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OSCILLATION OF NONLINEAR SECOND ORDER MATRIX DIFFERENTIAL EQUATIONS

N. Parhi* — P. Praharaj**

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ABSTRACT. In this paper, sufficient conditions are obtained for oscillation of all nontrivial, prepared, symmetric solutions of a class of nonlinear second order matrix differential equations of the form

$$(P(t)Y')' + Q(t)F(Y) = 0, \qquad t \geq 0,$$

and

$$Y'' + Q(t)F(Y) = 0, \qquad t \geq 0.$$

1. Introduction

In this paper, sufficient conditions are obtained for oscillation of all nontrivial, symmetric, prepared solutions of a class of nonlinear second order matrix differential equations of the form

$$(P(t)Y')' + Q(t)F(Y) = 0, \quad t \geq 0, \tag{1.1}$$

where $P(t)$ and $Q(t)$ are $n \times n$ real continuous symmetric matrix functions on $[0, \infty)$, $P(t)$ is positive definite, $F\colon M_n \to M_n$ and M_n is the vector space of all $n \times n$ real symmetric matrices. If $P(t) = I_n$, $n \times n$ identity matrix, then (1.1) takes the form

$$Y'' + Q(t)F(Y) = 0, \qquad t \geq 0. \tag{1.2}$$

The oscillation of Eqs. (1.1) and (1.2) must be studied separately since, unlike the scalar case, there is no oscillation preserving transformation of the independent variable that allows the passage between the two forms ([1]).

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Some authors ([7], [8], [9]) have obtained sufficient conditions for oscillation of solutions of (1.1) and (1.2) employing variational techniques. It seems that the work of H o w a r d [2] is the first one where the variational method is not used for the study of oscillation of solutions of (1.2). He has studied oscillatory behavior of nontrivial, prepared, symmetric solutions of (1.2). His major assumption that

$$K(t) = \int_{t_0}^{t} Q(s)\,\mathrm{d}s \qquad (t_0 > 0)$$

possesses the property D, viz,

$$\inf_{\xi}(\xi^* K(t)\xi) \to \infty \qquad \text{as} \quad t \to \infty,$$

where ξ represents a column vector of unit length, is not easy to verify. It seems that no example could be given in the paper due to this reason. In this paper some new and easily verifiable oscillation criteria are given for oscillation of nontrivial, prepared, symmetric solutions of a class of nonlinear matrix differential equations.

A solution $Y(t)$ of (1.1) is said to be nontrivial if $\det Y(t) \neq 0$ (determinant of $Y(t)$ is denoted by $\det Y(t)$) for at least one $t \in [0, \infty)$. A solution $Y(t)$ of (1.1) is said to be prepared or self-conjugate or conjoined if

$$Y^*(t)(P(t)Y'(t)) - (P(t)Y'(t))^* Y(t),$$

that is, if $P(t)Y'(t)Y^{-1}(t)$ is symmetric for $t \in [0, \infty)$. (The transpose of a matrix A is denoted by A^*.) A nontrivial, prepared solution $Y(t)$ of (1.1) is said to be oscillatory if, for every $t_0 \geq 0$, there exists a $t_1 > t_0$ such that $\det Y(t_1) = 0$, that is, $\det Y(t)$ has arbitrarily large zeros in $[0, \infty)$; otherwise, $Y(t)$ is called non-oscillatory. It may be noted that oscillation is defined through a prepared solution because it is possible (see [6]) that a nontrivial, nonprepared, nonoscillatory solution of a linear matrix differential equation exists. (The solution may be symmetric or not.) Indeed,

$$U(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

is a nontrivial, nonoscillatory solution (because $\det U(t) = 1 > 0$) of

$$Y'' + Y = 0, \qquad t \geq 0, \tag{1.3}$$

where Y is a 2×2 matrix. Since

$$U^*(t)U'(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$(U'(t))^* U(t) = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

then $U(t)$ is not prepared. We may note that $U(t)$ is not symmetric. As a second example, we may consider

$$V(t) = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}, \qquad t \geq 0.$$

It is a nontrivial, nonprepared, symmetric, nonoscillatory solution of (1.3) because $\det V(t) = -1 < 0$, $t \geq 0$ and

$$V(t)V'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad V'(t)V(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

On the other hand, Eq. (1.3) may admit a nontrivial, nonprepared, oscillatory solution. Indeed,

$$W(t) = \begin{bmatrix} \sin t & \cos t \\ 2\sin t & 3\cos t \end{bmatrix}, \qquad t \geq 0,$$

is such a solution of (1.3), because $\det W(t) = \sin t \cos t$ and

$$W^*(t)W'(t) = \begin{bmatrix} \sin t & 2\sin t \\ \cos t & 3\cos t \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ 2\cos t & -3\sin t \end{bmatrix} \tag{1.4}$$

$$= \begin{bmatrix} 5\sin t\cos t & -7\sin^2(t) \\ 7\cos^2 t & -10\sin t\cos t \end{bmatrix} \tag{1.5}$$

and

$$(W'(t))^*W(t) = \begin{bmatrix} \cos t & 2\cos t \\ -\sin t & -3\sin t \end{bmatrix} \begin{bmatrix} \sin t & \cos t \\ 2\sin t & 3\cos t \end{bmatrix} \tag{1.6}$$

$$= \begin{bmatrix} 5\sin t\cos t & 7\cos^2 t \\ -7\sin^2 t & -10\sin t\cos t \end{bmatrix} \tag{1.7}$$

imply that $W(t)$ is not prepared. Moreover, Eq. (1.3) admits a nontrivial, prepared, symmetric, oscillatory solution

$$Y(t) = \begin{bmatrix} \sin t & \cos t \\ \cos t & \sin t \end{bmatrix}.$$

It also admits a nontrivial, prepared, nonsymmetric, oscillatory solution

$$Y(t) = \begin{bmatrix} -\cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

There are differential equations which admit nontrivial, prepared, symmetric, nonoscillatory solutions. For example,

$$Y(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

is such a solution of the equation

$$(P(t)Y')' + Q(t)Y = 0, \qquad t \geq 0,$$

where

$$P(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{2t} \end{bmatrix} = Q(t).$$

2. Oscillation results

Some oscillation results are obtained in this section. We need the following condition in the sequel:

(C_1) Let $P^{-1}(t) \geq I_n$, $F(X)$ be a polynomial in X with real coefficients, $Q(t)$ be positive semidefinite and $F(X)X^{-1} \geq I_n$ for every nonsingular matrix $X \in M_n$.

Theorem 1. *Let* (C_1) *hold. If either*

(C_2) $\lim\limits_{T\to\infty} \frac{1}{T} \int\limits_0^T \left(\operatorname{tr} \int\limits_0^t Q(s)\,ds \right) dt = \infty$

or

(C_3) $\lim\limits_{T\to\infty} \frac{1}{T} \int\limits_0^T \left(\operatorname{tr} \int\limits_0^t Q(s)\,ds \right)^2 dt = \infty,$

then every nontrivial, prepared, symmetric solution of (1.1) *oscillates.*

Remark. If $Y(t)$ is a nontrivial, prepared, symmetric, nonoscillatory solution of (1.1), then there exists a $t_0 > 0$ such that $\det Y(t) \neq 0$ for $t \geq t_0$. Hence $Y^{-1}(t)$ exists for $t \geq t_0$ and $Y(t)Y^{-1}(t) = I_n$ implies that $(Y(t)Y^{-1}(t))' = 0$. Consequently $(Y^{-1}(t))' = -Y^{-1}(t)Y'(t)Y^{-1}(t)$. Setting

$$S(t) = -P(t)Y'(t)Y^{-1}(t), \qquad t \geq t_0, \tag{2.1}$$

we obtain

$$S'(t) = Q(t)R(t) + S(t)P^{-1}(t)S(t), \tag{2.2}$$

where $R(t) = F(Y(t))Y^{-1}(t)$. From (C_1) it follows that $R(t)$ is symmetric. Since $Y(t)$ is prepared, then $S(t)$ is symmetric. Indeed,

$$\begin{aligned} S^*(t) &= -(Y^{-1}(t))^*(P(t)Y'(t))^* \\ &= -(Y^*(t))^{-1}Y^*(t)(P(t)Y'(t))Y^{-1}(t) \\ &= -P(t)Y'(t)Y^{-1}(t) = S(t). \end{aligned}$$

Further, $Y'(t)P(t) = Y(t)P(t)Y'(t)Y^{-1}(t)$ as $Y(t)$ is symmetric and prepared. Hence

$$\begin{aligned}(Y'(t)P(t))' &= (Y(t)P(t)Y'(t)Y^{-1}(t))' \\ &= Y'(t)P(t)Y'(t)Y^{-1}(t) + Y(t)(P(t)Y'(t)Y^{-1}(t))' \\ &= Y'(t)P(t)Y'(t)Y^{-1}(t) + Y(t)(P(t)Y'(t))'Y^{-1}(t) \\ &\quad + Y(t)P(t)Y'(t)(Y^{-1}(t))' \\ &= Y'(t)P(t)Y'(t)Y^{-1}(t) - Y(t)Q(t)F(Y(t))Y^{-1}(t) \\ &\quad - Y(t)P(t)Y'(t)Y^{-1}(t)Y'(t)Y^{-1}(t) \\ &= Y'(t)P(t)Y'(t)Y^{-1}(t) - Y(t)Q(t)F(Y(t))Y^{-1}(t) \\ &\quad - Y'(t)P(t)Y'(t)Y^{-1}(t) \\ &= -Y(t)Q(t)F(Y(t))Y^{-1}(t)\end{aligned}$$

and $((P(t)Y'(t))')^* = -(Q(t)F(Y(t)))^*$, that is, $(Y'(t)P(t))' = -F(Y(t))Q(t)$ imply that

$$Y(t)Q(t)F(Y(t))Y^{-1}(t) = F(Y(t))Q(t),$$

that is, $Q(t)F(Y(t))Y^{-1}(t) = Y^{-1}(t)F(Y(t))Q(t) = F(Y(t))Y^{-1}(t)Q(t)$ because $F(Y(t))Y(t) = Y(t)F(Y(t))$. Hence $Q(t)R(t) = R(t)Q(t)$. Consequently, $Q(t)R(t)$ is symmetric.

We need the following lemmas for the proof of Theorem 1.

Lemma 2. *Let* (C_1) *hold. Then*

$$0 < \lim_{T\to\infty} \int_{t_0}^{T} \operatorname{tr}\left[S(t)P^{-1}(t)S(t)\right] \,\mathrm{d}t < \infty, \tag{2.3}$$

where $S(t)$ *is defined by* (2.1).

Proof. Integrating (2.2) from t_0 to t and then taking trace we obtain

$$\operatorname{tr} S(t) - \operatorname{tr} S(t_0) = \int_{t_0}^{t} \operatorname{tr}(Q(s)R(s))\,\mathrm{d}s + \int_{t_0}^{t} \operatorname{tr}(S(s)P^{-1}(s)S(s))\,\mathrm{d}s.$$

Further integration from t_0 to T yields

$$\frac{1}{T}\int_{t_0}^{T}\left(\int_{t_0}^{t}\operatorname{tr}(S(s)P^{-1}(s)S(s))\,\mathrm{d}s\right)\mathrm{d}t = \frac{1}{T}\int_{t_0}^{T}\operatorname{tr}S(t)\,\mathrm{d}t - \left(1-\frac{t_0}{T}\right)\operatorname{tr}S(t_0)$$
$$-\frac{1}{T}\int_{t_0}^{T}\left(\int_{t_0}^{t}\operatorname{tr}(Q(s)R(s))\,\mathrm{d}s\right)\mathrm{d}t.$$

Since $R(t) - I_n \geq 0$, $Q(t) \geq 0$ and $Q(t)$ commutes with $R(t)$, then $Q(t)(R(t) - I_n) \geq 0$ and hence $\operatorname{tr}(Q(t)R(t)) \geq \operatorname{tr} Q(t)$ for $t \geq t_0$. Thus

$$\int_{t_0}^{t}\operatorname{tr}(Q(s)R(s))\,\mathrm{d}s \geq \int_{t_0}^{t}\operatorname{tr}Q(s)\,\mathrm{d}s > 0.$$

Consequently,

$$\frac{1}{T}\int_{t_0}^{T}\left(\int_{t_0}^{t}\operatorname{tr}(S(s)P^{-1}(s)S(s))\,\mathrm{d}s\right)\mathrm{d}t < \frac{1}{T}\int_{t_0}^{T}\operatorname{tr}S(t)\,\mathrm{d}t - \left(1-\frac{t_0}{T}\right)\operatorname{tr}S(t_0). \tag{2.4}$$

As $S(t)P^{-1}(t)S(t) \geq S^2(t) \geq 0$ implies that $\operatorname{tr}(S(t)P^{-1}(t)S(t)) \geq 0$, then

$$\int_{t_0}^{t}\operatorname{tr}(S(s)P^{-1}(s)S(s))\,\mathrm{d}s$$

is an increasing function of t and hence

$$\lim_{t\to\infty}\int_{t_0}^{t}\operatorname{tr}(S(s)P^{-1}(s)S(s))\,\mathrm{d}s = \mu,$$

where $0 < \mu \leq \infty$. If $\mu = \infty$, then it may be shown that

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(\int_{t_0}^{t}\operatorname{tr}(S(s)P^{-1}(s)S(s))\,\mathrm{d}s\right)\mathrm{d}t = \infty$$

and hence from (2.4) it follows that

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\operatorname{tr}S(t)\,\mathrm{d}t = \infty.$$

Thus

$$\frac{1}{T}\int_{t_0}^{T} \operatorname{tr} S(t)\,\mathrm{d}t > -\left(1-\frac{t_0}{T}\right)\operatorname{tr} S(t_0)$$

for large T. Then (2.4) yields

$$\frac{1}{T}\int_{t_0}^{T}\left(\int_{t_0}^{t} \operatorname{tr}(S(s)P^{-1}(s)S(s))\,\mathrm{d}s\right)\mathrm{d}t < \frac{2}{T}\int_{t_0}^{T} \operatorname{tr} S(t)\,\mathrm{d}t \tag{2.5}$$

for large T. An application of the Cauchy-Schwarz inequality yields, for $T > T_1 > t_0$,

$$\begin{aligned}\left[\frac{1}{T}\int_{t_0}^{T} \operatorname{tr} S(t)\,\mathrm{d}t\right]^2 &\le \left[\frac{1}{T}\int_{t_0}^{T} (\operatorname{tr} S(t))^2\,\mathrm{d}t\right]\left[\frac{1}{T}\int_{t_0}^{T} 1^2\,\mathrm{d}t\right]\\ &\le \left(\frac{n}{T}\int_{t_0}^{T} \operatorname{tr} S^2(t)\,\mathrm{d}t\right)\left(1-\frac{t_0}{T}\right)\\ &\le \frac{n}{T}\int_{t_0}^{T} \operatorname{tr}(S(t)P^{-1}(t)S(t))\,\mathrm{d}t.\end{aligned}$$

Hence, from (2.5) it follows, for $T > T_1 > t_0$, that

$$\left[\frac{1}{T}\int_{t_0}^{T}\left(\int_{t_0}^{t} \operatorname{tr}(S(s)P^{-1}(s)S(s))\,\mathrm{d}s\right)\mathrm{d}t\right]^2 < \frac{4n}{T}\int_{t_0}^{T} \operatorname{tr}(S(t)P^{-1}(t)S(t))\,\mathrm{d}t. \tag{2.6}$$

Setting, for $T > T_1 > t_0$,

$$H(T) = \int_{t_0}^{T}\left(\int_{t_0}^{t} \operatorname{tr}(S(s)P^{-1}(s)S(s))\,\mathrm{d}s\right)\mathrm{d}t > 0,$$

we get

$$H'(T) = \int_{t_0}^{T} \operatorname{tr}(S(t)P^{-1}(t)S(t))\,\mathrm{d}t.$$

From (2.6) it follows that

$$\frac{H'(T)}{H^2(T)} > \frac{1}{4nT}.$$

Integrating the above inequality from T_1 to T and then taking limit as $T \to \infty$ we obtain

$$\infty = \frac{1}{4n} \lim_{T\to\infty} \ln\left(\frac{T}{T_1}\right) \le \frac{1}{H(T_1)} < \infty,$$

a contradiction. Hence $0 < \mu < \infty$. This completes the proof of the lemma. □

Lemma 3.

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \left(\operatorname{tr} \int_0^t Q(s)\,\mathrm{d}s \right) \mathrm{d}t = \infty$$

if and only if

$$\limsup_{T\to\infty} \frac{1}{T} \int_{t_0}^T \left(\operatorname{tr} \int_{t_0}^t Q(s)\,\mathrm{d}s \right) \mathrm{d}t = \infty$$

for every $t_0 \ge 0$.

The proof of the lemma is straight-forward and hence is omitted.

P r o o f o f T h e o r e m 1. Suppose (C_2) holds. Let $Y(t)$ be a nontrivial, symmetric, prepared, non-oscillatory solution of (1.1). Hence $\det Y(t) \ne 0$ for $t \ge t_0 > 0$. Consequently, $Y^{-1}(t)$ exists for $t \ge t_0$. Setting $S(t)$ as in (2.1), we obtain (2.2). Integrating it yields

$$S(t) - S(t_0) = \int_{t_0}^t Q(s)R(s)\,\mathrm{d}s + \int_{t_0}^t S(s)P^{-1}(s)S(s)\,\mathrm{d}s.$$

Hence

$$\begin{aligned}\operatorname{tr} S(t) - \operatorname{tr} S(t_0) &= \operatorname{tr} \int_{t_0}^t Q(s)R(s)\,\mathrm{d}s + \operatorname{tr} \int_{t_0}^t S(s)P^{-1}(s)S(s)\,\mathrm{d}s \\ &\ge \int_{t_0}^t \operatorname{tr}[Q(s)R(s)]\,\mathrm{d}s \ge \int_{t_0}^t \operatorname{tr} Q(s)\,\mathrm{d}s,\end{aligned}$$

since $R(t) - I_n \ge 0$, $Q(t) \ge 0$ and $R(t)Q(t) = Q(t)R(t)$. Then

$$\frac{1}{T} \int_{t_0}^T \operatorname{tr} S(t)\,\mathrm{d}t - \left(1 - \frac{t_0}{T}\right) \operatorname{tr} S(t_0) \ge \frac{1}{T} \int_{t_0}^T \left(\int_{t_0}^t \operatorname{tr} Q(s)\,\mathrm{d}s \right) \mathrm{d}t.$$

From Lemma 3 and the assumption (C_2) it follows that

$$\limsup_{T\to\infty} \frac{1}{T} \int_{t_0}^T \operatorname{tr} S(t)\,\mathrm{d}t = \infty.$$

Hence there exists a sequence $\{T_m\}$ such that $T_m \to \infty$ as $m \to \infty$ and

$$\lim_{m\to\infty} \frac{1}{T_m} \int_{t_0}^{T_m} \operatorname{tr} S(t)\,\mathrm{d}t = \infty. \tag{2.7}$$

The use of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\left[\frac{1}{T_m}\int_{t_0}^{T_m} \operatorname{tr} S(t)\,\mathrm{d}t\right]^2 &\le \left[\frac{1}{T_m}\int_{t_0}^{T_m} (\operatorname{tr} S(t))^2\,\mathrm{d}t\right]\left[\frac{1}{T_m}\int_{t_0}^{T_m} \mathrm{d}t\right] \\
&\le \left[\frac{n}{T_m}\int_{t_0}^{T_m} \operatorname{tr} S^2(t)\,\mathrm{d}t\right]\left[1-\frac{t_0}{T_m}\right] \\
&\le \frac{n}{T_m}\int_{t_0}^{T_m} \operatorname{tr} S^2(t)\,\mathrm{d}t \\
&\le \frac{n}{T_m}\int_{t_0}^{T_m} \operatorname{tr}(S(t)P^{-1}(t)S(t))\,\mathrm{d}t
\end{aligned}$$

since $P^{-1}(t) \ge I_n$. From (2.7) it follows that

$$\lim_{m\to\infty} \frac{1}{T_m}\int_{t_0}^{T_m} \operatorname{tr}(S(t)P^{-1}(t)S(t))\,\mathrm{d}t = \infty,$$

a contradiction to (2.3).

Suppose (C_3) holds. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} \left(\operatorname{tr}\int_{t_0}^{t} Q(s)\,\mathrm{d}s\right)^2 \mathrm{d}t = \infty.$$

Integrating (2.2) from t_0 to t , we obtain

$$\operatorname{tr} S(t) = \operatorname{tr}\int_{t_0}^{t} Q(s)R(s)\,\mathrm{d}s - \operatorname{tr}\int_{t}^{\infty} S(s)P^{-1}(s)S(s)\,\mathrm{d}s + L,$$

where

$$L = \operatorname{tr} S(t_0) + \operatorname{tr}\int_{t_0}^{\infty} S(s)P^{-1}(s)S(s)\,\mathrm{d}s$$

and $-\infty < L < \infty$, by Lemma 2. As $Q(t)(R(t) - I_n) \geq 0$, then

$$0 \leq \operatorname{tr} \int_{t_0}^{t} Q(s)\,\mathrm{d}s \leq \operatorname{tr} \int_{t_0}^{t} Q(s)R(s)\,\mathrm{d}s$$

$$\leq \operatorname{tr} S(t) + \operatorname{tr} \int_{t}^{\infty} S(s)P^{-1}(s)S(s)\,\mathrm{d}s - L.$$

Then $\left(\operatorname{tr} \int_{t_0}^{t} Q(s)\,\mathrm{d}s\right)^2 \leq 4(\operatorname{tr} S(t))^2 + 4\left(\operatorname{tr} \int_{t}^{\infty} S(s)P^{-1}(s)S(s)\,\mathrm{d}s\right)^2 + 2L^2$ and hence

$$\frac{1}{T}\int_{t_0}^{T}\left(\operatorname{tr}\int_{t_0}^{t} Q(s)\,\mathrm{d}s\right)^2 \mathrm{d}t$$

$$\leq \frac{4}{T}\int_{t_0}^{T}(\operatorname{tr} S(t))^2\,\mathrm{d}t + \frac{4}{T}\int_{t_0}^{T}\left(\operatorname{tr}\int_{t}^{\infty} S(s)P^{-1}(s)S(s)\,\mathrm{d}s\right)^2 \mathrm{d}t + 2L^2\left(1 - \frac{t_0}{T}\right).$$

As

$$\operatorname{tr}\int_{t_0}^{\infty} S(s)P^{-1}(s)S(s)\,\mathrm{d}s < \infty$$

and

$$\int_{t_0}^{T}(\operatorname{tr} S(t))^2\,\mathrm{d}t \leq n\int_{t_0}^{T}\operatorname{tr} S^2(t)\,\mathrm{d}t \leq n\int_{t_0}^{T}\operatorname{tr}(S(t)P^{-1}(t)S(t))\,\mathrm{d}t,$$

then

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}(\operatorname{tr} S(t))^2\,\mathrm{d}t = 0$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(\operatorname{tr}\int_{t}^{\infty} S(s)P^{-1}(s)S(s)\,\mathrm{d}s\right)^2 \mathrm{d}t = 0.$$

Hence

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(\operatorname{tr}\int_{t_0}^{t} Q(s)\,\mathrm{d}s\right)^2 \mathrm{d}t < \infty,$$

a contradiction.

Thus the proof of the theorem is complete. □

Remark. In general, the conditions (C_2) and (C_3) are independent. However, if

$$\left|\int_0^\infty q_{ij}(t)\,\mathrm{d}t\right| < \infty,$$

where $Q(t) = (q_{ij}(t))_{n\times n}$, then ($C_2$) implies ($C_3$). Indeed, $Q(t) \geq 0$ and $|\int_0^\infty q_{ij}(t)\,\mathrm{d}t| < \infty$ imply that $\int_0^t Q(s)\,\mathrm{d}s \geq 0$ for $t > 0$ and hence $\operatorname{tr}\int_0^t Q(s)\,\mathrm{d}s \geq 0$. Thus

$$\frac{1}{T}\int_0^T \left(\operatorname{tr}\int_0^t Q(s)\,\mathrm{d}s\right)\mathrm{d}t \leq \left(\frac{1}{T}\int_0^T \left(\operatorname{tr}\int_0^t Q(s)\,\mathrm{d}s\right)^2 \mathrm{d}t\right)^{1/2}.$$

Thus (C_2) implies (C_3).

Remark. It is possible to find symmetric matrices Y_0 and $\tilde{Y_0}$ such that $Y_0(P(t_0)\tilde{Y_0}) - (\tilde{Y_0}P(t_0))Y_0 = 0$. If $Y(t)$ is a symmetric solution of the initial value problem (1.1) and $Y(0) = Y_0$ and $Y'(0) = \tilde{Y_0}$ and if $Y(t)$ commutes with $Q(t)$, then $Y(t)$ is prepared because

$$[Y(t)(P(t)Y'(t)) - (Y'(t)P(t))Y(t)]' = 0$$

implies that

$$Y(t)(P(t)Y'(t)) - (Y'(t)P(t))Y(t) = C,$$

a constant matrix. The existence of a symmetric solution of (1.1) can be established by the suitable choice of fixed point theorems.

In order to obtain an example to illustrate Theorem 1, we consider following equations:

$$y''(t) + q_1(t)f(y(t))g(y'(t)) = 0 \tag{2.8}$$

and

$$(r(t)y'(t))' + p(t)y'(t) + q_2(t)f(y(t)) = 0, \tag{2.9}$$

$t \geq t_0 \geq 0$, where $f \in C((-\infty,\infty),(-\infty,\infty))$ with $yf(y) > 0$ for $y \neq 0$, $g \in C((-\infty,\infty),(-\infty,\infty))$ with $g(y) \geq K > 0$ for $y \neq 0$, p, q_1 and $q_2 \in C([t_0,\infty),(-\infty,\infty))$ with $q_1(t) \geq 0$ but $q_1(t) \not\equiv 0$ on any ray $[t_1,\infty)$, $t_1 \geq t_0$ and $r \in C^1([t_0,\infty),(0,\infty))$. A solution of (2.8)/(2.9) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called non-oscillatory.

Lemma 4. (see [5, Theorem 3.5]) *If $f(y)/y \geq \mu_0 > 0$ for $y \neq 0$, where μ_0 is a constant, then every solution of (2.8) is oscillatory provided that for each $b \geq t_0$ and for some $\lambda > 1$, the following two inequalities hold:*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_b^t (s-b)^\lambda K\mu_0 q_1(s)\,\mathrm{d}s > \frac{\lambda^2}{4(\lambda-1)}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_b^t (t-s)^\lambda K\mu_0 q_1(s)\,\mathrm{d}s > \frac{\lambda^2}{4(\lambda-1)}.$$

Lemma 5. (see [4, Corollary A] and [5]) *If*

$$\lim_{t\to\infty} t \int_t^{2t} q_2(s)\,\mathrm{d}s = \alpha > \alpha_0,$$

then every solution of (2.9) *is oscillatory, where* $\alpha_0 = 3 - 2\sqrt{2}$.

Example 1. Consider

$$Y'' + Y + Y^3 = 0, \qquad t \ge 0. \tag{2.10}$$

In this case, $P(t) = I_2$, $Q(t) = I_2$ and $F(X) = X + X^3$ for $X \in M_2$. For a nonsingular matrix $X \in M_2$, $F(X)X^{-1} = I_2 + X^2 \ge I_2$ since $X^2 \ge 0$. Hence (C_1) holds. Further, (C_2) holds because $\operatorname{tr} Q(t) = 2$. From Theorem 1 it follows that every non-trivial, symmetric, prepared solution of (2.10) is oscillatory. If $y_{11}(t)$ and $y_{22}(t)$ are nontrivial solutions of

$$x'' + x + x^3 = 0, \tag{2.11}$$

then

$$Y(t) = \begin{bmatrix} y_{11}(t) & 0 \\ 0 & y_{22}(t) \end{bmatrix}$$

is a non-trivial, symmetric solution of (2.10). Further, $Y(t)$ is prepared because $Y(t)Y'(t) = Y'(t)Y(t)$. Hence $Y(t)$ is oscillatory by Theorem 1, that is, $\det Y(t) = y_{11}(t)y_{22}(t)$ has arbitrarily large zeros. On the other hand, from Lemma 4/Lemma 5 it follows that $y_{11}(t)$ and $y_{22}(t)$ are oscillatory solutions of (2.11).

Example 2. Consider

$$(P(t)Y')' + Y + Y^3 = 0, \qquad t \ge 0, \tag{2.12}$$

where

$$P(t) = \begin{bmatrix} p_{11}(t) & 0 \\ 0 & p_{22}(t) \end{bmatrix},$$

p_{11} and $p_{22} \in C^1([0,\infty),(0,1])$. We may observe that $P(t)$ is a symmetric, positive definite matrix function on $[0,\infty)$. Hence $P^{-1}(t)$ exists and is given by

$$P^{-1}(t) = \begin{bmatrix} \frac{1}{p_{11}(t)} & 0 \\ 0 & \frac{1}{p_{22}(t)} \end{bmatrix}$$

As $p_{11}(t) \le 1$ and $p_{22}(t) \le 1$, then $P^{-1}(t) \ge I_2$. Thus (C_1) and (C_2) hold. From Theorem 1 it follows that every nontrivial, symmetric, prepared solution of (2.12) oscillates. In particular,

$$Y(t) = \begin{bmatrix} y_{11}(t) & 0 \\ 0 & y_{22}(t) \end{bmatrix}$$

oscillates, where $y_{11}(t)$ and $y_{22}(t)$ are nontrivial solutions of

$$(p_{11}(t)x')' + x + x^3 = 0$$

and

$$(p_{22}(t)x')' + x + x^3 = 0$$

respectively. On the other hand, $y_{11}(t)$ and $y_{22}(t)$ oscillate due to Lemma 5. Hence $\det Y(t) = y_{11}(t)y_{22}(t)$ is oscillatory, which confirms the assertion made above.

Remark. Consider

$$Y'' + Q(Y + Y^3) = 0, \qquad t \ge 0, \tag{2.13}$$

where

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

We may observe that Q is symmetric but not positive semi-definite because $x^*Qx = 2x_1^2 - x_2^2$. Hence Theorem 1 cannot be applied to (2.13). However, the following theorem can be applied.

Theorem 6. *Let $F(X)$ be a polynomial in X with real coefficients and $XF(X) > 0$ for $X \in M_n$. Let $F'(X) \ge I_n$, where $F'(X)$ stands for the derivative of the polynomial $F(X)$ with respect to X in the symbolic sense. Let* (C_3) *hold. If*

(C_4) $\liminf\limits_{T\to\infty} \frac{1}{T^p} \int\limits_0^T (T-t)^p \operatorname{tr} Q(t)\,\mathrm{d}t > -\infty,$

where $p > 1$ is an integer, then every nontrivial, symmetric, prepared solution of (1.2) *oscillates.*

Proof. If possible, let (1.2) admit a nontrivial, symmetric, prepared, non-oscillatory solution $Y(t)$ on $[0,\infty)$. Then there exists a $t_0 > 0$ such that $\det Y(t) \ne 0$ for $t \ge t_0$. Hence $Y^{-1}(t)$ exists for $t \ge t_0$. As $\det(Y(t)F(Y(t))) = \det Y(t) \det F(Y(t))$ and $Y(t)F(Y(t)) > 0$ implies that $\det(Y(t)F(Y(t))) > 0$, then $\det F(Y(t)) \ne 0$ and hence $(F(Y(t)))^{-1}$ exists for $t \ge t_0$. Setting

$$Z(t) = \int\limits_{t_0}^{t} [F(Y(s))]^{-1} Y'(s)\,\mathrm{d}s, \tag{2.14}$$

we obtain

$$Z'(t) = (F(Y(t)))^{-1}Y'(t).$$

Hence

$$\begin{aligned} Z''(t) &= (F(Y(t)))^{-1}Y''(t) + [(F(Y(t)))^{-1}]'Y'(t) \\ &= -(F(Y(t)))^{-1}Q(t)F(Y(t)) + [(F(Y(t)))^{-1}]'Y'(t). \qquad 2.15 \end{aligned}$$

As $(Y''(t))^* = -(Q(t)F(Y(t)))^*$ implies $Y''(t) = -F(Y(t))Q(t)$, then $Q(t)F(Y(t)) = F(Y(t))Q(t)$. Further,

$$F(Y(t))(F(Y(t)))^{-1} = I_n \implies [F(Y(t))(F(Y(t)))^{-1}]' = 0.$$

Hence

$$[(F(Y(t)))^{-1}]' = -(F(Y(t)))^{-1}F'(Y(t))Y'(t)(F(Y(t)))^{-1}.$$

Consequently, from (2.15) we have

$$\begin{aligned} Z''(t) &= -Q(t) - (F(Y(t)))^{-1}F'(Y(t))Y'(t)(F(Y(t)))^{-1}Y'(t) \\ &= -Q(t) - F'(Y(t))(F(Y(t)))^{-1}Y'(t)(F(Y(t)))^{-1}Y'(t) \\ &= -Q(t) - F'(Y(t))(Z'(t))^2 \end{aligned}$$

because $F(Y)$ is a polynomial in Y and Y commutes with itself imply tha $F'(Y(t))F(Y(t)) = F(Y(t))F'(Y(t))$, that is,

$$Z''(t) + Q(t) + F'(Y(t))(Z'(t))^2 = 0, \qquad t \geq t_0. \qquad (2.16$$

Since $Y(t)$ is prepared, then $Y(t)Y'(t) = Y'(t)Y(t)$, that is, $F(Y(t))Y'(t)$ $Y'(t)F(Y(t))$ and hence $Z(t)$ and $Z'(t)$ are symmetric. Indeed, from (2.14) it follows that

$$\begin{aligned} Z^*(t) &= \left[\int_{t_0}^{t} (F(Y(s)))^{-1}Y'(s)\,\mathrm{d}s\right]^* = \int_{t_0}^{t} [(F(Y(s)))^{-1}Y'(s)]^*\,\mathrm{d}s \\ &= \int_{t_0}^{t} (Y'(s))^*[(F(Y(s)))^{-1}]^*\,\mathrm{d}s = \int_{t_0}^{t} Y'(s)(F(Y(s)))^{-1}\,\mathrm{d}s \\ &= \int_{t_0}^{t} (F(Y(s)))^{-1}Y'(s)\,\mathrm{d}s = Z(t). \end{aligned}$$

Integrating (2.16) from t_0 to t and then taking the trace we get

$$\begin{aligned} \int_{t_0}^{t} \operatorname{tr} Q(s)\,\mathrm{d}s &= \operatorname{tr} Z'(t_0) - \operatorname{tr} Z'(t) - \int_{t_0}^{t} \operatorname{tr}[F'(Y(s))(Z'(s))^2]\,\mathrm{d}s \\ &= C_0(t) - \operatorname{tr} Z'(t), \end{aligned}$$

where

$$C_0(t) = \operatorname{tr} Z'(t_0) - \int_{t_0}^{t} \operatorname{tr}[F'(Y(s))(Z'(s))^2] \, ds.$$

We may note that

$F'(Y(t)) \geq I_n > 0$ and $(Z'(t))^2 \geq 0$ implies $\operatorname{tr}[F'(Y(t))(Z'(t))^2] \geq 0$ for $t \geq t_0$.

If possible, let

$$0 \leq \int_{t_0}^{\infty} \operatorname{tr}[F'(Y(s))(Z'(s))^2] \, ds = K_0 < \infty.$$

As

$$F'(Y(t)) - I_n \geq 0 \qquad \text{and} \qquad (Z'(t))^2 \geq 0$$

imply that

$$\operatorname{tr}[F'(Y(t))(Z'(t))^2] \geq \operatorname{tr}(Z'(t))^2,$$

then

$$\int_{t_0}^{t} \operatorname{tr}(Z'(s))^2 \, ds \leq \int_{t_0}^{t} \operatorname{tr}[F'(Y(s))(Z'(s))^2] \, ds < K_0.$$

Further,

$$\begin{aligned}\left(\int_{t_0}^{t} \operatorname{tr} Q(s) \, ds \right)^2 &= (C_0(t) - \operatorname{tr} Z'(t))^2 \\ &\leq 2[C_0^2(t) + (\operatorname{tr} Z'(t))^2] \\ &\leq 2[C_0^2(t) + n \operatorname{tr}(Z'(t))^2].\end{aligned}$$

Hence

$$\begin{aligned}(C_0(t))^2 &= \left[\operatorname{tr} Z'(t_0) - \int_{t_0}^{t} \operatorname{tr} \left(F'(Y(s))(Z'(s))^2 \right) \, ds \right]^2 \\ &\leq 2(\operatorname{tr} Z'(t_0))^2 + 2 \left[\int_{t_0}^{t} \operatorname{tr} \left(F'(Y(s))(Z'(s))^2 \right) \, ds \right]^2 \\ &\leq 2(\operatorname{tr} Z'(t_0))^2 + 2K_0^2 = L\end{aligned}$$

implies that

$$\frac{1}{T}\int_{t_0}^{T}\left(\int_{t_0}^{t}\operatorname{tr}Q(s)\,\mathrm{d}s\right)^2\mathrm{d}t \le 2L\Big(1-\frac{t_0}{T}\Big)+\frac{2n}{T}\int_{t_0}^{T}\operatorname{tr}(Z'(t))^2\,\mathrm{d}t$$
$$< 2L\Big(1-\frac{t_0}{T}\Big)+\frac{2nK_0}{T}.$$

Thus

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(\int_{t_0}^{t}\operatorname{tr}Q(s)\,\mathrm{d}s\right)^2\mathrm{d}t \le 2L<\infty,$$

a contradiction to (C_3). Hence

$$\int_{t_0}^{\infty}\operatorname{tr}\left[F'(Y(s))(Z'(s))^2\right]\,\mathrm{d}s=\infty.$$

However, this is true if and only if (see the remark below)

$$\lim_{T\to\infty}\frac{1}{T^p}\int_{t_0}^{T}(T-t)^p\operatorname{tr}\left[F'(Y(t))(Z'(t))^2\right]\,\mathrm{d}t=\infty,$$

where $p>1$ is an integer. Multiplying (2.16) through by $(T-t)^p$, $t_0<t<T$, integrating from t_0 to T and then taking trace we obtain

$$\int_{t_0}^{T}(T-t)^p\operatorname{tr}Q(t)\,\mathrm{d}t+\int_{t_0}^{T}(T-t)^p\operatorname{tr}\left[F'(Y(t))(Z'(t))^2\right]\,\mathrm{d}t$$
$$=(T-t_0)^p\operatorname{tr}Z'(t_0)-p\int_{t_0}^{T}(T-t)^{p-1}\operatorname{tr}Z'(t)\,\mathrm{d}t \tag{2.17}$$

An application of the Cauchy-Schwarz inequality yields

$$\left| \int_{t_0}^{T} (T-t)^{p-1} \operatorname{tr} Z'(t)\, dt \right|$$

$$\leq \int_{t_0}^{T} (T-t)^{p/2} |\operatorname{tr} Z'(t)| (T-t)^{p/2-1}\, dt$$

$$\leq \left[\int_{t_0}^{T} (T-t)^{p} (\operatorname{tr} Z'(t))^2\, dt \right]^{1/2} \left[\int_{t_0}^{T} (T-t)^{p-2}\, dt \right]^{1/2}$$

$$\leq \left[n \int_{t_0}^{T} (T-t)^{p} \operatorname{tr}(Z'(t))^2\, dt \right]^{1/2} \left[\int_{t_0}^{T} (T-t)^{p-2}\, dt \right]^{1/2}$$

$$\leq \left[n \int_{t_0}^{T} (T-t)^{p} \operatorname{tr}\left(F'(Y(t))(Z'(t))^2\right)\, dt \right]^{1/2} \left[\frac{(T-t_0)^{p-1}}{p-1} \right]^{1/2},$$

that is,

$$-\frac{1}{T^p} \int_{t_0}^{T} (T-t)^{p-1} \operatorname{tr} Z'(t)\, dt$$

$$\leq \left[\frac{n}{T^p} \int_{t_0}^{T} (T-t)^{p} \operatorname{tr}\left(F'(Y(t))(Z'(t))^2\right)\, dt \right]^{1/2} \left[\frac{(T-t_0)^{p-1}}{(p-1)T^p} \right]^{1/2}.$$

From (2.17) we obtain

$$\frac{1}{T^p} \int_{t_0}^{T} (T-t)^{p} \operatorname{tr} Q(t)\, dt$$

$$\leq \left(1 - \frac{t_0}{T}\right)^p \operatorname{tr} Z'(t_0) - \frac{1}{T^p} \int_{t_0}^{T} (T-t)^{p} \operatorname{tr}[F'(Y(t))(Z'(t))^2]\, dt$$

$$+ p \left[\frac{n}{T^p} \int_{t_0}^{T} (T-t)^{p} \operatorname{tr}(F'(Y(t))(Z'(t))^2)\, dt \right]^{1/2} \left[\frac{(T-t_0)^{p-1}}{(p-1)T^p} \right]^{1/2}.$$

Setting

$$f(T) = \left[\frac{1}{T^p}\int\limits_{t_0}^{T}(T-t)^p \operatorname{tr}(F'(Y(t))(Z'(t))^2)\,\mathrm{d}t\right]^{1\;2},$$

we observe that $\lim\limits_{T\to\infty} f(T) = \infty$ and

$$\begin{aligned}
&\frac{1}{T^p}\int\limits_{t_0}^{T}(T-t)^p \operatorname{tr} Q(t)\,\mathrm{d}t \\
&\quad\le \Big(1-\frac{t_0}{T}\Big)^p \operatorname{tr} Z'(t_0) - f^2(T) + p\,n^{1/2} f(T)\left[\frac{(T-t_0)^{p\;\;1}}{(p-1)T^p}\right]^{1\;2} \\
&\quad\le \Big(1-\frac{t_0}{T}\Big)^p \operatorname{tr} Z'(t_0) - f(T)\left(f(T) - pn^{1/2}\left[\frac{1}{(p-1)T}\Big(1-\frac{t_0}{T}\Big)^{I\;\;1}\right]^{1\;2}\right).
\end{aligned}$$

Hence

$$\liminf_{T\to\infty}\frac{1}{T^p}\int\limits_{t_0}^{T}(T-t)^p \operatorname{tr} Q(t)\,\mathrm{d}t = -\infty,$$

a contradiction to (C_4).

Remark.

(i)

$$\int\limits_{t_0}^{\infty}\operatorname{tr}(F'(Y(t))(Z'(t))^2)\,\mathrm{d}t = \infty$$

if and only if

$$\lim_{T\to\infty}\frac{1}{T^p}\int\limits_{t_0}^{T}(T-t)^p \operatorname{tr}(F'(Y(t))(Z'(t))^2)\,\mathrm{d}t - \infty.$$

(ii)

$$\liminf_{T\to\infty}\frac{1}{T^p}\int\limits_{0}^{T}(T-t)^p \operatorname{tr} Q(t)\,\mathrm{d}t > -\infty$$

implies that

$$\liminf_{T\to\infty}\frac{1}{T^p}\int\limits_{t_0}^{T}(T-t)^p \operatorname{tr} Q(t)\,\mathrm{d}t > -\infty \qquad \text{for every} \quad t_0 \ge 0.$$

Example 3. Consider (2.13). Since $F(X) = X + X^3$, then $XF(X) = X^2 + X^4 > 0$ and $F'(X) = I_n + 3X^2 > I_n$. Conditions (C_3) and (C_4) hold because $\operatorname{tr} Q = 2 + (-1) = 1$. If

$$Y(t) = \begin{bmatrix} y_{11}(t) & 0 \\ 0 & y_{22}(t) \end{bmatrix},$$

where $y_{11}(t)$ and $y_{22}(t)$ are non-trivial solutions of $x'' + 2x + 2x^3 = 0$ and $x'' - x - x^3 = 0$ respectively, then $Y(t)$ is a non-trivial, symmetric, prepared solution of (2.13). From Theorem 6 it follows that $Y(t)$ oscillates. On the other hand, from Lemma 4/Lemma 5 it follows that $y_{11}(t)$ is oscillatory. Hence $Y(t)$ is oscillatory, because $\det Y(t) = y_{11}(t)y_{22}(t)$.

THEOREM 7. *Let* (C_1) *hold. Let* $g(t)$ *be a positive, differentiable function on* $[0,\infty)$ *such that*

$$\lim_{t\to\infty} \int_0^t (g(s))^{-1}\,\mathrm{d}s = \infty$$

and

$$\lim_{t\to\infty} \left[\int_0^t \operatorname{tr} \left(g(s)Q(s) - \left(\frac{(g'(s))^2}{4g(s)} \right) I_n \right) \mathrm{d}s + \frac{n}{2}\, g'(t) \right] = \infty.$$

Then every nontrivial, prepared, symmetric solution of (1.1) *oscillates.*

The proof is similar to that of Theorem 1 and hence is omitted.

THEOREM 8. *Suppose that all the conditions of* Theorem 6 *hold, except* (C_3) *which is replaced by*

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \left(\int_0^t \operatorname{tr} Q(s)\,\mathrm{d}s \right) \mathrm{d}t = \infty.$$

Then every nontrivial, prepared, symmetric solution of (1.2) *oscillates.*

The proof is similar to that of Theorem 6 and hence is omitted.

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