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Discrete limit theorems for the Laplace transform of the Riemann zeta-function

R. Kačinskaitė and A. Laurinčikas

Abstract. In the paper discrete limit theorems in the sense of weak convergence of probability measures on the complex plane as well as in the space of analytic functions for the Laplace transform of the Riemann zeta-function are proved.

1. Introduction

Let $s = \sigma + it$ be a complex variable, $\zeta(s)$ denote the Riemann zeta-function, and let

$$L(s) = \int_0^\infty |\zeta(\frac{1}{2} + ix)|^2 \mathrm{e}^{-sx} \mathrm{d}x$$

be the Laplace transform of $|\zeta(\frac{1}{2} + ix)|^2$. In view of the estimate

$$\zeta(\frac{1}{2} + it) \ll t^{\frac{32}{205}+\varepsilon}, \quad t \geq t_0 > 0,$$

with every $\varepsilon > 0$, the integral defining $L(s)$ converges absolutely and uniformly on compact subsets of the half-plane $D = \{s \in \mathbb{C} : \sigma > 0\}$, and defines there an analytic function. Here, as usual, $\mathbb{C}$ denotes the complex plane.

The function $L(s)$ is an useful tool to study the mean square of the Riemann zeta-function

$$\int_0^T |\zeta(\frac{1}{2} + it)|^2 \mathrm{d}t,$$

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see, for example, [6], where $L(s)$ was applied in a new proof of the Atkinson formula [1], [5]. Therefore, the value distribution of the function $L(s)$ is an interesting and important problem of analytic number theory.

In [7] limit theorem in the sense of weak convergence of probability measures for $L(s)$ were proved. Let $\text{meas}\{A\}$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T^t(\ldots) = \frac{1}{T}\text{meas}\{t \in [0,T] : \ldots\},$$

where in place of dots a condition satisfied by t is to be written, and the sign t in ν_T^t indicates only that the measure is taken over $t \in [0,T]$. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Then in [7] two following statements were proved.

Theorem A. *Let $\sigma > 0$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\nu_T^t(L(\sigma+it) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $T \to \infty$.

Theorem B. *On $(H(D), \mathcal{B}(H(D)))$ there exists a probability measure $\widehat{P}$ such that the probability measure*

$$\nu_T^\tau(L(s+i\tau) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $\widehat{P}$ as $T \to \infty$.

Theorems A and B in a certain sense show the regularity of the behaviour of the function $L(s)$. In these theorems, the probability measures defined by continuous translations of $L(s)$ in the interval $[0,T]$ are considered. Therefore, Theorems A and B can be named as continuous limit theorems. It turns out that also discrete limit theorems for the function $L(s)$ can be considered. In theorems of such a kind, the weak convergence of probability measures defined by translations of $L(s)$ in some arithmetic progressions studied. Let, for non-negative integer N,

$$\mu_N(\ldots) = \frac{1}{N+1}\#\{0 \le m \le N : \ldots\},$$

where in place of dots a condition satisfied by m is to be written, and let h is a fixed positive number. Define the probability measure

$$P_{N,\sigma}(A) = \mu_N(L(\sigma+imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}).$$

Theorem 1. *Let $\sigma > 0$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the measure $P_{N,\sigma}$ converges weakly to P_σ as $N \to \infty$.*

Now let

$$\widehat{P}_N(A) = \mu_N(L(s+imh) \in A), \quad A \in \mathcal{B}(H(D)).$$

Theorem 2. *On $(H(D), \mathcal{B}(H(D)))$ there exists a probability measure $\widehat{P}$ such that the measure $\widehat{P}_N$ converges weakly to $\widehat{P}$ as $N \to \infty$.*

We begin the proof of Theorems 1 and 2 with discrete limit theorems for integrals over finite intervals.

2. Discrete limit theorems for integrals over finite intervals

For $a > 0$, let

$$L_a(s) = \int_0^a |\zeta(\frac{1}{2} + ix)|^2 \mathrm{e}^{-sx} \mathrm{d}x.$$

In this section we will prove limit theorems for the probability measures

$$P_{N,\sigma,a}(A) = \mu_N(L_a(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

and

$$\widehat{P}_{N,a} = \mu_N(L_a(s + imh) \in A), \quad A \in \mathcal{B}(H(D)).$$

Theorem 3. *On $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure $P_{\sigma,a}$ such that the measure $P_{N,\sigma,a}$ converges weakly to $P_{\sigma,a}$ as $N \to \infty$.*

Theorem 4. *On $(H(D), \mathcal{B}(H(D)))$ there exists a probability measure $\widehat{P}_a$ such that the measure $\widehat{P}_{N,a}$ converges weakly to $\widehat{P}_a$ as $N \to \infty$.*

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle, and let

$$\Omega_a = \prod_{u \in [0,a]} \gamma_u,$$

where $\gamma_u = \gamma$ for each $u \in [0, a]$. By the Tikhonov theorem, the torus Ω_a is a compact topological Abelian group. Note that Ω_a consists of all functions $f : [0, a] \to \gamma$. Define

$$Q_{N,a}(A) = \mu_N(\{\mathrm{e}^{imhu} : \ u \in [0, a]\} \in A), \quad A \in \mathcal{B}(\Omega_a).$$

The proof of Theorems 3 and 4 is based on the following lemma.

Lemma 5. *On $(\Omega_a, \mathcal{B}(\Omega_a))$ there exists a probability measure Q_a such that the probability measure $Q_{N,a}$ converges weakly to Q_a as $N \to \infty$.*

Proof. Let, as usual, $\mathbb{Z}$ denote the set of all integers. The dual group of Ω_a is isomorphic to

$$\oplus_{u \in [0,a]} \mathbb{Z}_u,$$

where $\mathbb{Z}_u = \mathbb{Z}$ for each $u \in [0, a]$. An element $\underline{k} = \{k_u, u \in [0, a]\}$ of $\oplus_{u \in [0,a]} \mathbb{Z}_u$, where only a finite number of integers k_u are non zero, acts on Ω_a by

$$\underline{k} \to \underline{x}^{\underline{k}} = \prod_{u \in [0,a]} x_u^{k_u},$$

where $\underline{x} = \{x_u : x_u \in \gamma, u \in [0, a]\}$. Hence the Fourier transform $g_{N,a}(\underline{k})$ of the probability measure $Q_{N,a}$ is of the form

$$\begin{aligned} g_{N,a}(\underline{k}) &= \int_{\Omega_a} \left(\prod_{u \in [0,a]} x_u^{k_u} \right) \mathrm{d}Q_{N,a} \\ &= \frac{1}{N+1} \sum_{m=0}^{N} \prod_{u \in [0,a]} \mathrm{e}^{imhuk_u} \end{aligned}$$

$$= \frac{1}{N+1}\sum_{m=0}^{N}\exp\{imh\sum_{u\in[0,a]}uk_u\}. \tag{1}$$

Here, as it was noted above, only a finite number of integers k_u are non zero.

It is easy to see that

$$\exp\{ih\sum_{u\in[0,a]}uk_u\}=1$$

if and only if there exists $r\in\mathbb{Z}$ such that

$$\sum_{u\in[0,a]}uk_u=\frac{2\pi r}{h}.$$

Therefore, in view of (1)

$$g_{N,a}(\underline{k}) = \begin{cases}1, & \text{if } \sum\limits_{u\in[0,a]}uk_u=\frac{2\pi r}{h} \text{ for some } r\in\mathbb{Z},\\ \frac{1}{N+1}\frac{1-\exp\{i(N+1)h\sum_{u\in[0,a]}uk_u\}}{1-\exp\{ih\sum_{u\in[0,a]}uk_u\}}, & \text{otherwise.}\end{cases}$$

Consequently,

$$\lim_{N\to\infty}g_{N,a}(\underline{k})=\begin{cases}1, & \text{if } \sum_{u\in[0,a]}uk_u=\frac{2\pi r}{h} \text{ for some } r\in\mathbb{Z},\\ 0, & \text{otherwise.}\end{cases}$$

Hence, by the continuity theorem for probability measures on locally compact topological groups, see, for example, [4], it follows that the probability measure $Q_{N,a}$ converges weakly to a probability measure Q_a defined by the Fourier transform

$$g(\underline{k})=\begin{cases}1, & \text{if } \sum_{u\in[0,a]}uk_u=\frac{2\pi r}{h} \text{ for some } r\in\mathbb{Z},\\ 0, & \text{otherwise.}\end{cases}$$

Proof of Theorem 3. Define, for $y_x\in\Omega_a$,

$$\widehat{y}_x=\begin{cases}y_x, & \text{if } y_x \text{ is integrable over } [0,a],\\ \text{arbitrary integrable function} \in\Omega_a, & \text{otherwise,}\end{cases}$$

and let the function $f_\sigma:\ \Omega_a\to\mathbb{C}$ be given by the formula

$$f_\sigma(y_x)=\int_0^a|\zeta(\frac{1}{2}+ix)|^2\mathrm{e}^{-\sigma x}\widehat{y_x}^{-1}\mathrm{d}x,\quad y_x\in\Omega_a.$$

Then by the definition of $\widehat{y}_x$ and the Lebesgue theorem of bounded convergence the function f_σ is continuous, moreover,

$$f_\sigma(\mathrm{e}^{imhx})=\int_0^a|\zeta(\frac{1}{2}+ix)|^2\mathrm{e}^{-(\sigma+imh)x}\mathrm{d}x=L_a(\sigma+imh).$$

Therefore, $P_{N,\sigma,a}=Q_{N,a}f_\sigma^{-1}$, and by Theorem 5.1 of [2] and Lemma 5 we obtain that the probability measure $P_{N,\sigma,a}$ converges weakly to $Q_af_\sigma^{-1}$ as $N\to\infty$.

Proof of Theorem 4. We argue similarly to the proof of Theorem 3. Define $\widehat{f} : \Omega_a \to H(D)$ by the formula

$$\widehat{f}(y_x) = \int_0^a |\zeta(\frac{1}{2} + ix)|^2 \mathrm{e}^{-sx} \widehat{y_x}^{-1} \mathrm{d}x, \quad \widehat{y}_x \in \Omega_a.$$

Then $\widehat{f}$ is a continuous function, and

$$\widehat{f}(\mathrm{e}^{imhx}) = \int_0^a |\zeta(\frac{1}{2} + ix)|^2 \mathrm{e}^{-(s+imh)x} \mathrm{d}x = L_a(s + imh).$$

Hence we have again that $\widehat{P}_{N,a} = Q_{N,a} \widehat{f}^{-1}$, and the theorem follows in the same way as Theorem 3.

3. Proof of Theorem 1

Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Let (S, ϱ) be a separable metric space with a metric ϱ, and let $Y_n, X_{1n}, X_{2n}, \ldots$ be the S-valued random elements defined on a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$. We will use the following statement.

Lemma 6. *Suppose that* $X_{kn} \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X_k$ *for each* k, *and that* $X_k \underset{k\to\infty}{\longrightarrow} X$. *If, for every* $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(\varrho(X_{kn}, Y_n) \geq \varepsilon) = 0,$$

then $Y_n \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X$.

The lemma is Theorem 4.2 of [2] where its proof is given.

Proof of Theorem 1. By Theorem 3 the probability measure $P_{N,\sigma,a}$ converges weakly to some probability measure $P_{\sigma,a}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $N \to \infty$. On a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ define a random variable θ_N by

$$\mathbb{P}(\theta_N = hm) = \frac{1}{N+1}, \quad m = 0, 1, \ldots, N,$$

and put

$$X_{N,a} = X_{N,a}(\sigma) = L_a(\sigma + i\theta_N).$$

Then we have that

$$X_{N,a} \underset{N\to\infty}{\xrightarrow{\mathcal{D}}} X_a, \tag{2}$$

where $X_a = X_a(\sigma)$ is a complex-valued random variable with the distribution $P_{\sigma,a}$. By the Chebyshev inequality we find, for $M > 0$,

$$\mathbb{P}(|X_{N,a}| > M) \leq \frac{1}{(N+1)M} \sum_{m=0}^{N} |L_a(\sigma + imh)|.$$

Since the integral for $L(s)$ converges absolutely for $\sigma > 0$, hence we have that

$$\limsup_{N\to\infty} \mathbb{P}(|X_{N,a}| > M) \leq \frac{1}{M} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} |L_a(\sigma + imh)| \leq R < \infty. \tag{3}$$

Let $\varepsilon > 0$ be arbitrary, and $M = R\varepsilon^{-1}$. Then (3) shows that

$$\limsup_{N\to\infty} \mathbb{P}(|X_{N,a}| > M) \le \varepsilon. \tag{4}$$

The function $v: \mathbb{C} \to \mathbb{R}$ given by $v(z) = |z|$, $z \in \mathbb{C}$, is continuous, therefore Theorem 5.1 of [2] and (2) yield

$$|X_{N,a}| \xrightarrow[N\to\infty]{\mathcal{D}} |X_a|.$$

This and (4) show that

$$\mathbb{P}(|X_a| > M) \le \varepsilon. \tag{5}$$

Now we recall some notions and results of Prokhorov's theory. A family of probability measures $\{P\}$ on $(S, \mathcal{B}(S))$ is called tight if, for every $\varepsilon > 0$, there exists a compact subset K of S such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$, and $\{P\}$ is relatively compact if every sequence of $\{P\}$ contains a weakly convergent subsequence. By Prokhorov's theory, see, for example, [2], if the family $\{P\}$ is tight, then it is also relatively compact.

The set $K_\varepsilon = \{s \in \mathbb{C} : |s| \le M\}$ is compact, and in view of (5)

$$\mathbb{P}(X_a \in K_\varepsilon) \ge 1 - \varepsilon,$$

or, by the definition of X_a,

$$P_{\sigma,a}(K_\varepsilon) \ge 1 - \varepsilon$$

for all $a > 0$. This shows that the family of probability measures $\{P_{\sigma,a}\}$ is tight, and therefore, it is relatively compact. Let $\{P_{\sigma,a_1}\} \subset \{P_{\sigma,a}\}$ be such that P_{σ,a_1} converges weakly to some probability measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $a_1 \to \infty$. Then we have that

$$X_{a_1} \xrightarrow[a_1\to\infty]{\mathcal{D}} P_\sigma. \tag{6}$$

Moreover, by the definition of $L_a(s)$, for $\sigma > 0$,

$$\lim_{a\to\infty} L_a(s) = L(s), \tag{7}$$

uniformly in t. Therefore, for $\sigma > 0$,

$$\lim_{a\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} |L(\sigma + imh) - L_a(\sigma + imh)| = 0.$$

Hence by Chebyshev's inequality, for every $\varepsilon > 0$ and $\sigma > 0$,

$$\begin{aligned} &\lim_{a\to\infty} \limsup_{N\to\infty} \mu_N(|L(\sigma + imh) - L_a(\sigma + imh)| \ge \varepsilon) \\ &\le \lim_{a\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{m=0}^{N} |L(\sigma + imh) - L_a(\sigma + imh)| = 0. \end{aligned} \tag{8}$$

Now define

$$X_N = X_N(\sigma) = L(\sigma + i\theta_N).$$

Relations (2), (6) and (8) show that all hypotheses of Lemma 6 are satisfied. Therefore,

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P_\sigma,$$

and this is equivalent to the assertion of Theorem 1.

4. Proof of Theorem 2

For the proof of Theorem 2 we need a metric on $H(D)$ which induces its topology. It is known, see, for example, [3], that there exists a sequence $\{K_n\}$ of compact subsets of D such that

$$D = \bigcup_{n=1}^{\infty} K_n,$$

$K_n \subset K_{n+1}$, $n = 1, 2, \ldots$, and if K is a compact of D, then $K \subseteq K_n$ for some n. For $f, g \in H(D)$, define

$$\varrho(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\varrho_n(f,g)}{1+\varrho_n(f,g)},$$

where

$$\varrho_n(f,g) = \sup_{s \in K_n} |f(s) - g(s)|.$$

Then it is not difficult to see that ϱ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta.

Proof of Theorem 2. We preserve the notation of the proof of Theorem 1. Define

$$\widehat{X}_{N,a} = \widehat{X}_{N,a}(s) = L_a(s + i\theta_N).$$

Then by Theorem 4 we have that

$$\widehat{X}_{N,a} \xrightarrow[N\to\infty]{\mathcal{D}} \widehat{X}_a, \tag{9}$$

where $\widehat{X}_a = \widehat{X}_a(s)$ is an $H(D)$-valued random element with the distribution $\widehat{P}_a$. For $M_n > 0$,

$$\mathbb{P}(\sup_{s\in K_n} |\widehat{X}_{N,a}(s)| > M_n) \leq \frac{1}{M_n(N+1)} \sum_{m=0}^{N} \sup_{s\in K_n} |L_a(s + imh)|.$$

The integral for $L(s)$ converges uniformly on compact subsets of D, therefore hence we obtain that

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}(\sup_{s\in K_n} |\widehat{X}_{N,a}(s)| > M_n) \\
&\leq \frac{1}{M_n} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s\in K_n} |L_a(s + imh)| \leq R_n < \infty.
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary and $M_n = R_n 2^n \varepsilon^{-1}$. Then the last inequality yields

$$\limsup_{N\to\infty} \mathbb{P}(\sup_{s\in K_n} |\widehat{X}_{N,a}(s)| > M_n) \leq \frac{\varepsilon}{2^n}, \quad n = 1, 2, \ldots. \tag{10}$$

Consider a function $v: H(D) \to \mathbb{R}$ defined by

$$v(f) = \sup_{s\in K_n} |f(s)|, \quad f \in H(D).$$

Clearly, the function v is continuous, therefore by (9) and Theorem 5.1 of [2]

$$\sup_{s\in K_n} |\widehat{X}_{N,a}(s)| \xrightarrow[N\to\infty]{\mathcal{D}} \sup_{s\in K_n} |\widehat{X}_a(s)|.$$

This together with (10) gives

$$\mathbb{P}(\sup_{s\in K_n} |\widehat{X}_a(s)| > M_n) \leq \frac{\varepsilon}{2^n}, \quad n = 1, 2, \dots. \tag{11}$$

Define

$$H_\varepsilon = \{f \in H(D) : \sup_{s\in K_n} |f(s)| \leq M_n, \quad n = 1, 2, \dots\}.$$

Then by the compactness principle the set H_ε is a compact subset of $H(D)$, and by (11)

$$\mathbb{P}(\widehat{X}_a(s) \in H_\varepsilon) \geq 1 - \varepsilon,$$

or, by the definition of $\widehat{X}_a$,

$$\widehat{P}_a(H_\varepsilon) \geq 1 - \varepsilon$$

for all $a > 0$. This shows that the family of probability measures $\{\widehat{P}_a\}$ is tight, therefore it is relatively compact. Let $\{\widehat{P}_{a_1}\} \subset \{\widehat{P}_a\}$ be such that $\widehat{P}_{a_1}$ converges weakly to some probability measure $\widehat{P}$ on $(H(D), \mathcal{B}(H(D)))$ as $a_1 \to \infty$. This implies

$$\widehat{X}_{a_1} \xrightarrow[a_1\to\infty]{\mathcal{D}} \widehat{P}. \tag{12}$$

By (7), for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{a\to\infty} \limsup_{N\to\infty} \mu_N(\varrho(L(s+imh), L_a(s+imh)) \geq \varepsilon) \\ &\leq \lim_{a\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{m=0}^{N} \varrho(L(s+imh), L_a(s+imh)) = 0. \end{aligned} \tag{13}$$

Let

$$\widehat{X}_N = \widehat{X}_N(s) = L(s + i\theta_N).$$

The space $H(D)$ is separable, therefore by (9), (12) and (13), and Lemma 6 we obtain that

$$\widehat{X}_N \xrightarrow[N\to\infty]{\mathcal{D}} \widehat{P},$$

and Theorem 2 is proved.

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