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Mathematica Slovaca, Vol. 57 (2007), No. 4, [333]--338

Persistent URL: http://dml.cz/dmlcz/136959

VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0027-2
Math. Slovaca **57** (2007), No. 4, 333–338

RELATIVE PURITY OVER NOETHERIAN RINGS

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(Communicated by Tibor Katriňák)

ABSTRACT. In this note we are going to show that if M is a left module over a left noetherian ring R of the infinite cardinality $\lambda \geq |R|$, then its injective hull $E(M)$ is of the same size. Further, if M is an injective module with $|M| \geq (2^\lambda)^+$ and $K \leq M$ is its submodule such that $|M/K| \leq \lambda$, then K contains an injective submodule L with $|M/L| \leq 2^\lambda$. These results are applied to modules which are torsionfree with respect to a given hereditary torsion theory and generalize the results obtained by different methods in author's previous papers: [*A note on pure subgroups*, Contributions to General Algebra 12. Proceedings of the Vienna Conference, June 3–6, 1999, Verlag Johannes Heyn, Klagenfurt, 2000, pp. 105–107], [*Pure subgroups*, Math. Bohem. **126** (2001), 649–652].

In this paper R denotes an associative ring with identity, which is usually left noetherian, and R-mod stands for the category of all unitary left R-modules. As usual, for a submodule K of the module M and for any element $x \in M$ the *annihilator (left) ideal* $(K : x)$ of R consists of all elements $r \in R$ with $rx \in K$. Dualizing the notion of the injective envelope of a module ([7]) H. Bass [1] investigated the projective cover of a module and he characterized the class of so called *perfect rings* over which every module has a projective cover. By a *projective cover* of a module M it is meant an epimorphism $\varphi\colon F \to M$ with F projective and such that the kernel K of φ is *superfluous* in F in the sense that the equality $K + L = F$ implies $L = F$ whenever L is a submodule of F.

2000 Mathematics Subject Classification: Primary 16P40, 16D50, 16D90, 18E40.
Keywords: left noetherian ring, injective module, injective hull, hereditary torsion theory, relative purity.
This research has been partially supported by the Grant Agency of the Charles University, grant #GAUK 301-10/203115/B-MAT/MFF and also by the institutional grant MSM 113 200 007.

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Recently, the general theory of covers has been studied intensively. If $\mathcal{G}$ is an *abstract class* of modules (i.e. $\mathcal{G}$ is closed under isomorphic copies) then a homomorphism $\varphi\colon G \to M$ with $G \in \mathcal{G}$ is called a *$\mathcal{G}$-precover* of the module M if for each homomorphism $f\colon F \to M$ with $F \in \mathcal{G}$ there is $g\colon F \to G$ such that $\varphi g = f$. A $\mathcal{G}$-precover of M is said to be a *$\mathcal{G}$-cover* if every endomorphism f of G with $\varphi f = \varphi$ is the automorphism of G. It is well-known (see e.g. 11]) that an epimorphism $\varphi\colon F \to M$, F projective, is a projective cover of the module M if and only if it is a $\mathcal{P}$-cover of M, where $\mathcal{P}$ denotes the class of all projective modules. Denoting by $\mathcal{F}$ the class of all flat modules, the Enochs' conjecture ([8]), whether every module over any associative ring with identity has an $\mathcal{F}$-cover, has been recently solved in affirmative independently by E. Enochs and L. Bican with R. El Bashir in the common paper [4].

In the general theory of precovers several types of purities are used. In some cases (see e.g. [5], [6], [10], [11]) the existence of pure submodules in the kernels of some homomorphisms plays an important role. Using the general theory of covers, in [6] the main result of this note appears as a corollary. However, the direct proof presented here is of some interest because the existence of non-zero pure submodules of "large" flat modules contained in submodules with "small" factors is sufficient for the existence of flat covers (see [6] and [4]).

In my previous paper [2] I proved that if λ is an infinite cardinal, then for any torsionfree abelian group F of the size $|F| \geq (2^\lambda)^+$ and any its subgroup K such that F/K is p-primary and $|F/K| \leq \lambda$, the subgroup K contains a non-zero subgroup L pure in F. This result was extended in [3] to the case when $F K$ is an arbitrary torsion group of the size at most λ, but the lower bound for the size of F is $(\nu^{\aleph_0})^+$, where ν is the first cardinal with $\lambda_i < \nu$, and λ_i are given by $\lambda_0 = \lambda$ and $\lambda_{i+1} = 2^{\lambda_i}$ for every $i - 0, 1, \ldots$.

The purpose of this note is to solve this problem for modules over left noetherian rings, which are torsionfree with respect to a given hereditary torsion theory for the category R-mod. As a consequence we obtain that in the abelian groups category the estimation $(2^\lambda)^+$ valid for the p-primary case is good enough for the general case. Moreover, we shall see that the submodule L of K can be found "large" in the sense that $|F/L| \leq 2^\lambda$. As a by-product we shall also prove that for any module M over a left noetherian ring R of the size $|M| \geq \max(|R|, \aleph_0)$ the injective envelope $E(M)$ of M is of the same size as M.

Theorem 1. *Let R be a left noetherian ring, $\mu = \max(|R|, \aleph_0)$. If M is an arbitrary module then*

(i) $|E(M)| = |M|$ *whenever* $|M| \geq \mu$;

(ii) $|E(M)| \leq \mu$ *whenever* $|M| < \mu$.

Proof.

(i) Proving indirectly, let us suppose that $|E(M)| > |M|$ for some $M \in R$-mod with $|M| \geq \mu$. For each finite subset $L = \{a_1, \dots, a_m\}$ we fix an order on L and consequently we shall consider L as a finite sequence. Now for each left ideal I of R we fix a finite set G_I of generators of I and to each element $x \in \overline{M}_0 = E(M) \setminus M$ we associate the finite sequence $S_x = \{a_1 x, \dots, a_m x\} \subseteq M$ in such a way that $I_x = (M : x)$ and $G_{I_x} = \{a_1, \dots, a_m\}$ (as a sequence). Further, we define the equivalence relation $\sim_0$ on the set $\overline{M}_0$ by setting $x \sim_0 y$ if and only if $(M : x) = I_x = I_y = (M : y)$ and $S_x = S_y$ (as sequences, again). Obviously, there is at most $|M|$ different sequences of the form S_x and consequently there exists an equivalence class M_0 under $\sim_0$ having $|E(M)|$ elements. Finally, we put $x_0 = 0$, we select an element $x_1 \in M_0$ arbitrarily and we shall continue by the induction. Assume that for some $n < \omega$ we have constructed the subsets $M_0, M_1, \dots, M_n$ of $E(M)$ and the elements $x_0, x_1, \dots, x_{n+1}$ in such a way that $M_0 \supseteq M_1 \supseteq \dots \supseteq M_n$, $x_0 = 0$, $x_{i+1} \in M_i$, $|M_i| = |E(M)|$ for each $i = 0, 1, \dots, n$ and $I_{x-x_i} = I_{y-x_i}$, $S_{x-x_i} = S_{y-x_i}$ for all $x, y \in M_i$ and all $i = 0, 1, \dots, n$. Setting $\overline{M}_{n+1} = \{x \in M_n : \; x - x_{n+1} \notin M\}$ we obviously get the set of the size $|E(M)|$ and we define the equivalence relation $\sim_{n+1}$ on the set $\overline{M}_{n+1}$ by setting $x \sim_{n+1} y$ if and only if $I_{x-x_{n+1}} = I_{y-x_{n+1}}$ and $S_{x-x_{n+1}} = S_{y-x_{n+1}}$. By the same argument as in the case $n = 0$ we obtain the existence of an equivalence class M_{n+1} under $\sim_{n+1}$ having $|E(M)|$ elements and we finally select an element $x_{n+2} \in M_{n+1}$ arbitrarily. To finish the proof, let $n \in \{1, 2, \dots\}$ and $x \in M_n \subseteq M_{n-1}$ be arbitrary. Then $x \sim_n x_{n+1}$ yields $I_{x-x_n} = I_{x_{n+1}-x_n} = I$ and for $G_I = \{a_1, \dots, a_m\}$ it is $a_i(x - x_n) = a_i(x_{n+1} - x_n)$, $i = 1, \dots, m$. Thus, for each $r \in I$, $r = \sum_{i=1}^{m} r_i a_i$, we have $r(x - x_{n+1}) = r(x - x_n) - r(x_{n+1} - x_n) = \sum_{i=1}^{m} r_i a_i (x - x_n) - \sum_{i=1}^{m} r_i a_i (x_{n+1} - x_n) = 0$. On the other hand, $x - x_{n+1} \notin M$ and so there is an element $s \in I_{x-x_{n+1}} = (M : (x - x_{n+1}))$ with $0 \neq s(x - x_{n+1}) \in M$, M being essential in $E(M)$. We have thus proved that $I_{x-x_n} \subsetneqq I_{x-x_{n+1}}$ for each $n = 1, 2, \dots$, which contradicts the hypothesis that the ring R is left noetherian.

(ii) If the ring R is infinite, then $E(R) \oplus E(M) \cong E(R \oplus M)$ is of the size μ by (i), while for R finite $E(R^{(\omega)} \oplus M)$ is of the size μ again, and the assertion follows easily. $\square$

For the sake of completeness we sketch the proof of the following technical lemma, the proof of which can be found in [5]. Recall, that two homomorphisms $f\colon F \to M$ and $g\colon G \to M$ are called *M-equivalent*, if there is an isomorphism $\pi\colon F \to G$ such that $g\pi = f$.

Lemma 2. *Let $F = \bigoplus\limits_{\delta\in D} F_\delta$ be a direct sum of modules and let $f\colon F \to M$ be an arbitrary homomorphism. Then there is a subset $D' \subseteq D$ such that $F = U \oplus V$, where $U = \bigoplus\limits_{\delta\in D'} F_\delta$, $V \subseteq \operatorname{Ker} f$ and for $\delta, \varepsilon \in D'$, $\delta \neq \varepsilon$, the restrictions $f\big|_{F_\delta}$ and $f\big|_{F_\varepsilon}$ are not M-equivalent.*

Proof. Denoting $f_\delta = f\big|_{F_\delta}$ for each $\delta \in D$, we can define the equivalence relation $\sim$ on the set D in such a way that we put $\delta \sim \varepsilon$ if and only if f_δ and f are M-equivalent. Let D' be any representative set of equivalence classes under $\sim$. If $\delta \sim \varepsilon$, then there is an isomorphism $\pi_{\varepsilon\delta}\colon F_\delta \to F_\varepsilon$ such that $f_\varepsilon\pi_{\varepsilon\delta} = f_\delta$. Setting $G_{\varepsilon\delta} = \{x - \pi_{\varepsilon\delta}(x) : x \in F_\delta\}$ and $D'_\delta = \{\varepsilon \in D : \varepsilon \sim \delta, \ \varepsilon \neq \delta\}$, it is a routine to check that $F \quad \bigoplus\limits_{\delta\in D'} \Big(F_\delta \oplus \Big(\bigoplus\limits_{\varepsilon\in D'_\delta} G_{\varepsilon\delta}\Big)\Big)$, the inclusion $V -$ $\bigoplus\limits_{\delta\in D'} \Big(\bigoplus\limits_{\varepsilon\in D'_\delta} G_{\varepsilon\delta}\Big) \subseteq \operatorname{Ker} f$ is obvious and the proof is complete. $\square$

Theorem 3. *Let R be a left noetherian ring and let $\lambda \geq \mu = \max(|R|, \aleph_0)$ be an arbitrary cardinal number. If M is an injective module with $|M| \geq (2^\lambda)^+$ and if K is its submodule such that $|M/K| \leq \lambda$, then there exists a submodule $L \subseteq K$ of M such that L is injective and $|M/L| \leq 2^\lambda$.*

Proof. Since R is left noetherian, we can decompose the module M into a direct sum $M = \bigoplus\limits_{\delta\in D} M_\delta$ of injective hulls of cyclic modules. By Lemma 2 there is a subset $D' \subseteq D$ such that $M = U \oplus V$, where $V \subseteq \operatorname{Ker}\pi$, $\pi\colon M \to M \ K$ being the canonical projection, and $U = \bigoplus\limits_{\delta\in D'} M_\delta$, where $\pi\big|_{M_\delta}$ and $\pi\big|_{M_\varepsilon}$ are not M/K-equivalent whenever $\delta, \varepsilon \in D'$, $\delta \neq \varepsilon$. There is at most μ left ideals of R, the ring R being left noetherian, hence at most μ different cyclic modules and consequently, by Theorem 1, at most $(\lambda^\mu)^\mu \leq \lambda^\lambda = 2^\lambda$ homomorphisms from injective hulls of cyclic modules into M/K. So, it suffices to put $L = V$, since in this case we have $L \subseteq K$ and $|M/L| = |U| \leq 2^\lambda$. $\square$

Recall, that a *hereditary torsion theory* $\sigma = (\mathcal{T}, \mathcal{F})$ for the category R-mod consists of two abstract classes $\mathcal{T}$ and $\mathcal{F}$, the *σ-torsion class* and the *σ-torsionfree class*, respectively, such that $\operatorname{Hom}(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$, the class $\mathcal{T}$ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class $\mathcal{F}$ is closed under submodules, extensions and arbitrary direct products and for each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

THEOREM 4. *Let $\sigma = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for the category R-mod over a left noetherian ring R. If $\lambda \geq \mu = \max(|R|, \aleph_0)$ is an arbitrary cardinal number and $F \in \mathcal{F}$ is an arbitrary module such that $|F| \geq (2^\lambda)^+$, then every submodule $K \leq F$ such that $|F/K| \leq \lambda$ contains a submodule L with $F/L \in \mathcal{F}$ and $|F/L| \leq 2^\lambda$.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d, "\pi"'] & E(F) \arrow[d, "g"] \\ F/K \arrow[r, "\iota"] & E(F/K) \end{tikzcd}$$

where j, ι are inclusions, π is the canonical projection and the existence of a homomorphism g making the square commutative follows from the injectivity of $E(F/K)$. Since $|E(F)| \geq (2^\lambda)^+$ and $|E(F/K)| \leq \lambda$ by Theorem 1, it follows from Theorem 3 that there exists an injective submodule $V \subseteq \operatorname{Ker} g$ such that $|E(F)/V| \leq 2^\lambda$. Setting $L = V \cap F$, we obviously have $L \subseteq K$, $F/L \cong (F+V)/V \subseteq E(F)/V$ yields $|F/L| \leq 2^\lambda$ and $F/L \in \mathcal{F}$, V being a direct summand of $E(F)$. □

As a special case we obtain the following generalization of our previous results proved in [2] and [3].

COROLLARY 5. *Let λ be an infinite cardinal and let F be a torsionfree abelian group such that $|F| \geq (2^\lambda)^+$. If $K \leq F$ is any subgroup with $|F/K| \leq \lambda$, then there exists a pure subgroup L of F contained in K and such that $|F/L| \leq 2^\lambda$.*

Proof. Obvious. □

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Received 11. 11. 2004

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