], some questions related to Conjecture 2 are discussed for the fractional parts $\{\xi b^n\}$ and for the distances to the nearest integer $||\xi b^n||$, where $b \geqslant 2$ is an integer and $n = 0, 1, 2, \dots$.

In [5], Conjecture 2 was proved for $b = 2, 3, 4, 5, 6$. However, there is a key difference in the proof of Conjecture 2 for $b \in \{2, 3, 4, 6\}$ and for $b = 5$. In order to explain this, we recall that any finite set of prime numbers $S = S(b)$ is called *unavoidable with respect to* b if, for any $\xi > 0$, the sequence $[\xi b^n]$, $n = 0, 1, 2, \dots$, contains infinitely many elements divisible by at least one prime number $p \in S(b)$.

For $b = 2$, using the fact that $u_n < 2 - 1 = 1$ for infinitely many $n \in \mathbb{N}$, we find that there are infinitely many $n \in \mathbb{N}$ for which $u_n = 0$. Thus there are infinitely many even numbers among $[\xi 2^n]$, $n \in \mathbb{N}$, and so the set $\{2\}$ is unavoidable with respect to $b = 2$. In a similar manner, we showed in [5] that the set $\{2, 3\}$ is unavoidable with respect to $b = 3$ and to $b = 4$, and that the set $\{2, 3, 5\}$ is unavoidable with respect to $b = 6$.

On the other hand, although Conjecture 2 was proved in [5] for $b = 5$ too, we showed there that no finite unavoidable set exists with respect to $b = 5$. Namely, for any finite set of primes S, there is a real number $\xi = \xi(S) > 0$ such that each number $[\xi 5^n]$, where $n \geqslant 0$ is an integer, is not divisible by any $p \in S$. We conjecture that no finite unavoidable set exists with respect to b if $b \in \mathbb{N} \setminus \{2, 3, 4, 6\}$.

Conjecture 3. *If $b \in \mathbb{N} \setminus \{2, 3, 4, 6\}$ then no finite unavoidable set exists with respect to b.*

Conjecture 3 clearly holds for $b = 1$, because then, for any finite set of primes S, we can take $\xi = p$, where $p \notin S$ is a prime number. (Then the number $[\xi 1^n] = p$ is relatively prime to any element of S.) As we said above, the case $b = 5$ of Conjecture 3 was established in [5]. The next theorem extends this result from $b = 5$ to those $b \in \mathbb{N}$ for which the number $b - 1$ is not square-free, that is, $b - 1$ is divisible by a square of an integer > 1.

Theorem 4. *If $b \geqslant 3$ is an integer for which $b - 1$ is not square-free then no finite unavoidable set exists with respect to b.*

Note that the theorem is applicable to $b = 10$, because $10 - 1 = 3^2$. This shows that the interesting case $b = 10$ of Conjectures 1 and 2 cannot be proved by the method of unavoidable sets. A simple reduction modulo $2, 3, 5, 7$ and 11 shows, however, that if the numbers $u_0 10^n + u_1 10^{n-1} + \dots + u_n$, $n = 0, 1, 2, \dots$, are all prime then, from certain place, the word $u_0 u_1 u_2 u_3 \dots$ contains the blocks of digits 933, 99333, 9939333, 993939333, 99393939333 and 9939393939333 only.

3. Proofs

We first prove that Conjecture 1 is equivalent to Conjecture 2. Fix $b \geqslant 2$. In order to show that Conjecture 1 $\Rightarrow$ Conjecture 2, we suppose that there is a positive number ξ' such that there are only finitely many composite numbers in the set $[\xi' b^n]$, $n \in \mathbb{N}$. Then there is an integer n_0 such that the numbers $[\xi' b^n]$, $n = n_0, n_0 + 1, n_0 + 2, \dots$, are all prime. On replacing ξ' by $\xi = \xi' b^{n_0}$ we obtain that the numbers $[\xi b^n]$, $n = 0, 1, 2, \dots$, are all prime. Writing ξ by the sum of $[\xi] = u_0$ and $\{\xi\} = u_1 b^{-1} + u_2 b^{-2} + u_3 b^{-3} + \dots$, where $u_1, u_2, u_3, \dots \in \{0, 1, \dots, b-1\}$ and no $n \in \mathbb{N}$ exists for which $u_n = u_{n+1} = \dots = b - 1$, we obtain that the numbers

$$[\xi b^n] = u_0 b^n + u_1 b^{n-1} + \dots + u_{n-1} b + u_n$$

are prime for each nonnegative integer n. Hence the word $u_0 u_1 u_2 u_3 \dots$ is an infinite right truncatable prime in base b, a contradiction with Conjecture 1.

In order to prove that Conjecture 2 implies Conjecture 1, we consider all possible words $u_0 u_1 u_2 u_3 \dots$, where $u_0 \geqslant 0$ is an integer and $u_1, u_2, u_3, \dots \in \{0, 1, \dots, b - 1\}$. Without loss of generality we can assume that $u_j > 0$ for at least one $j \geqslant 0$. Suppose first that the word $u_0 u_1 u_2 u_3 \dots$ has no *suffix* (end) of the form $(b-1)^\infty = (b-1)(b-1)(b-1)\dots$. Then $u_0 + u_1 b^{-1} + u_2 b^{-2} + \dots$ is the b-adic expansion of, say $\xi > 0$. By Conjecture 2, it follows that the number $[\xi b^n] = u_0 b^n + \dots + u_{n-1} b + u_n$ is composite for infinitely many $n \in \mathbb{N}$, so the word $u_0 u_1 u_2 u_3 \dots$ is not an infinite right truncatable prime in base b.

Alternatively, suppose that $u_0u_1u_2u_3\dots$ is of the form $u_0\dots u_{m-1}(b-1)^\infty$. Set $v_0 = u_0b^{m-1} + u_1b^{m-2} + \dots + u_{m-1}$. We need to show that then the word $v_0(b-1)^\infty$ is not an infinite right truncatable prime in base b. If it would be so, then the numbers $x_0 = v_0$, $x_j = bx_{j-1} + b - 1$, where $j = 1, 2, \dots$, would all be prime numbers. Fix any $j > 0$ for which $x_j > 1$. Let $q > 1$ be a prime divisor of x_j. From $x_j = bx_{j-1} + b - 1$ we see that q does not divide b. It follows that

$$x_{j+q-1} = bx_{j+q-2} + b - 1 = b^2x_{j+q-3} + b^2 - 1 = \dots = b^{q-1}x_j + b^{q-1} - 1$$

is divisible by q, because $q|x_j$ and $q|(b^{q-1}-1)$, by Fermat's little theorem. Thus x_{j+q-1} is composite. Hence the word $v_0(b-1)^\infty$ is not an infinite right truncatable prime in base b. This completes the proof of the implication Conjecture 2 $\Rightarrow$ Conjecture 1.

Proof of Theorem 4: If $b-1$, where $b \geqslant 3$, is not square-free, then there is a prime number q such that $b = q^2k+1$, where $k \in \mathbb{N}$. Let S be an arbitrary finite set of primes, and let $P = P(S) = \prod_{p \in S\setminus\{q\}} p$. Take $s \in \mathbb{N}$ such that $sP = q^2\ell + q + 1$, where $\ell \in \mathbb{N}$. Set $\xi = sP/q$.

Then, since sPb^n modulo q is equal to 1, we obtain that $[\xi b^n] = [sPb^n/q] = (sPb^n - 1)/q$. Furthermore, since sPb^n modulo q^2 is equal to $q+1$, we deduce that $(sPb^n-1)/q$ modulo q is equal to 1. Hence $[\xi b^n]$ is not divisible by q for each $n = 0, 1, 2, \dots$.

Similarly, for each integer $n \geqslant 0$, the integer $[\xi b^n] = (sPb^n - 1)/q$ is not divisible by $p \in S \setminus \{q\}$. Indeed, if it would be so, then $sPb^n - 1$ would be divisible by p, which is not the case, by the definition of P. This completes the proof of the theorem.

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