FC-modules with an application to cotorsion pairs

Yonghua Guo

Abstract. Let R be a ring. A left R-module M is called an FC-module if $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a flat right R-module. In this paper, some homological properties of FC-modules are given. Let n be a nonnegative integer and $\mathcal{FC}_n$ the class of all left R-modules M such that the flat dimension of M^+ is less than or equal to n. It is shown that $(^\perp(\mathcal{FC}_n^\perp), \mathcal{FC}_n^\perp)$ is a complete cotorsion pair and if R is a ring such that $\mathrm{fd}((_RR)^+) \leq n$ and $\mathcal{FC}_n$ is closed under direct sums, then $(\mathcal{FC}_n, \mathcal{FC}_n^\perp)$ is a perfect cotorsion pair. In particular, some known results are obtained as corollaries.

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1. Introduction

Throughout this note, R is an associative ring with identity and all modules are unitary. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. The left R-module category is denoted by $_R\mathcal{M}$. The projective (resp., injective, flat) dimension of M is denoted by $\mathrm{pd}(M)$ (resp., $\mathrm{id}(M)$, $\mathrm{fd}(M)$). The symbol $\mathcal{P}_n$ (resp., $\mathcal{I}_n$, $\mathcal{F}_n$) stands for the class of all left R-modules with projective (resp., injective, flat) dimension less than or equal to a fixed nonnegative integer n.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a $\mathcal{C}$-*preenvelope* of M [9] if for any homomorphism $f\colon M \to F'$ where $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. A $\mathcal{C}$-preenvelope $\phi : M \to F$ is said to be a $\mathcal{C}$-*envelope* if every endomorphism $g : F \to F$ such that $g\phi = \phi$ is an isomorphism. Following [9, Definition 7.1.6], a monomorphism $\alpha : M \to C$ with $C \in \mathcal{C}$ is said to be a *special* $\mathcal{C}$*-preenvelope* of M if $\mathrm{coker}(\alpha) \in {}^\perp\mathcal{C}$. Dually we have the definitions of a (*special*) $\mathcal{C}$*-precover* and a $\mathcal{C}$*-cover*. Special $\mathcal{C}$-preenvelopes (resp. special $\mathcal{C}$-precovers) are obviously $\mathcal{C}$-preenvelopes (resp., $\mathcal{C}$-precovers). If every R-module has a $\mathcal{C}$-(pre)envelope (resp., $\mathcal{C}$-(pre)cover), we say that $\mathcal{C}$ is (*pre*)*enveloping* (resp., (*pre*)*covering*).

A pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a *cotorsion pair* (or *cotorsion theory*) [9, 12] if $\mathcal{F}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{F}$, where $\mathcal{F}^\perp = \{C : \mathrm{Ext}^1_R(F, C) = 0 \text{ for all } F \in \mathcal{F}\}$, and ${}^\perp\mathcal{C} = \{F : \mathrm{Ext}^1_R(F, C) = 0 \text{ for all } C \in \mathcal{C}\}$. A cotorsion pair $(\mathcal{F}, \mathcal{C})$

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is called *complete* (resp., *perfect*) provided that every R-module has a special $\mathcal{C}$-preenvelope and a special $\mathcal{F}$-precover (resp., a $\mathcal{C}$-envelope and an $\mathcal{F}$-cover).

In what follows, we write $wD(R)$ for the weak dimension of the ring R. Recall that a left R-module M is called *FP-injective* (or *absolutely pure*) [18] if $\mathrm{Ext}_R^1(N,M)=0$ for all finitely presented left R-modules N. A ring R is called *right IF-ring* [14] if every injective right R-module is flat.

For unexplained concepts and notations, we refer the reader to [1], [9].

2. Some results on FC-modules

Following Ramamurthi [16] we call an R-module M an FC-module if M^+ is a flat R-module on the opposite side.

Let $\mathcal{FI}=\{M \mid M$ is an FP-injective left R-module$\}$ and $\mathcal{FC}_n=\{M \mid M$ is a left R-modules with $\mathrm{fd}(M^+)\le n\}$, thus $\mathcal{FC}_0=\{M\in {}_R\mathcal{M} \mid M$ is an FC-module$\}$.

We note that if M is an FC-module then M is FP-injective (Proposition 2.1).

Proposition 2.1. *Let M be a left R-module. Consider the following statements:*

(1) *M is an FC-module;*
(2) *$M^+\twoheadrightarrow S^+$ is a flat precover for every submodule S of M;*
(3) *there exists a pure exact sequence $0\to M\to N\to L\to 0$ with $N\in\mathcal{FC}_0$;*
(4) *M is FP-injective.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4). And (4) $\Rightarrow$ (3) holds in case R is a left coherent ring.

Proof: (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are trivial.

(1) $\Rightarrow$ (2) For a flat right R-module F, $(F\otimes_R M)^+\to(F\otimes_R S)^+\to 0$ is exact, equivalently, $\mathrm{Hom}_R(F,M^+)\to\mathrm{Hom}_R(F,S^+)\to 0$ is exact. So $M^+\twoheadrightarrow S^+$ is a flat precover.

(3) $\Rightarrow$ (1) Let $0\to M\to N\to L\to 0$ be a pure exact sequence with $N\in\mathcal{FC}_0$. $0\to L^+\to N^+\to M^+\to 0$ is split by [11, Theorem 3.1]. Thus M^+ is flat since N^+ is flat.

(1) $\Rightarrow$ (4) Since $0\to M\to M^{++}$ is a pure embedding and M^{++} is injective, M is FP-injective by [18, Proposition 2.6].

If R is left coherent, then (4) $\Rightarrow$ (1) follows from [4, Theorem 1]. □

Remark 2.2. Given an exact sequence $F\xrightarrow{f}N\longrightarrow 0$ with F flat, in general, $f:F\longrightarrow N$ need not be a flat precover. For example, $\mathbb{Z}\xrightarrow{\pi}\mathbb{Z}_2\longrightarrow 0$ is exact, and $\mathbb{Z}\xrightarrow{\pi}\mathbb{Z}_2$ is not a flat precover.

It is not true in general that a submodule of an FC$_n$-module is an FC$_n$-module. However, we have the following proposition.

Proposition 2.3. *Let R be a ring. If S is a pure submodule of a right FC$_n$-module M, then S and M/S are FC$_n$-modules.*

Proof: Since S is a pure submodule of M, $0\to(M/S)^+\to M^+\to S^+\to 0$ is a split exact sequence by [11, Theorem 3.1]. Hence $\mathrm{fd}(S^+)\le n$ and $\mathrm{fd}((M/S)^+)\le n$. □

Let $\mathcal{C}$ be a class of modules. $\mathcal{C}$ is called *coresolving* [12, Definition 2.2.8(ii)], provided that $\mathcal{C}$ is closed under extensions, $\mathcal{I}_0 \subset \mathcal{C}$ and $C \in \mathcal{C}$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence such that $A, B \in \mathcal{C}$.

Theorem 2.4. *Let R be a ring. Then the following are equivalent:*

(1) *R is left coherent;*
(2) *$\mathcal{FI}$ is coresolving;*
(3) *$\mathcal{FC}_0$ is coresolving;*
(4) *$\mathcal{I}_0 \subseteq \mathcal{FC}_0$.*

Proof: Since $\mathcal{FI}$ is closed under extensions and $\mathcal{I}_0 \subseteq \mathcal{FI}$, (1) $\Leftrightarrow$ (2) follows from [6, Theorem 1.5].

(1) $\Rightarrow$ (3) By [4, Theorem 1], $\mathcal{FC}_0 = \mathcal{FI}$ since R is left coherent. Therefore $\mathcal{FC}_0$ is coresolving by (2).

(3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1) It is enough to prove $\mathcal{FC}_0 = \mathcal{FI}$ by [4, Theorem 1]. By Proposition 2.1, we have $\mathcal{FC}_0 \subseteq \mathcal{FI}$. For any $F \in \mathcal{FI}$, there is a pure short exact sequence $0 \to F \to E \to C \to 0$ with E injective. Hence $F \in \mathcal{FC}_0$ by Proposition 2.1. It follows that $\mathcal{FC}_0 = \mathcal{FI}$, as desired. □

Remark 2.5. If R is not a left coherent ring, then there exists an injective right R-module M such that M is not an FC-module by Theorem 2.4.

Corollary 2.6. *R is left coherent if and only if every left R-module has a monomorphic $\mathcal{FC}_0$-preenvelope.*

Proof: If R is left coherent, then $\mathcal{FI} = \mathcal{FC}_0$. By [10, Corollary 1.4], every left R-module has a monomorphic $\mathcal{FC}_0$-preenvelope. On the other hand, if every left R-module has a monomorphic $\mathcal{FC}_0$-preenvelope, then every injective left R-module is an FC-module. Hence, R is left coherent by Theorem 2.4. □

Proposition 2.7. *Let R be a ring. Then the following are equivalent:*

(1) *R is a right IF-ring;*
(2) *$\mathcal{F}_0 \subseteq \mathcal{FC}_0$;*
(3) *$\mathcal{P}_0 \subseteq \mathcal{FC}_0$.*

Proof: (1) $\Rightarrow$ (2) Let F be a flat left R-module. Since F^+ is injective as a right R-module, F^+ is flat and hence F is an FC-module.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) follows from [5, Theorem 1(4)]. □

Remark 2.8. The conditions in Proposition 2.7 are equivalent to $\mathcal{F}_n \subseteq \mathcal{FC}_0$ by [7, Theorem 3.5] for every positive integer n.

Corollary 2.9. *Let R be a ring. If R is a two-sided IF-ring, then R is two-sided coherent. Moreover, commutative IF-rings are coherent.*

A coherent ring need not be an IF-ring. $\mathbb{Z}$ is not an IF-ring since $\mathbb{Q}/\mathbb{Z}$ is injective (divisible) but not flat ($\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$). It is an open question whether a right IF-ring is left coherent [14, P442]. The next theorem gives a partial answer to this question.

Theorem 2.10. *Let R is a right IF-ring. If $\mathrm{fd}(E^{++}) < \infty$ for every injective left R-module E, then R is left coherent.*

Proof: Let E be an injective left R-module. Note that $\mathrm{id}(E^{+++}) = \mathrm{fd}(E^{++}) < \infty$ by hypothesis, and so E^{+++} is flat by [5, Proposition 4]. Since E^{+} is a pure submodule of E^{+++}, E^{+} is flat. Thus R is left coherent by Theorem 2.4. □

Proposition 2.11. *The following are equivalent for a commutative ring R:*

(1) *R is an IF-ring;*
(2) *M is flat if and only if M is an FC-module;*
(3) *$\mathcal{F}_0 = \mathcal{FC}_n$ for any integer $n \geq 0$.*

Proof: It follows from Proposition 2.7 and the proof of Theorem 2.10. □

Remark 2.12. If R is a coherent and self-injective commutative ring, then R is an IF-ring by Proposition 2.7. According to above propositions, in this ring, an R-module is flat if and only if it is FP-injective. Hence [3, Theorem 12] allows us to get examples of rings over which every finitely presented module has an FP-injective envelope but not every module has an FP-injective envelope.

Proposition 2.13. *The following are equivalent for a ring R:*

(1) *R is von Neumann regular;*
(2) *every left R-module is an FC-module;*
(3) *M^{+} is an FC-module for every pure injective right R-module M.*

Proof: (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (2) For any left R-module N, N^{+} is pure injective right R-module. Therefore N^{++} is an FC-module. Since N is a pure submodule of N^{++}, N is an FC-module by Proposition 2.1.

(2) $\Rightarrow$ (1) For any left R-module M, let $f : F \to M$ be a flat cover of M. Then F^{+} is injective and the exact sequence $0 \to M^{+} \to F^{+} \to (\mathrm{Ker}(f))^{+} \to 0$ is split since $(\mathrm{Ker}(f))^{+}$ is flat by assumption. Thus M^{+} is injective, and hence M is flat. □

Proposition 2.14. *Let R a commutative ring such that $wD(R_{\mathfrak{p}}) < \infty$ for each prime ideal $\mathfrak{p}$ of R. The following are equivalent:*

(1) *R is von Neumann regular;*
(2) *every R-module has a monomorphic flat envelope;*
(3) *R is an IF-ring such that very R-module has an $\mathcal{FC}_0$-envelope.*

PROOF: (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) If every R-module has a monomorphic flat envelope, then R is an IF-ring. Now by using [2, Theorem 9], we get that $wD(R) \leq 2$. Hence R is von Neumann regular by [5, Proposition 5].

(1) $\Rightarrow$ (3) follows from Proposition 2.13.

(3) $\Rightarrow$ (2) By Proposition 2.11, every R-module has a flat envelope. Since every injective module is flat, the flat envelope must be monomorphic. □

3. An application to cotorsion pairs

We begin with the following

Proposition 3.1. *For a family $\{F_i\}$ of right R-modules, if ΠF_i is a right FC_n-module, then $\oplus F_i$ is a right FC_n-module.*

PROOF: The result follows since $\oplus F_i$ is a pure submodule of ΠF_i. □

Remark 3.2. By [17, Corollary 3.5(c)], if a class $\mathcal{G}$ of modules over a ring is closed under pure submodules, then $\mathcal{G}$ is preenveloping if and only if it is closed under direct products. If a class $\mathcal{F}$ is closed under pure quotient modules, then $\mathcal{F}$ is precovering if and only if it is covering if and only if $\mathcal{F}$ is closed under direct sums by [13, Theorem 2.5]. From Proposition 3.1, we know that if $\mathcal{FC}_n$ is preenveloping, then $\mathcal{FC}_n$ is covering. Moreover, $\mathcal{FC}_n$ is a Kaplansky class by [13, Proposition 3.2].

Lemma 3.3. *$\mathcal{FC}_n$ is covering if and only if $\mathcal{FC}_n$ is closed under direct sums.*

PROOF: This follows from Proposition 2.3 and [13, Theorem 2.5]. □

Corollary 3.4. *For a left coherent ring R, every left R-module has an FP-injective cover.*

Theorem 3.5. *$({}^{\perp}(\mathcal{FC}_n^{\perp}), \mathcal{FC}_n^{\perp})$ is a complete cotorsion pair. Moreover, if R is a ring such that $\mathrm{fd}(({}_RR)^+) \leq n$ and $\mathcal{FC}_n$ is closed under direct sums, then $(\mathcal{FC}_n, \mathcal{FC}_n^{\perp})$ is a perfect cotorsion pair.*

PROOF: Let E be a right R-module with $\mathrm{fd}(E^+) \leq n$. By [9, Lemma 5.3.12], if $\mathrm{Card}\, R \leq \aleph_\beta$, then, for each $x \in E$, there is a pure submodule $S \subseteq E$ with $x \in S$ such that $\mathrm{Card}\, S \leq \aleph_\beta$ (simply let $N = Rx$ and $f = \mathrm{id}_N$ in the lemma). By Proposition 2.3, $S \in \mathcal{FC}_n$ and $E/S \in \mathcal{FC}_n$. So we can write E as a union of a continuous chain $(E_\alpha)_{\alpha<\lambda}$ of pure submodules of E such that $\mathrm{Card}\, E_0 \leq \aleph_\beta$ and $\mathrm{Card}(E_{\alpha+1}/E_\alpha) \leq \aleph_\beta$ whenever $\alpha + 1 < \lambda$. Moreover $E_0 \in \mathcal{FC}_n$ and $E_{\alpha+1}/E_\alpha \in \mathcal{FC}_n$. By [9, Theorem 7.3.4], we see that if C is a right R-module such that $\mathrm{Ext}^1(E_0, C) = 0$ and $\mathrm{Ext}^1(E_{\alpha+1}/E_\alpha, C) = 0$ whenever $\alpha + 1 < \lambda$, then $\mathrm{Ext}^1(E, C) = 0$. So if Y is a set of representatives of all right R-modules $G \in \mathcal{FC}_n$ with $\mathrm{Card}\, G \leq \aleph_\beta$, then $C \in \mathcal{FC}_n^{\perp}$ if and only if $\mathrm{Ext}^1(G, C) = 0$ for all $G \in Y$. But then this just says that the given cotorsion pair $({}^{\perp}(\mathcal{FC}_n^{\perp}), \mathcal{FC}_n^{\perp})$ is cogenerated by the set Y. Hence $({}^{\perp}(\mathcal{FC}_n^{\perp}), \mathcal{FC}_n^{\perp})$ is a complete cotorsion pair by [8, Theorem 10].

By Proposition 2.3 and hypothesis, $\mathcal{FC}_n$ is closed under direct limits. Since $R \in \mathcal{FC}_n$, we may assume $R \in Y$. So the class ${}^\perp(\mathcal{FC}_n^\perp)$ consists of direct summands of Y-filtered modules by [12, Corollary 3.2.4]. By an induction on the length of the Y-filtration, we get that ${}^\perp(\mathcal{FC}_n^\perp) = \mathcal{FC}_n$. Therefore, $(\mathcal{FC}_n, \mathcal{FC}_n^\perp)$ is perfect by [12, Corollary 2.3.7]. □

Corollary 3.6 ([15, Theorem 3.4(1)])**.** *For a left coherent ring R with FP-*id$({}_RR) \leq n$, $(\mathcal{FI}_n, \mathcal{FI}_n^\perp)$ *is a perfect cotorsion pair.*

Corollary 3.7 ([12, Theorem 4.1.13])**.** *Let R be a left noetherian ring. Then $\mathfrak{C}_n = ({}^\perp(\mathcal{I}_n^\perp), \mathcal{I}_n^\perp)$ is a complete cotorsion pair. Moreover, if* id$({}_RR) \leq n$, *then $\mathfrak{C}_n = (\mathcal{I}_n, \mathcal{I}_n^\perp)$ is a perfect cotorsion pair.*

Let $\mathcal{C}$ be a class of modules. Then $\mathcal{C}$ is *definable* [12, Definition 3.1.9] provided that $\mathcal{C}$ is closed under direct limits, direct products and pure submodules.

Theorem 3.8. *If R is a right IF-ring such that $\mathcal{FC}_n$ is closed under direct products, then $\mathcal{FC}_n$ is definable and $(\mathcal{FC}_n, \mathcal{FC}_n^\perp)$ is a perfect cotorsion pair.*

Proof: By hypothesis and Proposition 3.1, $\mathcal{FC}_n$ is closed under direct sums. Thus $\mathcal{FC}_n$ is definable by Proposition 2.3 and $(\mathcal{FC}_n, \mathcal{FC}_n^\perp)$ is a perfect cotorsion pair by Theorem 3.5. □

Remark 3.9. If R is a ring such that $(\mathcal{FC}_0, \mathcal{FC}_0^\perp)$ is a cotorsion pair, then R is a right IF-ring.

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Department of Mathematics, Guangxi Normal University, Guilin, Guangxi 541004, China
and
Department of Mathematics, Nanjing University, Nanjing 210093, China
Email: gyonghua@yahoo.com

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