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Mathematica Slovaca, Vol. 57 (2007), No. 2, [129]--140

Persistent URL: `http://dml.cz/dmlcz/136942`

VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0004-9
Math. Slovaca, **57** (2007), No. 2, 129–140

SEPARATING POINTS OF MEASURES ON EFFECT ALGEBRAS

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(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. We prove the existence of separating points for every countable family of nonatomic σ-additive modular measures on a σ-complete lattice ordered effect algebra.

Introduction

A question which has been studied in classical Measure Theory is to establish when a family M of measures on a Boolean ring R admits a separating point, i.e. a point $a \in R$ such that $\mu(a) \neq \nu(a)$ for all pair (μ, ν) of different members of M. In particular, in [B-W] the authors proved that every countable family of atomless σ-additive measures on a σ-complete Boolean ring admits a dense G_δ-set of separating points with respect to a suitable topology. In [A-B-C], it is proved that a similar result also holds for modular functions on orthomodular lattices. Then a natural question which arises is the validity of such result in the more general context of modular measures on effect algebras.

This paper is devoted to the answer, affirmative, to the previous question (see (3.1)).

The paper is organized as follows: in Section 2 we study some properties of modular measures on lattice ordered effect algebras and of their generated uniformities that are involved in the proof of the main result and in Section 3 we prove the main result.

We recall that effect algebras have been introduced by D. J. Foulis and M. K. Bennett in 1994 (see [B-F]) for modelling unsharp measurement in

2000 Mathematics Subject Classification: Primary 28B05, 06C15.
Keywords: modular function, effect algebra, separating point.

Springer

a quantum mechanical system. They are a generalization of many structures which arises in quantum physics (see [B-C]) and in Mathematical Economics (see [E-Z], [M], [P], [B-K]), in particular of orthomodular lattices in non-commutative measure theory and MV-algebras in fuzzy measure theory. For a study of effect algebras we refer to the bibliography of [D-P].

1. Preliminaries

An *effect algebra* $(L, \oplus, 0, 1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for every $a, b, c \in L$:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(2) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ are defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(3) For every $a \in L$, there exists a unique $a^{\perp} \in L$ such that $a \oplus a^{\perp}$ is defined and $a \oplus a^{\perp} = 1$.
(4) If $a \oplus 1$ is defined, then $a = 0$.

In every effect algebra a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a.

Moreover we can define a binary relation on L by $a \leq b$ if and only if there exists $c \in L$ such that $c \oplus a = b$ and $\leq$ is a partial ordering in L with 0 as smallest element. We say that two elements $a, b \in L$ are *orthogonal*, and we write $a \perp b$, if $a \oplus b$ exists. Then $a \perp b$ if and only if $a \leq b^{\perp}$.

Moreover, for every $a \in L$, we have $a^{\perp} = 1 \ominus a$.

If $(L, \leq)$ is a lattice, we say that the effect algebra is a *lattice ordered effect algebra* or a *D-lattice*.

Effect algebras are a common generalization of orthomodular posets and MV-algebras. For a study, we refer to [D-P].

It is helpful to recall from [D-P] the following basic results.

PROPOSITION (1.1). *Assume a, b, c elements of an effect algebra L.*

(1) *If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.*
(2) *$a \ominus a = 0$, $a \ominus 0 = a$ and $a \oplus 0 = a$.*
(3) *If $a \leq b$, then $b \ominus a = 0$ if and only if $b = a$.*
(4) *If $a \leq b$, then $b = a \oplus (b \ominus a)$.*
(5) *If $a \perp b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$.*
(6) *If $a \leq b \leq c^{\perp}$, then $a \oplus c \leq b \oplus c$.*
(7) *If $a \perp b$ and $a \perp c$, then $a \oplus b = a \oplus c$ if and only if $b = c$.*
(8) *If $a \leq b \leq c$, then $a \perp c \ominus b$ and $a \oplus (c \ominus b) = c \ominus (b \ominus a)$.*
(9) *If L is a D-lattice, $a \leq c$ and $b \leq c$, then $c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$.*

In what follows, we denote by L a D-lattice and by $(G,+)$ a topological Abelian group.

For $a, b \in L$ with $a \le b$, we set $[a,b] = \{c \in L : a \le c \le b\}$. Moreover we set $\Delta = \{(a,b) \in L \times L : a = b\}$. We say that L is *σ-complete* if every countable set in L has supremum and infimum.

If $a_1, \dots, a_n \in L$, we inductively define $a_1 \oplus \dots \oplus a_n = (a_1 \oplus \dots \oplus a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, \dots, a_n\}$ of L is *orthogonal* if $a_1 \oplus \dots \oplus a_n$ exists. For a sequence $\{a_n\}$, we say that it is *orthogonal* if, for every positive integer n, $\bigoplus\limits_{i \le n} a_i$ exists. If, moreover, $\sup\limits_n \bigoplus\limits_{i \le n} a_i$ exists, we set $\bigoplus\limits_n a_n = \sup\limits_n \bigoplus\limits_{i \le n} a_i$.

For $a, b \in L$, we set $a \Delta b = (a \vee b) \ominus (a \wedge b)$. By [D-P, (1.8.2)], we have $a \Delta b = ((a \vee b) \ominus a) \vee ((a \vee b) \ominus b)$. Hence, if $b \le a$, we get $a \Delta b = a \ominus b$ since $a \ominus a = 0$.

A function $\mu \colon L \to G$ is said to be a *measure* if, for every $a, b \in L$, with $a \le b$, $\mu(b \ominus a) = \mu(b) - \mu(a)$. It is easy to see that μ is a measure if and only if, for every $a, b \in L$, with $a \perp b$, $\mu(a \oplus b) = \mu(a) + \mu(b)$. We say that μ is *σ-additive* if, for every orthogonal sequence $\{a_n\}$ in L such that $\bigoplus\limits_n a_n$ exists, $\mu\Big(\bigoplus\limits_n a_n\Big) = \sum\limits_{n=1}^{\infty} \mu(a_n)$. μ is said to be *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.

Every measure on an MV-algebra (and therefore every T_∞-valuation on a clan of fuzzy sets in the sense of [B-K]) is a modular function, and every modular function on an orthomodular lattice is a measure.

2. D-uniformities and modular measures

An essential tool to obtain the main result of the paper is to use the topological structure generated by modular measures on D-lattices. For this reason the present section is devoted to the study of some properties of modular measures and of their generated uniformities, which are involved in the results of the next section.

By [F-T], every G-valued modular function μ on any lattice generates a *lattice uniformity* $\mathcal{U}(\mu)$, i.e. a uniformity which makes the lattice operations $\vee$ and $\wedge$ uniformly continuous. By [A-B, (3.2)], for a G-valued modular measure μ on a D-lattice L, $\mathcal{U}(\mu)$ is a D-uniformity, i.e. $\mathcal{U}(\mu)$ makes also the operation $\oplus$ and $\ominus$ of L uniformly continuous, and a base of $\mathcal{U}(\mu)$ is the family of the sets $\{(a,b) \in L \times L : \mu([0, a \Delta b]) \subseteq W\}$, where W is a neighbourhood of 0 in G. Hence a base of neighbourhoods of 0 in $\mathcal{U}(\mu)$ is the family consisting of the sets $\{a \in L:\ \mu([0,a]) \subseteq W\}$, where W is a neighbourhood of 0 in G.

Moreover it is helpful to recall that, by [A-B, 4.1], a lattice uniformity $\mathcal{U}$ on L is a D-uniformity if and only if, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \ominus \Delta \subseteq U$ and $\Delta \ominus V \subseteq U$. Moreover, if $\mathcal{U}$ is a D-uniformity, then for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \oplus \Delta \subseteq U$. By [A-V, (2.4)], a base of $\mathcal{U}$ is the family of sets

$$F^{\Delta} = \{(a,b) \in L \times L:\ a\Delta b \in F\},$$

where F is a neighbourhood of 0 in $\mathcal{U}$.

PROPOSITION (2.1). *If $\mu\colon L \to G$ is a modular measure and $\mathcal{U}$ is a D-uniformity, then the following conditions are equivalent:*

(1) *μ is uniformly continuous.*
(2) *μ is continuous.*
(3) *$\mathcal{U}(\mu) \leq \mathcal{U}$.*

Proof. The equivalence of (1) and (3) holds because $\mathcal{U}(\mu)$ is the weakest lattice uniformity which makes μ uniformly continuous (see [W2, 3.1]).

The equivalence of (1) and (2) has been proved in [A, 3.5]. □

A lattice uniformity $\mathcal{U}$ on L is said to be *exhaustive* if every monotone sequence in L is a Cauchy sequence in $\mathcal{U}$. A G-valued modular function μ on L is said to be exhaustive if $\mathcal{U}(\mu)$ is exhaustive.

PROPOSITION (2.2). *Let $\mathcal{U}$ be a D-uniformity. Then the following conditions are equivalent:*

(1) *$\mathcal{U}$ is exhaustive.*
(2) *Every increasing sequence in L is Cauchy in $\mathcal{U}$.*
(3) *Every orthogonal sequence in L converges to 0 in $\mathcal{U}$.*

Proof.

(1) is trivial.

(2) $\Longrightarrow$ (3): Let $\{a_n\}$ be an orthogonal sequence in L and F a neighbourhood of 0 in $\mathcal{U}$. For each positive integer n, set $b_n = \bigoplus_{i\leq n} a_i$. By (1.1)-5, $\{b_n\}$ is an increasing sequence and therefore $b_n \to 0$ in $\mathcal{U}$. By (2.1), we can find a positive integer ν such that, for each $n > \nu$, $b_{n+1} \ominus b_n = b_{n+1}\Delta b_n \in F$. By (1.1)-4 we get $b_{n+1} = b_n \oplus (b_{n+1} \ominus b_n)$. On the other hand, we have $b_{n+1} = b_n \oplus a_n$. Therefore, by (1.1)-7, we obtain $a_n = b_{n+1} \ominus b_n \in F$ for each $n > \nu$. Hence $\{a_n\}$ converges to 0 in $\mathcal{U}$.

(3) $\Longrightarrow$ (1):

(i) As first step, we prove that, for every increasing sequence $\{a_n\}$ in L and for every $U \in \mathcal{U}$, there exists a positive integer ν such that, for every $n > \nu$, $(a_n, a_{n+1}) \in U$.

Let $\{a_n\}$ be an increasing sequence in L and $U \in \mathcal{U}$. By (2.1), we can choose a neighbourhood F of 0 in $\mathcal{U}$ such that $F^\Delta \subseteq U$. For each positive integer n, set $b_n = a_{n+1} \ominus a_n$. By [A-B, 2.4], $\{b_n\}$ is an orthogonal sequence in L. By (3), we can find a positive integer ν such that, for each $n > \nu$, $b_n \in F$. Therefore we get $(a_n, a_{n+1}) \in F^\Delta \subseteq U$.

(ii) We prove that every increasing sequence in L is Cauchy in $\mathcal{U}$.

Let $\{a_n\}$ be an increasing sequence in L and suppose that $\{a_n\}$ is not Cauchy. Then we can inductively obtain subsequences $\{b_n\}$ and $\{c_n\}$ of $\{a_n\}$ and $U \in \mathcal{U}$ such that, for each n, $b_n \le c_n \le b_{n+1}$ and $(b_n, c_n) \notin U$. Set $\{d_1, \dots, d_n\} = \{b_1, c_1, b_2, c_2, \dots\}$. Then $\{d_n\}$ in an increasing sequence in L and, for each n, we can find $m > n$ such that $(d_m, d_{m+1}) \notin U$, a contradiction with (i).

(iii) We prove that every decreasing sequence is Cauchy in $\mathcal{U}$.

Let $\{a_n\}$ be a decreasing sequence in L. Then $\{a_n^\perp\}$ is an increasing sequence in L. By (ii), $\{a_n^\perp\}$ is Cauchy in $\mathcal{U}$. Let $U, V \in \mathcal{U}$ be such that $\Delta \ominus V \subseteq U$ and choose a positive integer ν such that, for each $n > \nu$, $(a_n^\perp, a_{n+1}^\perp) \in V$. Then, for each $n > \nu$, we obtain $(a_n, a_{n+1}) = (1,1) \ominus (a_n^\perp, a_{n+1}^\perp) \in \Delta \ominus V \subseteq U$. □

COROLLARY (2.3). *Let $\mu\colon L \to G$ be a modular measure. Then μ is exhaustive if and only if, for every orthogonal sequence $\{a_n\}$ in L, $\{\mu(a_n)\}$ converges to 0 in $\mathcal{U}$.*

P r o o f . It is trivial that the condition is necessary since μ is continuous with respect to $\mathcal{U}(\mu)$.

Conversely, suppose that $\mu(a_n) \to 0$ for every orthogonal sequence $\{a_n\}$ in L. Suppose that there exists an orthogonal sequence $\{a_n\}$ in L which not converges to 0 in $\mathcal{U}(\mu)$. Then we can find a neighbourhood W of 0 in G such that, for each positive integer ν, there exists $n > \nu$ such that $\mu([0, a_n])$ is not contained in W. Hence we can obtain an orthogonal sequence $\{b_n\}$ in L with $\mu(b_n) \notin W$ for each n, a contradiction. Then μ is exhaustive by (2.2). □

A lattice uniformity $\mathcal{U}$ is said to be *nonatomic* if, for every $a \in L$ and every neighbourhood F of 0 in $\mathcal{U}$, there exists an orthogonal finite family $\{a_1, \dots, a_n\}$ in L with $\bigoplus_{i \le n} a_i = a$ and $a_i \in F$ for each $i \le n$.

A modular function $\mu\colon L \to G$ is said to be nonatomic if $\mathcal{U}(\mu)$ is nonatomic.

PROPOSITION (2.4). *Let $\mathcal{U}$ be a D-uniformity. Then the following conditions are equivalent:*

(1) *$\mathcal{U}$ is nonatomic.*

(2) *For every neighbourhood F of 0 in $\mathcal{U}$, there exists an orthogonal finite family $\{a_1, \dots, a_n\}$ in L such that $\bigoplus_{i \le n} a_i = 1$ and $a_i \in F$ for each $i \le n$.*

(3) *$\mathcal{U}$ is chained, i.e. for every $a, b \in L$ with $a \le b$ and every $U \in \mathcal{U}$, there exist $a_1, \dots, a_{n-1}$ in L such that $a = a_0 \le a_1 \le \dots \le a_n = b$ and $(a_{i-1}, a_i) \in U$ for each $i \le n$.*

P r o o f.

(1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3):

(i) As first step, we prove (3) with $a = 0$ and $b = 1$. Let $U, V \in \mathcal{U}$ be such that $\Delta \oplus V \subseteq U$. Choose an orthogonal family $\{a_1, \dots, a_n\}$ in L with $\bigoplus_{i \le n} a_i = 1$ and $(0, a_i) \in V$ for each $i \le n$. Set $b_0 = 0$ and, for each $i \in \{1, \dots, n\}$, $b_i = \bigoplus_{j \le i} a_j$. Then we have $0 = b_0 \le b_1 \le \dots \le b_n = 1$ and, for each $i \in \{1, \dots n\}$,

$$(b_{i-1}, b_i) = \Big(\bigoplus_{i \le n-1} a_i, \bigoplus_{i \le n-1} a_i \Big) \oplus (0, a_i) \in \Delta \oplus V \subset U.$$

(ii) Now we prove (3). Let $a, b \in L$ with $a \le b$ and $U \in \mathcal{U}$. Choose $V, V' \in \mathcal{U}$ with $V' \wedge \Delta \subseteq V$ and $V \vee \Delta \subseteq U$. By (i), we can find $b_1, \dots, b_{n-1}$ in L with $0 = b_0 \le b_1 \le \dots \le b_n = 1$ and $(b_{i-1}, b_i) \in V'$ for each $i \le n$. Set, for each $i \le n$, $c_i = (b_i \wedge b) \vee a$. Then we have $a = c_0 \le c_1 \le \dots \le c_n = b$ and, for each $i \le n$,

$$(c_{i-1}, c_i) = (b_{i-1} \wedge b, b_i \wedge b) \vee (a, a) \in (V' \wedge \Delta) \vee \Delta \subseteq V \vee \Delta \subseteq U.$$

(3) $\Longrightarrow$ (1): Let $a \in L$ and F be a neighbourhood of 0 in $\mathcal{U}$. Choose $U, V \in \mathcal{U}$ such that $V \ominus \Delta \subseteq U$ and $U(0) \subseteq F$. Choose $a_1, \dots, a_{n-1}$ in L such that $0 = a_0 \le a_1 \le \dots \le a_n = a$, and $(a_{i-1}, a_i) \in V$ for each $i \in \{1, \dots, n\}$. Set $b_0 = 0$ and, for each $i \in \{1, \dots, n\}$, $b_i = a_i \ominus a_{i-1}$. By [A-B, (2.1)], $\{b_0, \dots, b_n\}$ is orthogonal and $b_0 \oplus \dots \oplus b_n = a$. Moreover we have, for each $i \le n$, $(0, b_i) = (a_{i-1}, a_i) \ominus (a_{i-1}, a_{i-1}) \in V \ominus \Delta \subseteq U$, from which $b_i \in F$. $\square$

We recall that, for a G-valued modular measure μ on L, a base of neighbourhoods of 0 in $\mathcal{U}(\mu)$ is the family of sets $\{a \in L : \mu([0, a]) \subset W\}$, where W is a neighbourhood of 0 in G. From this fact and (2.4), we immediately obtain the following result (see also [B, 2.6]).

COROLLARY (2.5). *Let $\mu\colon L \to G$ be a modular measure. Then the following conditions are equivalent:*

(1) *μ is nonatomic.*
(2) *For every $a \in L$ and every neighbourhood W of 0 in G, there exists an orthogonal family $\{a_1, \dots, a_n\}$ in L such that $\bigoplus a_i = a$ and $\mu([0, a_i]) \subseteq W$ for each $i \le n$.*
(3) *For every neighbourhood W of 0 in G, there exists an orthogonal family $\{a_1, \dots, a_n\}$ in L such that $\bigoplus\limits_{i \le n} a_i = 1$ and $\mu([0, a_i]) \subseteq W$ for each $i \le n$.*

A modular function $\mu\colon L \to G$ is said to be *atomless* if, for every $a \in L$ with $\mu(a) \ne 0$, there exists $b < a$ such that $\mu(b) \ne 0$ and $\mu(b) \ne \mu(a)$.

We prove that, as for measures on Boolean algebras, nonatomic and atomless are equivalent for σ-additive modular measures.

We need the following definitions and result.

A lattice L' is said to be *dense in itself* if, for every $a, b \in L'$ with $a < b$, there exists $c \in L'$ with $a < c < b$, and *atomless* if, for every $a \in L'$ with $a \ne 0$, there exists $b \in L'$ with $0 < b < a$.

LEMMA (2.6). *Let L' be an effect algebra. Then L' is dense in itself if and only if is atomless.*

P r o o f. If L' is dense in itself, then it is trivial that L' is atomless.

Conversely, suppose that L' is atomless. Let $a, b \in L'$ with $a < b$, and set $c = b \ominus a$. By (1.1)-3, $c \ne 0$. Choose $d \in L'$ with $0 < d < c$. Then $d \perp a$ by (1.1)-1. Moreover, by (1.1)-5 and 7, we get $d \oplus a \ge a$ and $d \oplus a \ne a$ and, by (1.1)-4, we have $b = a \oplus c$. Therefore $d \oplus a \le b$ by (1.1)-6, and $d \oplus a \ne b$, otherwise we would have $c = b \ominus a = (d \oplus a) \ominus a = d$ by (1.1)-5. Hence L' is dense in itself. □

In the next result we use that, by [A-B, 2.4], a modular measure $\mu\colon L \to G$ is σ-additive if and only if μ is *σ-order-continuous* (σ-o.c.), i.e., for every increasing (decreasing) sequence $\{a_n\}$ in L such that $a = \sup\limits_n a_n$ ($a = \inf\limits_n a_n$) exists, $\mu(a_n) \to \mu(a)$.

For a modular function $\mu\colon L \to G$, we set $N(\mu) = \bigcap\{U : U \in \mathcal{U}(\mu)\}$ (see notations before (2.1)). By [W1, 1.2.2, 1.2.4], $N(\mu)$ is a lattice congruence, the quotient $L/N(\mu)$ is a lattice and the quotient uniformity $\hat{\mathcal{U}}$ generated by $\mathcal{U}(\mu)$ is a lattice uniformity. Moreover, if μ is a modular measure, it is easy to see that $\hat{L}$ is a D-lattice and $\hat{\mathcal{U}}$ is a D-uniformity.

PROPOSITION (2.7). *Suppose that G is metrizable, L is σ-complete and $\mu\colon L \to G$ is a σ-additive modular measure. Then μ is nonatomic if and only if μ is atomless.*

P r o o f . Suppose that μ is nonatomic. Let $a \in L$ with $\mu(a) \neq 0$. Since G is Hausdorff, we can find a neighbourhood W of 0 in G such that $\mu(a) \notin W$. Choose an orthogonal family $\{a_1, \dots, a_n\}$ with $\bigoplus_{i \leq n} a_i = a$ and $\mu([0, a_i]) \subseteq W$ for each $i \leq n$. Since $\mu(a) = \sum_{i \leq n} \mu(a_i)$, we can find $j \leq n$ with $\mu(a_j) \neq 0$. Since $\mu(a_j) \in W$, we get $\mu(a_j) \neq \mu(a)$.

Now suppose that μ is atomless. By (2.4) we have to prove that $\mathcal{U}(\mu)$ is chained. Set $\hat{L} = L/N(\mu)$ and denote by $\hat{\mathcal{U}}$ the quotient uniformity generated by $\mathcal{U}(\mu)$. By [W2, 5.8], if $(\hat{L}, \hat{\mathcal{U}})$ is complete and $\mathcal{U}$ is exhaustive, then $\mathcal{U}(\mu)$ is chained if and only if $\hat{L}$ is dense in itself. Now, since μ is atomless, $\hat{L}$ is atomless and therefore dense in itself by (2.6). Moreover, since μ is σ-o.c. by [A-B, (2.4)] and L is σ-complete, then $\mathcal{U}(\mu)$ is exhaustive by [W2, 3.5] and [W1, 8.1.2]. Finally, since $\hat{\mathcal{U}}$ has a countable base because G is metrizable, by [W1, 8.1.4, 6.3] we obtain that $(\hat{L}, \hat{\mathcal{U}})$ is complete. Therefore $\mathcal{U}(\mu)$ is chained, i.e. μ is nonatomic. □

PROPOSITION (2.8). *Let $\{\mathcal{U}_\alpha : \alpha \in A\}$ be a family of nonatomic lattice uniformities on L. Then $\mathcal{U} = \sup_{\alpha \in A} \mathcal{U}_\alpha$ is nonatomic.*

P r o o f . Let $a \in L$ and F a neighbourhood of 0 in $\mathcal{U}$. Choose $U \in \mathcal{U}$, $\alpha_1, \dots, \alpha_n \in A$ and $U_i \in \mathcal{U}_{\alpha_i}$ such that $U(0) \subseteq F$ and $\bigcap_{i \leq n} U_i \subseteq U$. By [W1, 1.1.3], for each $i \leq n$ we can find $V_i \in \mathcal{U}_{\alpha_i}$ such that $V_i \subseteq U_i$ and

$$[a \wedge b, a \vee b] \times [a \wedge b, a \vee b] \subseteq V_i \qquad \text{whenever} \quad (a, b) \in V_i . \tag{$*$}$$

We prove by induction the existence of an orthogonal family $\{a_1, \dots, a_r\}$ in $\bigcap_{i \leq n} V_i(0) \subseteq U(0) \subseteq F$ with $a = \bigoplus_{i \leq r} a_i$.

By induction, suppose that there exists an orthogonal family $\{a_1, \dots, a_s\}$ in $\bigcap_{i \leq n-1} V_i(0)$ with $a = \bigoplus_{i \leq s} a_i$. For each $i \leq s$, we can find an orthogonal family $\{b_{ji} : j \leq r_i\}$ in $V_n(0)$ with $\bigoplus_{j \leq r_i} b_{ji} = a_i$. Since $b_{ji} \leq a_i$, by $(*)$ we get $b_{ji} \in \bigcap_{j \leq n} V_j(0)$ and, if we denote by $c_1, \dots, c_t$ the elements of $\{b_{ji} : j \leq r_i, \ i \leq s\}$, we obtain as before that $\bigoplus_{i \leq t} c_i$ exists and $\bigoplus_{i \leq s} c_i = a$. Hence $\mathcal{U}$ is nonatomic. □

COROLLARY (2.9). *If $\mu\colon L \to G$ and $\nu\colon L \to G$ are nonatomic modular measures, then $\lambda = \mu + \nu$ is nonatomic, too.*

P r o o f . Recall that, for a modular measure μ, a base of $\mathcal{U}(\mu)$ is the family consisting of the sets $\big\{(a, b) \in L \times L : \ \mu([0, a \triangle b]) \subseteq W\big\}$, where W is a neighbourhood of 0 in G. From this, we immediately obtain that $\mathcal{U}(\lambda) \leq \mathcal{U}(\mu) \vee \mathcal{U}(\nu)$. By (2.8), we obtain that $\mathcal{U}(\lambda)$ is nonatomic and then λ is nonatomic, too. □

3. Separating points of modular measures

If M is a family of G-valued measures on L, we say that an element $a \in L$ is a *separating point* for M if $\mu(a) \neq \nu(a)$ for every $\mu, \nu \in M$ with $\mu \neq \nu$.

The aim of this section is to prove the following result.

THEOREM (3.1). *Let M be a family of G-valued nonatomic modular measures on L. Suppose that M is finite, or that M is countable, L is σ-complete and every $\mu \in M$ is σ-additive. Then the set of the separating points of M is a dense G_δ-set with respect to $\mathcal{U} = \sup\{\mathcal{U}(\mu) : \mu \in M\}$ (and therefore is non-empty).*

To prove (3.1), we need the following notations and results.

For a modular measure $\mu \colon L \to G$, we set

$$I(\mu) = \{a \in L : \mu([0,a]) = \{0\}\},$$
$$N(\mu) = \{(a,b) \in L \times L : \mu([0, a\triangle b]) = \{0\}\}.$$

By [A-B, (4.2)], we have $N(\mu) = \bigcap\{U : U \in \mathcal{U}(\mu)\}$ and $I(\mu) = \bigcap\{F : F$ neighbourhood of 0 in $\mathcal{U}(\mu)\}$ (see notations before (2.1)). Moreover, by [W2, 2.5, 3.1], we also have $N(\mu) = \{(a,b) \in L \times L : \mu$ is constant on $[a \wedge b, a \vee b]\}$.

PROPOSITION (3.2). *Let $\mu \colon L \to G$ be a modular measure and $\mathcal{U}$ a D-uniformity. Then the following conditions are equivalent:*

(1) *The set $\{a \in L : \mu(a) \neq 0\}$ is dense in $(L, \mathcal{U})$.*
(2) *$I(\mu)$ is not open in $(L, \mathcal{U})$.*

Proof. Set $I_0 = \{a \in L : \mu(a) = 0\}$.

(1) $\Longrightarrow$ (2): Suppose that $I(\mu)$ is open. We prove that I_0 is open, a contradiction with (1).

Let $a \in I_0$ and set $U_a = \{b \in L : a\triangle b \in I(\mu)\}$. Since $I(\mu)$ is open in $(L, \mathcal{U})$, by [A-V, (2.4)] we get that U_a is a neighbourhood of a in $\mathcal{U}$. We prove that $U_a \subseteq I_0$. Let $b \in U_a$ and $U, V \in \mathcal{U}(\mu)$ be such that $\Delta \oplus V \subseteq U$. By (1.1)-4, we have $a \vee b = (a \wedge b) \oplus (a\triangle b)$. Then, since $a\triangle b \in I(\mu) = \bigcap\{F :$ F neighbourhood of 0 in $\mathcal{U}(\mu)\}$, we get

$$(a \wedge b, a \vee b) = (a \wedge b, a \wedge b) \oplus (0, a\triangle b) \in \Delta \oplus V \subseteq U.$$

Since $U \in \mathcal{U}(\mu)$ is arbitrary, we get $(a \wedge b, a \vee b) \in N(\mu)$. Then μ is constant on $[a \wedge b, a \vee b]$. In particular, we have $\mu(b) = \mu(a) = 0$, since $a \in I_0$. Therefore $b \in I_0$.

(2) $\Longrightarrow$ (1): Let $U \in \mathcal{U}$. Choose symmetric $V, V', V'' \in \mathcal{U}$ such that $V \subseteq V' \subseteq V''$, $V' \vee \Delta \subseteq U$, $V'' \oplus \Delta \subseteq U$ and $[a \wedge b, a \vee b] \times [a \wedge b, a \vee b] \subseteq V$ whenever $(a,b) \in V$ (see [W1, 1.1.3]).

(i) As first step, we prove that there exists $d \in V(0)$ such that $\mu(d) \neq 0$.

Suppose that $\mu(d) = 0$ for each $d \in V(0)$. By the choice of V, we get $[0,d] \subseteq V(0)$ for each $d \in V(0)$ and therefore $V(0) \subseteq I(\mu)$. We prove that $I(\mu)$ is open in $(L,\mathcal{U})$, a contradiction with (2).

Let $a \in I(\mu)$ and set $U_a = \{b \in L : a\Delta b \in V(0)\}$. By [A-V, (2.4)], U_a is a neighbourhood of a in $\mathcal{U}$. We prove that $U_a \subseteq I(\mu)$.

Let $b \in U_a$ and $c \leq b$. Since $a\Delta b = ((a \vee b) \ominus a) \vee ((a \vee b) \ominus b)$ and $a\Delta b \in V(0) \subseteq I(\mu)$, we get $(a \vee b) \ominus a \in I(\mu)$. Since $(a \vee c) \ominus a \leq (a \vee b) \ominus a$, we have $\mu((a \vee c) \ominus a) = 0$. On the other hand, $\mu((a \vee c) \ominus a) = \mu(a \vee c) - \mu(a) = \mu(a) + \mu(c) - \mu(a \wedge c) - \mu(a) = \mu(c) - \mu(a \wedge c)$. Then $\mu(c) = \mu(a \wedge c) = 0$, since $a \in I(\mu)$. Since $c \leq b$ is arbitrary, we get $b \in I(\mu)$.

(ii) Now we prove that $L \setminus I_0$ is dense in $(L,\mathcal{U})$. Let $a \in I_0$. By (i), we can choose $d \in V(0)$ such that $\mu(d) \neq 0$. Set $b = (a \vee d) \ominus a$. Then $\mu(b) = \mu(a \vee d) - \mu(a) = \mu(d) - \mu(a \wedge d)$. Since $\mu(d) \neq 0$, we get either

$$\mu(b) \neq 0 \quad \text{or} \quad \mu(a \wedge d) \neq 0. \tag{$*$}$$

Now by (1.1)-8 we get that $a \perp b$ and $a \oplus b = (a \vee d) \ominus (a \ominus a) = a \vee d$. Hence $\mu(a \vee d) = \mu(a \oplus b) = \mu(a) + \mu(b) = \mu(b)$ since $a \in I_0$. Moreover, if we set $c = a \ominus (a \wedge d)$, we have $\mu(c) = \mu(a) - \mu(a \wedge d) = -\mu(a \wedge d)$. By $(*)$ we get either $a \vee d \notin I_0$ or $c \notin I_0$. On the other hand, since $[0,d] \subseteq V(0) \subseteq V'(0) \subseteq V''(0)$, we have $(a, a \vee d) \in \Delta \vee V' \subseteq U$ and, by (1.1)-4, $(a,c) = (a \wedge d, 0) \oplus (c,c) \in V'' \oplus \Delta \subseteq U$. Then, in any case, we obtain an element $e \notin I_0$ such that $(a,e) \in U$. □

Corollary (3.3). *Let $\mathcal{U}$ be a D-uniformity on L and M a family of G-valued $\mathcal{U}$-continuous modular measures on L. Then the following assertions are true:*

1. *If M is finite, then $A_M = \{a \in L : \mu(a) \neq 0$ for all $\mu \in M\}$ is a dense open set in $(L,\mathcal{U})$ if and only if, for every $\mu \in M$, $I(\mu)$ is not open in $(L,\mathcal{U})$.*
2. *If M is countable and $(L,\mathcal{U})$ is a Baire space, then A_M is a dense G_δ-set in $(L,\mathcal{U})$ if and only if, for every $\mu \in M$, $I(\mu)$ is not open in $(L,\mathcal{U})$.*

Proof. First suppose that A_M is dense in $(L,\mathcal{U})$. Then, for every $\mu \in M$, the set $A_\mu = \{a \in L : \mu(a) \neq 0\}$ is dense in $(L,\mathcal{U})$. Therefore, by (3.2), $I(\mu)$ is not open in $(L,\mathcal{U})$.

Conversely, suppose that, for every $\mu \in M$, $I(\mu)$ is not open in $(L,\mathcal{U})$. Then, by (3.2), for every $\mu \in M$, A_μ is dense in $(L,\mathcal{U})$. Moreover, for every $\mu \in M$, A_μ is open in $\mathcal{U}(\mu)$ since μ is continuous with respect to $\mathcal{U}(\mu)$, and therefore A_μ is also open in $\mathcal{U}$ by (2.1). Then, if M is finite, A_M is a finite intersection of dense open sets and therefore is a dense open set in $(L,\mathcal{U})$. If M is countable, then A_M is a countable intersection of dense open sets, hence is open and dense in $(L,\mathcal{U})$, because $(L,\mathcal{U})$ is a Baire space. □

THEOREM (3.4). *Let M be a family of G-valued nonatomic non-zero modular measures on L. Then the following assertions are true:*

(1) *If M is finite, then the set $A_M = \{a \in L : \ \mu(a) \neq 0$ for every $\mu \in M\}$ is a dense open set with respect to $\mathcal{U} = \sup\{\mathcal{U}(\mu) : \ \mu \in M\}$.*

(2) *If M is countable, L is σ-complete and every $\mu \in M$ is σ-additive, then A_M is a dense G_δ-set with respect to $\mathcal{U} = \sup\{\mathcal{U}(\mu) : \ \mu \in M\}$.*

Proof. We prove that, for every $\mu \in M$, $I(\mu)$ is not open in $(L,\mathcal{U})$.

Suppose that there exists $\mu \in M$ such that $I(\mu)$ is open in $(L,\mathcal{U})$. Let $a \in L$. Since $\mathcal{U}$ is nonatomic by (2.8), we can find an orthogonal family $\{a_1, \dots, a_n\}$ in $I(\mu)$ with $a = \bigoplus_{i \leq n} a_i$. Therefore $\mu(a) = \sum_{i=1}^{n} \mu(a_i) = 0$. Since a is arbitrary, we get $\mu = 0$, a contradiction. Now the conclusion follows by (3.3), since, if L is σ-complete and every $\mu \in M$ is σ-additive, then $(L,\mathcal{U})$ is a Baire space by [W2, 3.5] and [W1, 3.1.5]. □

Now we can prove (3.1).

Proof of (3.1). We apply (3.4) to the family $M' = \{\mu - \nu : \ \mu, \nu \in M, \mu \neq \nu\}$, recalling that the elements of M' are nonatomic by (2.9). □

REFERENCES

[A] AVALLONE, A.: *Lattice uniformities on orthomodular structures*, Math. Slovaca **51** (2001), 403 419.

[A-B-C] AVALLONE, A. BARBIERI, G. CILIA, R.: *Control and separating points of modular functions*, Math. Slovaca **49** (1999), 155 182.

[A-B] AVALLONE, A.—BASILE, A.: *On a Marinacci uniqueness theorem for measures*, J. Math. Anal. Appl. **286** (2003), 378–390.

[A-V] AVALLONE, A.—VITOLO, P.: *Lattice uniformities on effect algebras*, Internat. J. Theoret. Phys. (To appear).

[A-B1] AVALLONE, A. BARBIERI, G.: *Liapunov measures on effect algebras*, Comment. Math. Univ. Carolin. **44** (2003), 389–397.

[B] BARBIERI, G.: *Liapunov's theorem for measures on D-posets*, Internat. J. Theoret. Phys. **43** (2004), 1613 1623.

[B-F] BENNETT, M. K.—FOULIS, D. J.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1331 1352.

[B-W] BASILE, A. WEBER, H.: *Topological Boolean rings of first and second category. Separating points for a countable family of measures*, Rad. Mat. **2** (1986), 113 125.

[B-K] BUTNARIU, D. KLEMENT, P.: *Triangular Norm-Based Measures and Games with Fuzzy Coalitions*, Kluwer Acad. Publ., Dordrecht, 1993.

[C-D-M] CIGNOLI, R.—D'OTTAVIANO, I. M. L.—MUNDICI, D.: *Algebraic Foundations of Many-Valued Reasoning*, Kluwer Acad. Publ., Dordrecht, 2000.

[D-P] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*, Kluwer Acad. Publ./Ister Science, Dordrecht/Bratislava, 2000.

[F-T] FLEISCHER, I.—TRAYNOR, T.: *Equivalence of group-valued measure on an abstract lattice*, Bull. Polish Acad. Sci. Math. **28** (1980), 549–556.

[P] PAP, E.: *Pseudo-additive measures and their applications.* In: Handbook of Measure Theory, Vol. I, II, North-Holland, Amsterdam, 2002, pp. 1403–1468.

[P-P] PTÁK, P.—PULMANNOVÁ, S.: *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publ., Dordrecht, 1991.

[W1] WEBER, H.: *Uniform lattices I: A generalization of topological Riesz space and topological Boolean rings; Uniform lattices II*, Ann. Mat. Pura Appl. (4) **160; 165** (1991; 1993), 347–370; 133–158.

[W2] WEBER, H.: *On modular functions*, Funct. Approx. Comment. Math. **24** (1996), 35–52.

Received February 28, 2005
Revised May 24, 2005

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