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STRONG LAWS OF LARGE NUMBERS FOR WEIGHTED SUMS OF $\tilde{\rho}$-MIXING RANDOM VARIABLES

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ABSTRACT. Strong laws are established for linear statistics that are weighted sums of a $\tilde{\rho}$-mixing random sample. The results obtained generalize the results of Baxter et al. [*SLLN for weighted independent indentically distributed random variables*, J. Theoret. Probab. **17** (2004), 165–181] to $\tilde{\rho}$-mixing random variables.

1. Introduction

Given nonempty sets $S, T \subset \mathcal{N}$, define $\mathcal{F}_S = \sigma(X_k, k \in S)$, and the maximal correlation coefficient $\tilde{\rho}_n = \sup \operatorname{corr}(f, g)$ where the supremum is taken over all (S, T) with $\operatorname{dist}(S, T) \geq n$ and all $f \in L_2(\mathcal{F}_S)$, $g \in L_2(\mathcal{F}_T)$ and where $\operatorname{dist}(S, T) = \inf_{x \in S,\, y \in T} |x - y|$.

Definition 1. A sequence of random variables $\{X_n, n \geq 1\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ is called *$\tilde{\rho}$-mixing* if there exists $k \in \mathbb{N}$, such that $\tilde{\rho}(k) < 1$.

As for $\tilde{\rho}$-mixing sequences of random variables, one can refer to Bryc and Smolenski (1993), who found bounds for the moments of partial sums for a sequence of random variables satisfying

$$\lim_{n \to \infty} \tilde{\rho}(n) < 1,$$

to Peligrad (1996) for CLT, Peligrad (1998) for invariance principles, Peligrad and Gut (1999) for the Rosenthal type maximal inequality, Yang

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(1998) for the moment inequalities and strong law of large numbers, and to Utev and Peligrad (2003) for invariance principles of nonstationary sequences.

As for independent random variables, let $\{X, X_i, i \geq 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants. The almost sure (a.s.) limiting behavior of weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ was studied by many authors (see, Baxter, 2004; Sung, 2001; Bai and Cheng, 2000; Choi and Sung, 1987; Cuzick, 1995; Wu, 1999). Recently Baxter (2004) proved the following strong laws of large numbers (see Theorem A).

Theorem A. *Let $\{X, X_i, i \geq 1\}$ be a sequence of i.i.d. random variables satisfying $EX = 0$ and $E|X| < \infty$. And let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants satisfying $A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty$, $A_{\alpha,n} = \sum_{i=1}^{n} |a_{ni}|^\alpha / n$ for some $\alpha > 1$. Then we have*

$$\frac{1}{n}\sum_{i=1}^{n} a_{ni}X_i \to 0 \ \ a.s..$$

The main purpose of this paper is to establish the Marcinkiewicz-Zygmund strong laws for linear statistics of $\tilde{\rho}$-mixing sequences of random variables. The results obtained generalize the results of Baxter et al. [2] to $\tilde{\rho}$-mixing random variables.

2. The Marcinkiewicz-Zygmund strong laws

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$, and $a_n \ll b_n$ will mean $a_n = O(b_n)$.

In order to prove our results, we need the concept of complete convergence and Lemma 2.1 bellow. The concept of complete convergence see the following.

Definition 2. (see [8]) Let $\{X, X_n; n \geq 1\}$ be a sequence of random variables, if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty$$

holds, we call $\{X_n, n \geq 1\}$ *completely converging to X.*

As for complete convergence, let $\{X, X_n, n \geq 1\}$ be a sequence of independent indentically distribution random variables (i.i.d) random variables and denote $S_n - \sum_{i-1}^{n} X_i$. The Hsu-Robbins-Erdös law of large numbers ([8], [7]) states that

$$\forall\varepsilon > 0, \quad \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty$$

is equivalent to $EX = 0$ and $EX^2 < \infty$.

This is a foundamental theorem in probability theory and has been intensively investigated by many authors in the past decades. See in Petrov (1995), Chow (1997) and Stout (1974), for example. Many extensions of Hsu-Robbins-Erdös law of large numbers have appeared since in various directions.

Lemma 2.1. ([17]) *Let $\{X_i, i \geq 1\}$ be a $\tilde{\rho}$-mixing sequence of random variables, $EX_i \quad 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C \quad C(p)$, such that*

$$E \max_{1\leq k\leq n} \Big| \sum_{i-1}^{k} X_i \Big|^p \leq C\Big\{ \sum_{i=1}^{n} E|X_i|^p + \Big(\sum_{i=1}^{n} EX_i^2 \Big)^{p/2} \Big\}.$$

Lemma 2.2. ([13, p. 84]) *Let $\{X_i, i \geq 1\}$ be a sequence of independent random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$, such that*

$$E \max_{1\leq k\leq n} \Big| \sum_{i=1}^{k} X_i \Big|^p \leq C\Big\{ \sum_{i-1}^{n} E|X_i|^p + \Big(\sum_{i=1}^{n} EX_i^2 \Big)^{p/2} \Big\}.$$

Our main result is:

Theorem 2.1. *Let $\{X, X_i, i \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing identically distributed random variables satisfying $EX = 0$ and $E|X| < \infty$. And let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants satisfying $A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty$, $A_{\alpha,n} = \sum_{i=1}^{n} |a_{ni}|^\alpha / n$ for some $\alpha \geq 2$. Let $T_n = \sum_{i=1}^{n} a_{ni} X_i, n \geq 1$, then we have*

$$\forall\varepsilon > 0, \quad \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j<n} |T_j| > \varepsilon n \Big) < \infty. \tag{2.1}$$

Proof. For all $i \geq 1$, define $X_i^{(n)} = X_i I(|X_i| \leq n) + nI(X_i > n) - nI(X_i < -n)$, $T_j^{(n)} = \sum_{i=1}^{j}(a_{ni}X_i^{(n)} - Ea_{ni}X_i^{(n)})$, then $\forall \varepsilon > 0$,

$$\begin{aligned}
&P\Big(\max_{1\leq j\leq n}|T_j| > \varepsilon n\Big)\\
&\leq P\Big(\max_{1\leq j\leq n}|X_j| > n\Big) + P\Big(\max_{1\leq j\leq n}\Big|T_j^{(n)} + \sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\Big| > \varepsilon n\Big)\\
&\leq P\Big(\max_{1\leq j\leq n}|X_j| > n\Big) + P\Big(\max_{1\leq j\leq n}|T_j^{(n)}| > \varepsilon n - \max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\Big|\Big). \quad (2.2)
\end{aligned}$$

First we show that

$$n^{-1}\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\Big| \to 0, \qquad \text{as} \quad n \to \infty. \tag{2.3}$$

By $\sum_{i=1}^{n}|a_{ni}|^\alpha = O(n)$ and Hölder inequality, for all $1 \leq k \leq \alpha$, then

$$\sum_{i=1}^{n}|a_{ni}|^k \leq \Big(\sum_{i=1}^{n}|a_{ni}|^{k\frac{\alpha}{k}}\Big)^{\frac{k}{\alpha}}\Big(\sum_{i=1}^{n}1\Big)^{\frac{\alpha-k}{\alpha}} \leq Cn. \tag{2.4}$$

Using $EX = 0$, (2.4), Markov inequality and $E|X| < \infty$, when $n \to \infty$, then

$$\begin{aligned}
&n^{-1}\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\Big|\\
&\leq n^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|X_i| > n) + \sum_{i=1}^{n}|a_{ni}|P(|X_i| > n)\\
&\ll n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X|I(|X| > n) + nP(|X| > n)\\
&\leq CE|X|I(|X| > n) + nP(|X| > n) \to 0. \qquad (2.5)
\end{aligned}$$

From (2.5), we have that (2.3) is true.

From (2.2) and (2.3), it follows that for n large enough

$$P\Big(\max_{1\leq j\leq n}|T_j| > \varepsilon n\Big) \leq \sum_{j=1}^{n}P(|X_j| > n) + P\Big(\max_{1\leq j\leq n}|T_j^{(n)}| > \frac{\varepsilon}{2}n\Big).$$

Hence we need only to prove that

$$\begin{aligned} I &=: \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P(|X_j| > n) < \infty, \\ II &=: \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1 \le j \le n} |T_j^{(n)}| > \frac{\varepsilon}{2} n \Big) < \infty. \end{aligned} \tag{2.6}$$

From the fact that $E|X| < \infty$, it follows easily that

$$\begin{aligned} I &= \sum_{n=1}^{\infty} n^{-1} n P(|X| > n) \\ &= \sum_{n=1}^{\infty} P(|X| > n) \\ &\le E|X| + 1 < \infty. \end{aligned} \tag{2.7}$$

By Lemma 2.1, it follows that

$$\begin{aligned} II &\le C \sum_{n=1} n^{-1} n^{-2} E \max_{1 \le j \le n} |T_j^{(n)}|^2 \\ &\le C \sum_{n=1}^{\infty} n^{-3} \sum_{j=1}^{n} E|a_{nj} X_j^{(n)}|^2 \\ &= C \sum_{n=1}^{\infty} n^{-3} \Big\{ \sum_{i=1}^{n} |a_{ni}|^2 E X^2 I(|X| \le n) + n^2 \sum_{i=1}^{n} |a_{ni}|^2 P(|X| > n) \Big\} \\ &\ll \sum_{n=1}^{\infty} n^{-3} n E X^2 I(|X| \le n) + \sum_{n=1}^{\infty} n^{-1} n P(|X| > n) \\ &= \sum_{n=1}^{\infty} n^{-2} \sum_{k=1}^{n} E X^2 I(k-1 < |X| \le k) + \sum_{n=1}^{\infty} P(|X| > n) \\ &= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} n^{-2} E X^2 I(k-1 < |X| \le k) + \sum_{n=1}^{\infty} P(|X| > n) \\ &\le \sum_{k=1}^{\infty} k^{-1} k^2 P(k-1 < \ X| \le k) + E|X| + 1 \\ &= \sum_{k=1}^{\infty} k P(k-1 < |X| \le k) + E|X| + 1 \\ &\le 2(E|X| + 1) < \infty. \end{aligned} \tag{2.8}$$

Now we complete the prove of Theorem 2.1. □

Corollary 2.1. *Under the conditions of* Theorem 2.1,

$$\lim_{n\to\infty}\frac{|T_n|}{n}=0\ \ a.s..$$

P r o o f. By (2.1), we have

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1}P\Big(\max_{1\le j\le n}|T_j|>\varepsilon n\Big)\\
&= \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} n^{-1}P\Big(\max_{1\le j\le n}|T_j|>\varepsilon n\Big)\\
&\ge \frac{1}{2}\sum_{i=1}^{\infty}P\Big(\max_{1\le j\le 2^i}|T_j|>\varepsilon 2^{i+1}\Big).
\end{aligned}$$

By Borel-Cantelli Lemma, we have

$$P\Big(\max_{1\le j\le 2^i}|T_j|>\varepsilon 2^{i+1}\ \ i.o.\Big)=0.$$

Hence

$$\lim_{i\to\infty}\max_{1\le j\le 2^i}\frac{|T_j|}{2^i}=0\ \text{ a.s.}$$

and using

$$\max_{2^{i-1}\le n<2^i}\frac{|T_n|}{n}\le\max_{1\le j\le 2^i}\frac{|T_j|}{2^i},$$

we have

$$\lim_{n\to\infty}\frac{|T_n|}{n}=0\ \text{ a.s..}$$

Remark 2.1. Corollary 2.1 generalizes the result of B a x t e r et al. [2] to $\tilde{\rho}$-mixing random variables.

Theorem 2.2. *Let $\{X, X_i, i\ge 1\}$ be a sequence of independent identically distributed random variables satisfying $EX=0$ and $E|X|<\infty$. And let $\{a_{ni}, 1\le i\le n, n\ge 1\}$ be a triangular array of constants satisfying $A_\alpha=\limsup_{n\to\infty}A_{\alpha,n}<\infty$, $A_{\alpha,n}=\sum_{i=1}^{n}|a_{ni}|^\alpha/n$ for some $\alpha\ge 2$. Let $T_n=\sum_{i=1}^{n}a_{ni}X_i$, $n\ge 1$, then we have*

$$\forall\varepsilon>0,\quad \sum_{n=1}^{\infty}n^{-1}P\Big(\max_{1\le j\le n}|T_j|>\varepsilon n\Big)<\infty. \tag{2.9}$$

P r o o f. Using Lemma 2.2 instead of Lemma 2.1, the proof of Theorem 2.2 is similar to the proof of Theorem 2.1. □

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