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SOME NEW SEQUENCE SPACES DEFINED BY LACUNARY SEQUENCES

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(*Communicated by Pavel Kostyrko*)

ABSTRACT. It is natural to expect that lacunary almost convergence must be related to the some concept of lacunary almost bounded variations in the some view as almost convergence is related to almost bounded variation. The purpose of this paper is to examine this new concept in some details. Some inclusion theorems have been established.

1. Introduction

Let s be the set of all sequences real and complex and ℓ_∞, c and c_0 respectively be the Banach spaces of bounded, convergent and null sequences $x = (x_n)$ normed by $\|x\| = \sup\limits_n |x_n|$. Let D be the shift operator on s. That is,

$$Dx = \{x_n\}_{n=1}^{\infty}, \ D^2 x = \{x_n\}_{n=2}^{\infty}, \ \dots$$

and so on. It is evident that D is a bounded linear operator on ℓ_∞ onto itself and that $\|D^k\| = 1$ for every k.

It may be recalled that *Banach limit* L is a non-negative linear functional on ℓ_∞ such that L is invariant under the shift operator, that is, $L(Dx) = L(x)$ for all $x \in \ell_\infty$ and that $L(e) = 1$ where $e = (1, 1, 1, \dots)$, (see, Banach [1]).

A sequence $x \in \ell_\infty$ is called *almost convergent* (see, Lorentz [2]) if all Banach Limits of x coincide.

Let $\hat{c}$ denote the set of all almost convergent sequences. Lorentz [2] proved that

$$\hat{c} = \Big\{x : \ \lim_m d_{mn}(x) \ \text{exists uniformly in } n\Big\}$$

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where

$$d_{mn}(x) = \frac{x_n + x_{n+1} + \cdots + x_{n+m}}{m+1}$$

Let $\theta = (k_r)$ be the sequence of positive integers such that

i) $k_0 = 0$ and $0 < k_r < k_{r+1}$

ii) $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$.

Then θ is called a *lacuanry sequence*. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r (see, Freedman et al [5]).

Recently, Das and Mishra [3] defined M_θ, the set of almost lacunary convergent sequences, as follows:

$$M_\theta = \Big\{x : \text{ there exists } l \text{ such that uniformly in } i \geq 0, \lim_{r\to\infty} \tfrac{1}{h_r} \sum_{k\in I_r} (x_{k+i} - l) = 0\Big\}.$$

It is natural to expect that lacunary almost convergence must be related to some concept $\overset{\wedge}{BV}_\theta$ in the same view as almost convergence is related to the concept of $\overset{\wedge}{BV}$. $\overset{\wedge}{BV}$ denotes the set of all sequences of almost bounded variation and a sequence in $\overset{\wedge}{BV}_\theta$ will mean a sequence of lacunary almost bounded variation.

The main object of this paper is to study this new concept in some details. Also a new sequence space $\overset{\widehat{\wedge}}{BV}_\theta$ which is apparently more generals than $\overset{\wedge}{BV}_\theta$ naturally comes up for investigation and is considered along with $\overset{\wedge}{BV}_\theta$.

We may remark here that the concept $\overset{\wedge}{BV}$ of almost bounded variation have been recently introduced and investigated by Nanda and Nayak [4] as follows:

$$\overset{\wedge}{BV} = \Big\{x : \sum_m |t_{mn}(x)| \text{ converges uniformly in } n\Big\}$$

where

$$t_{mn}(x) = \frac{1}{m(m+1)} \sum_{v=1}^{m} v\,(x_{n+v} - x_{n+v-1}).$$

Put

$$t_{rn} = t_{rn}(x) = \frac{1}{h_r+1} \sum_{j=k_{r-1}+1}^{k_r+1} x_{j+n}.$$

Then write $r, n > 0$,

$$\varphi_{rn}(x) = t_{rn}(x) - t_{r-1n}(x).$$

When $r > 1$, straightforward calculation shows that

$$\varphi_{rn}(x) = \varphi_{rn} = \frac{1}{h_r(h_r+1)} \sum_{u=1}^{h_r} u\left(x_{k_{r-1}+u+1+n} - x_{k_{r-1}+u+n}\right).$$

Now write

$$\overset{\wedge}{BV_\theta} = \left\{x : \sum_r |\varphi_{rn}(x)| \text{ converges uniformly in } n\right\}$$

and

$$\overset{\widehat{\wedge}}{BV_\theta} = \left\{x : \sup_n \sum_r |\varphi_{rn}(x)| < \infty\right\}.$$

Here and afterwards summation without limits sum from 1 to ∞.

2. Main results

We have the following theorem.

Theorem 1. *$\overset{\wedge}{BV_\theta} \subset \overset{\widehat{\wedge}}{BV_\theta}$ for every θ.*

Proof. Suppose that $x \in \overset{\wedge}{BV_\theta}$ and write φ_{rn} for $\varphi_{rn}(x)$. We have to show that $\sum_r |\varphi_{rn}|$ is bounded. By the definition of $\overset{\wedge}{BV_\theta}$ there exists an integer R such that, for all n,

$$\sum_{r \geq R+1} |\varphi_{rn}| \leq 1.$$

Therefore it follows that for $r \geq R+1$, and all n

$$|\varphi_{rn}| \leq 1.$$

It is enough to show that, for fixed r, φ_{rn} is bounded in n. Let $r \geq 2$ be fixed. A straightforward calculation shows that

$$x_{k_r+1+n} - x_{k_r+n} = (h_r+1)\varphi_{rn} - (h_r-1)\varphi_{r-1n}.$$

Hence for any fixed $r > R+1$, $x_{k_r+1+n} - x_{k_r+n}$ is bounded and so φ_{rn} is bounded for all r and n.

This completes the proof. $\square$

Remark 1. It is now a pertinent question, whether $\overset{\widehat{\wedge}}{BV}_\theta \subset \overset{\wedge}{BV}_\theta$, that is whether $\overset{\wedge}{BV}_\theta = \overset{\widehat{\wedge}}{BV}_\theta$. We are not able to answer this question and it remains open.

We have:

Theorem 2. *$\overset{\wedge}{BV}_\theta$ is Banach space normed by*

$$\|x\| = \sup_n \sum_r |\varphi_{rn}| \tag{2.1}$$

Proof. Because of Theorem 1, (2.1) is meaningful. It is routine verification that $\overset{\wedge}{BV}_\theta$ is a normed linear space. To show that $\overset{\wedge}{BV}_\theta$ is complete in its norm topology, let $\{x^i\}_{i=0}^{\infty}$ be a Cauchy sequence in $\overset{\wedge}{BV}_\theta$. Then $\{x_n^i\}_{i=0}^{\infty}$ is a Cauchy sequence in $\mathcal{C}$ for each n. Therefore $x_n^i \to x_n$ (say). Put $x = \{x_n\}_{i=0}^{\infty}$. We now show that $x \in \overset{\wedge}{BV}_\theta$ and $\|x^i - x\| \to 0$. Since $\{x^i\}$ is a Cauchy sequence in $\overset{\wedge}{BV}_\theta$, given $\varepsilon > 0$, there exists N such that for $i, j \geq N$,

$$\sum_r \left|\varphi_{rn}\left(x^i - x^j\right)\right| < \varepsilon$$

for all n. Therefore for any M and $i, j \geq N$;

$$\sum_{r=0}^{M} \left|\varphi_{rn}\left(x^i - x^j\right)\right| < \varepsilon$$

for all n. Now taking limit as $j \to \infty$ and then as $M \to \infty$, we get for $i > N$

$$\sum_r \left|\varphi_{rn}\left(x^i - x\right)\right| < \varepsilon \tag{2.2}$$

for all n.

Thus $x^i - x \in \overset{\wedge}{BV}_\theta$ and therefore by linearity $x \in \overset{\wedge}{BV}_\theta$. Also (2.2) implies that $\|x^i - x\| < \varepsilon$ $(i \geq N)$.

This completes the proof. □

THEOREM 3. *If* $\liminf q_r > 1$, $\overset{\wedge}{BV} \subset \overset{\wedge}{BV}_\theta$.

Proof. Let $x \in \overset{\wedge}{BV}$. Since $\liminf q_r > 1$, $q_r > 1 + \delta$ for $\delta > 0$. Now we have

$$\begin{aligned} & \frac{1}{h_r(h_r+1)} \sum_{j=k_{r-1}+1}^{k_r} (j - k_{r-1})(x_{j+n+1} - x_{j+n}) \\ = & \frac{1}{h_r(h_r+1)} \sum_{j=k_{r-1}+1}^{k_r} j(x_{j+n+1} - x_{j+n}) \\ & - \frac{k_{r-1}}{h_r(h_r+1)} \sum_{j=k_{r-1}+1}^{k_r} j(x_{j+n+1} - x_{j+n}) \\ \leq & \frac{1}{h_r(h_r+1)} \sum_{j=k_{r-1}+1}^{k_r} j(x_{j+n+1} - x_{j+n}). \end{aligned}$$

By using property of lacunary sequence, we have

$$\begin{aligned} & \frac{1}{h_r(h_r+1)} \sum_{j=k_{r-1}+1}^{k_r} j(x_{j+n+1} - x_{j+n}) \\ = & \frac{1}{h_r(h_r+1)} \left[\sum_{j=1}^{k_r} j(x_{j+n+1} - x_{j+n}) - \sum_{j=1}^{k_{r-1}} j(x_{j+n+1} - x_{j+n}) \right] \\ = & \frac{(k_r+1)k_r}{h_r(h_r+1)} \frac{1}{(k_r+1)k_r} \sum_{j=1}^{k_r} j(x_{j+n+1} - x_{j+n}) \\ & - \frac{(k_{r-1}+1)k_{r-1}}{h_r(h_r+1)} \frac{1}{(k_{r-1}+1)k_{r-1}} \sum_{j=1}^{k_{r-1}} j(x_{j+n+1} - x_{j+n}) \end{aligned}$$

We now have

$$\begin{aligned} \frac{(k_r+1)k_r}{h_r(h_r+1)} & = \frac{k_r}{h_r}\left(\frac{k_r+1}{h_r+1}\right) = \frac{k_r}{h_r}\left(\frac{k_r+1}{k_r - k_{r-1}+1}\right) \\ & = \frac{k_r}{h_r}\left(\frac{1}{\frac{k_r-k_{r-1}+1}{k_r+1}}\right) = \frac{k_r}{h_r}\left(\frac{1}{1-\frac{k_{r-1}}{k_r+1}}\right). \end{aligned}$$

Since $k_r < k_r + 1$ and $1 - \frac{k_{r-1}}{k_r+1} > 0$, we have

$$\frac{(k_r+1)\,k_r}{h_r\,(h_r+1)} \leq \frac{k_r}{h_r}\left(\frac{1}{1-\frac{k_{r-1}}{k_r}}\right) = \left(\frac{1}{1-\frac{k_{r-1}}{k_r}}\right)\left(\frac{1}{1-\frac{k_{r-1}}{k_r}}\right).$$

Since $q_r > 1+\delta$, we get

$$\frac{(k_r+1)\,k_r}{h_r\,(h_r+1)} \leq \left(\frac{1+\delta}{\delta}\right)^2.$$

Again

$$\begin{aligned}\frac{(k_{r-1}+1)\,k_{r-1}}{h_r\,(h_r+1)} &= \frac{k_{r-1}}{h_r}\left(\frac{k_{r-1}+1}{h_r+1}\right) = \frac{k_{r-1}}{h_r}\left(\frac{k_{r-1}+1}{k_r-k_{r-1}+1}\right)\\ &= \frac{k_{r-1}}{h_r}\left(\frac{1}{\frac{k_r-k_{r-1}+1}{k_{r-1}+1}}\right)\\ &= \frac{k_{r-1}}{h_r}\left(\frac{1}{\frac{k_r+2}{k_{r-1}+1}-1}\right).\end{aligned}$$

Since $k_{r-1}+1 < k_r+2$ for every $r \in \mathbb{N}$, $1 < \frac{k_r+2}{k_{r-1}+1}$. Hence we obtain

$$\begin{aligned}\frac{(k_{r-1}+1)\,k_{r-1}}{h_r\,(h_r+1)} &= \frac{k_{r-1}}{h_r}\left(\frac{1}{\frac{k_r+2}{k_{r-1}+1}-1}\right) = \left(\frac{1}{\frac{k_r+2}{k_{r-1}+1}-1}\right)\left(\frac{1}{\frac{k_r}{k_{r-1}}-1}\right)\\ &\leq \left(\frac{1}{\delta}\right)^2.\end{aligned}$$

Consequently, it can be seen that, for all n and $r \geq 2$,

$$\begin{aligned}&\sum_r\left|\frac{1}{h_r\,(h_r+1)}\sum_{j=k_{r-1}+1}^{k_r}(j-k_{r-1})\,(x_{j+n+1}-x_{j+n})\right|\\ \leq\ &\left(\frac{1+\delta}{\delta}\right)^2\sum_r\left|\frac{1}{k_r\,(k_r+1)}\sum_{j=1}^{k_r} j\,(x_{j+n+1}-x_{j+n})\right|\\ &+\left(\frac{1}{\delta}\right)^2\sum_r\left|\frac{1}{k_{r-1}\,(k_{r-1}+1)}\sum_{j=1}^{k_{r-1}} j\,(x_{j+n+1}-x_{j+n})\right|.\end{aligned}$$

Since each of sums on left converges to any limit uniformly in n, the sum on right converges also to any limit uniformly in n. So we get $x \in \widehat{BV}_\theta$. This completes the proof. $\square$

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