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Dedicated to Professor Beloslav Riečan on the occasion of his 70th birthday

SPECTRAL RESOLUTIONS FOR σ-COMPLETE LATTICE EFFECT ALGEBRAS

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ABSTRACT. Recent results of the author about the existence of spectral measures for elements of σ-MV-algebras are applied to σ-complete lattice ordered effect algebras. It is shown that every element of a σ-complete lattice ordered effect algebra E admits a spectral measure that does not depend on the block to which the element belongs, and every element is uniquely defined by its spectral measure. Further, every σ-additive state defined on the orthomodular σ-lattice of sharp elements $\mathrm{Sh}(E)$ uniquely extends to a σ-additive state on the whole effect algebra, and pure states extend to pure states. To every element a in E, there is the smallest sharp element dominating it, and this sharply dominating element is contained in every block to which a belongs. Finally, it is shown that an effect-algebra commutator of two elements is sharply dominated by the commutator of their corresponding spectral measures, considered as real observables on $\mathrm{Sh}(E)$.

1. Definitions and known results

In this section, we give a review of recent results about the existence of spectral resolutions for elements of σ-MV-algebras and Dedekind σ-complete ℓ-groups, which are analogous to the spectral resolutions of self-adjoint operators on a Hilbert space [28], [29]. These results are based on generalized Loomis-Sikorski theorems for σ-MV-algebras and Dedekind σ-complete Abelian ℓ-groups, obtained by Dvurečenskij and Mundici, independently ([7],

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[27]), an on the theory of compressible groups recently developed by Foulis in his pioneering papers [12], [13], [14], [15]. A useful tool is also the Butnariu and Klement theorem ([2]).

For more details about MV-algebras see [5], effect algebras [9], orthomodular σ-lattices [35], [31], Abelian ℓ-groups [18].

Most of the results mentioned in this section are known, up to Theorem 1.8, which yields a characterization of regular spectral representations of σ-MV-algebras.

An *effect algebra*, introduced in [11] (see [25] and [17] for equivalent definitions) as an abstraction of the effect algebra of Hilbert-space effects (self-adjoint operators T on a Hilbert space such that $0 \le T \le I$) for the purposes of quantum mechanics, is an algebraic structure $(E; \dot{+}, 0, 1)$, where $\dot{+}$ is a partially defined binary operation and 0 and 1 are constants, such that the following axioms hold:

(E1) $a \dot{+} b = b \dot{+} a$;
(E2) $a \dot{+} (b \dot{+} c) = (a \dot{+} b) \dot{+} c$;
(E3) for every $a \in E$ there is a unique $a' \in E$ such that $a \dot{+} a' = 1$;
(E4) $a \dot{+} 1$ is defined iff $a = 0$.

The equalities in (E1) and (E2) mean that if one side is defined, so is the other and the equality holds. An effect algebra is partially ordered by the relation $a \le b$ iff there is c such that $a \dot{+} c = b$. The element c is then uniquely defined. This enables us to introduce another partial binary operation $\dot{-}$ by $b \dot{-} a = c$ iff $a \dot{+} c = b$, so that $b \dot{-} a$ is defined iff $a \le b$. In particular $a' = 1 \dot{-} a$. In the ordering $\le$, 1 is the largest and 0 is the smallest element in E. We also have that $a \dot{+} b$ exists iff $a \le b'$, and $a \dot{+} b = (b' \dot{-} a)'$. We say that a and b are *orthogonal* if $a \le b'$, i.e., iff $a \dot{+} b$ is defined. More generally, we say that elements $a_1, a_2, \dots, a_n$ are orthogonal if $a_1 \dot{+} a_2 \dot{+} \dots \dot{+} a_n$ exists in E, where the later element is defined recurrently.

An important example of effect algebras is obtained in the following way. Let (G, u) be an Abelian group with strong unit u. The unit interval $\{g \in G : 0 \le g \le u\} = [0, u]$ endowed with the operation $\dot{+}$ such that $a \dot{+} b$ is defined iff $a + b \le u$, and then $a \dot{+} b = a + b$, and $a' = u - a$ becomes an effect algebra with the same ordering as that inherited from (G, u). Effect algebras arising this way are called *interval effect algebras*.

In particular, if we take (G, u) as the group of all self-adjoint operators on a Hilbert space H and u as the identity operator I, then the interval $[0, I]$ of self-adjoint operators between the null and identity operator (so called Hilbert space effects) forms an effect algebra.

Another important case is obtained by considering the interval $[0, u]$ of a lattice ordered group G. It can be proved that in this case the corresponding effect algebra can be organized into an MV-algebra. We recall that an *MV-algebra* is an algebraic structure $(M;\oplus,^*,0,1)$ consisting of a nonempty set M, a binary operation $\oplus$, a unary operation * and two constants 0 and 1 satisfying the following axioms:

(M1) $a\oplus b=b\oplus a$;
(M2) $a\oplus(b\oplus c)=(a\oplus b)\oplus c$;
(M3) $a\oplus a^*=1$;
(M4) $a\oplus 0=a$;
(M5) $a^{**}=a$;
(M6) $0^*=1$;
(M7) $a\oplus 1=1$;
(M8) $(a^*\oplus b)^*\oplus b=(a\oplus b^*)^*\oplus a$.

Boolean algebras coincide with MV-algebras satisfying the additional condition $x\oplus x=x$. A routine computation ([5]) shows that the axiomatization is equivalent to the original one due to Chang [3], where MV-algebras were introduced as an algebraic basis for many-valued logic. A prototypical MV-algebra is given by the real unit interval $[0,1]=\{x\in\mathbb{R}:\ 0\le x\le 1\}$ equipped with the operations $x^*=1-x$, $x\oplus y=\min\{1,x+y\}$. Chang's completeness theorem ([4]) states that if an equation holds in $[0,1]$, then the equation holds in every MV-algebra. An MV-algebra is ordered by the relation $x\le y$ iff $x^*\oplus y=1$. This ordering makes M a distributive lattice with smallest element 0 and largest element 1. Suprema and infima in M are given by

$$x\vee y=(x^*\oplus y)^*\oplus y\,,\qquad x\wedge y=(x^*\vee y^*)^*\,.$$

An additional binary relation $\odot$ is defined by $x\odot y=(x^*\oplus y^*)^*$. An element $a\in M$ is *idempotent* if $a\oplus a=a$. The set of all idempotents in M will be denoted by $\mathcal{B}(M)$. With the operations $\vee$, $\wedge$ and * inherited from M, $\mathcal{B}(M)$ is a Boolean algebra.

Let (G,u) be an Abelian ℓ-group (additively written) with strong unit u. Let

$$\Gamma(G,u):=\{x\in G:\ 0\le x\le u\}=[0,u]$$

be the unit interval of G equipped with the operations $x^*=u-x$, $x\oplus y=u\wedge(x+y)$, $x\odot y=0\vee(x+y-1)$. For any morphism $\lambda\colon(G,u)\to(G',u')$, let $\Gamma(\lambda)$ be the restriction of λ to $[0,u]$. Then Γ is a categorical equivalence between Abelian ℓ-groups with strong unit and MV-algebras ([26]).

An MV-algebra that is a σ-lattice is called a *σ-MV-algebra.* σ-MV-algebras are in categorical equivalence with Dedekind σ-complete ℓ-groups, and moreover, the set of all idempotents in a σ-MV-algebra is a Boolean σ-algebra ([9]).

Every MV-algebra M becomes an effect algebra (with the same ordering) if we restrict the operation $\oplus$ to pairs a,b such that $a\le b^*$. Conversely, in [6] it is proved that an effect algebra E can be organized into an MV-algebra (with the same ordering) iff E is a lattice and for every $a, b \in E$ there holds

$$(a \vee b) \dot{-} a = b \dot{-} (a \wedge b) .$$

The MV-algebra operation $\oplus$ is defined by $a \oplus b = a \dot{+} (a' \wedge b)$ and $a^* = a'$. Another characterization of effect algebras that can be organized into MV-algebras (so-called MV-effect algebras, [16]), is the following: an effect algebra E is an MV-effect algebra iff E is a lattice, and satisfies the Riesz decomposition properties: $a \le (b \dot{+} c) \implies a = b_1 \dot{+} c_1$, where $b_1 \le b$, $c_1 \le c$.

We recall that an effect algebra is *σ-complete* iff the following condition is satisfied: For every sequence $(a_i)_{i \in \mathbb{N}}$ such that every finite subsequence is orthogonal, the element $\bigoplus_i a_i := \sup_{n \in \mathbb{N}} \oplus_{i \le n} a_i$ is defined. A lattice effect algebra is σ-complete iff it is a σ-lattice ([22]).

Let $(E; \dot{+}, 1, 0)$ be an effect algebra. Two elements $a, b \in L$ are *compatible*, written $a \leftrightarrow b$, if there are elements a_1, b_1, c in E such that $a_1 \dot{+} b_1 \dot{+} c$ is defined and $a = a_1 \dot{+} c$, $b = b_1 \dot{+} c$. If E is lattice ordered, then by [6], $a \leftrightarrow b$ iff $(a \vee b) \dot{-} b = a \dot{-} (a \wedge b)$. By [33], every maximal set of pairwise compatible elements (so-called *block*) in a lattice effect algebra E is a sub-effect algebra and a sub-lattice of E, which can be organized into an MV-algebra. In addition, every lattice effect algebra is a set-theoretical union of its blocks. This result generalizes the well-known result in orthomodular lattices, where blocks are maximal Boolean subalgebras, [35], [31]. We recall that orthomodular lattices can be characterized as lattice effect algebras with the additional identity $a \wedge a' = 0$.

A *state* on an effect algebra E is a mapping $m \colon E \to [0,1]$ such that $m(a \dot{+} b) = m(a) + m(b)$ whenever $a \perp b$, and $m(1) = 1$. A state is *σ-additive* if $m(a_n) \to m(a)$ whenever $a_n \nearrow a$. A state m on E is *pure* if it cannot be expressed as a convex combination of other states, i.e., if $m(a) = \alpha m_1(a) + (1-\alpha) m_2(a)$, $a \in E$, $0 < \alpha < 1$, then $m_1 = m_2 = m$.

States (and also σ-additive states) on MV-algebras coincide with states on the corresponding effect algebras. Pure states on MV-algebras correspond to MV-algebra-homomorphisms to the MV-algebra $[0,1]$ ([9; Theorem 7.1.1]).

The following notion is a direct generalization of a σ-algebra of sets ([9; p. 452]). A *tribe of fuzzy sets* on a set $\Omega \ne \emptyset$ is a nonempty system $\mathcal{T} \subseteq [0,1]^\Omega$ such that

(T1) $1_\Omega \in \mathcal{T}$;
(T2) if $a \in \mathcal{T}$, then $1_\Omega - a \in \mathcal{T}$;
(T3) $(a_n)_{n=1}^\infty \subseteq \mathcal{T}$ entails

$$\min\Big(\sum_{n=1}^{\infty} a_n, 1 \Big) \in \mathcal{T} .$$

Elements of $\mathcal{T}$ are called *fuzzy subsets* of Ω. Elements of $\mathcal{T}$ which are characteristic functions are called *crisp subsets* of Ω.

Denote by $\mathcal{B}(\mathcal{T}) = \{A \subset \Omega : \chi_A \in \mathcal{T}\}$ the system of all crisp subsets in $\mathcal{T}$. Then $\mathcal{B}(\mathcal{T})$ is a σ-algebra of subsets of Ω. By the Butnariu-Klement theorem ([2], [32; Theorem 8.1.12]), if $f \in \mathcal{T}$, then f is $\mathcal{B}(\mathcal{T})$-measurable and moreover, for every σ-additive state m on $\mathcal{T}$ we have

$$m(f) = \int_{\Omega} f(\omega)\, \mathrm{d}\mu(\omega)\,, \tag{1}$$

where $\mu(A) = m(\chi_A)$, $A \in \mathcal{B}(\mathcal{T})$, is a probability measure.

The following generalization of the Loomis-Sikorski theorem was proved in [7] and [27], independently (see also [1] for a different proof).

THEOREM 1.1. *For every σ-complete MV-algebra M there exist a tribe $\mathcal{T}$ of fuzzy subsets of a set Ω and an MV-σ-homomorphism h from $\mathcal{T}$ onto M.*

We note that the set Ω in Theorem 1.1 is the set of all extremal states of M which is a basically disconnected compact Hausdorff space. In addition, the σ-homomorphism h maps $\mathcal{B}(\mathcal{T})$ onto $\mathcal{B}(M)$ ([8]).

In view of the Butnariu and Klement theorem and the above generalization of the Loomis-Sikorski theorem, we obtain that for every $a \in M$, the mapping $\Lambda_a \colon \mathcal{B}([0,1]) \to \mathcal{B}(M)$ from the σ-algebra of Borel subsets of the interval $[0,1]$ of reals to the Boolean σ-algebra of idempotents of M, defined by $\Lambda_a(D) := h \circ f_a^{-1}(D)$, where $f_a \in \mathcal{T}$ is such that $h(f_a) = a$, is a σ-homomorphism (of Boolean σ-algebras).

The following theorems have been proved in [28].

THEOREM 1.2. ([28; Theorem 3.1]) *Let M be a σ-MV-algebra. To every $a \in M$ a σ-homomorphism $\Lambda_a \colon \mathcal{B}([0,1]) \to \mathcal{B}(M)$ can be constructed such that the map $a \mapsto \Lambda_a$ is one-to-one and for every σ-additive state m on M we have*

$$m(a) = \int_0^1 \lambda\, m(\Lambda_a(\mathrm{d}\lambda))\,. \tag{2}$$

Equation (2) enables us to prove the following statement.

THEOREM 1.3. ([28; Theorem 3.3]) *Let M be a σ-MV-algebra. Every probability measure on the Boolean σ-algebra $\mathcal{B}(M)$ of idempotent elements in M uniquely extends to a σ-additive state on M.*

We call any σ-homomorphism Λ_a satsfying conditions of Theorem 1.2 a *spectral measure* or a *spectral resolution* of a, and the mapping $a \mapsto \Lambda_a$ is called a *spectral representation* of M.

In general, there might exist several mappings $a \mapsto \Lambda_a$ which satisfy statements of Theorem 1.2. In accordance with [28], let us call a triple $(X, \mathcal{T}, h)$, where $\mathcal{T}$ is a tribe of functions on X, and $h\colon \mathcal{T} \to M$ is a surjective MV-σ-homomorphism, a *representation* of M. A representation is called *regular* if $h(f) = 0$ iff $h\big(N(f)\big) = 0$, where $N(f) = \{x \in X : f(x) > 0\}$ is the support of f. The representation constructed in the proof of the Loomis-Sikorski theorem is a regular representation, which is called the *canonical representation* of M. It was proved in [28] that every regular representation induces a mapping $a \to \Lambda_a$ that satisfies statements of Theorem 1.2. We call any such σ-homomorphism Λ_a a *regular spectral measure* or a *regular spectral resolution* of a, and the mapping $a \mapsto \Lambda_a$ is called a *regular spectral representation* of M.

The mapping $a \mapsto \Lambda_a$, where Λ_a is constructed via the Loomis-Sikorski representation of M, is called the *canonical spectral representation* of M, and the σ-homomorphism Λ_a is called the *canonical spectral measure* of the element a.

Using categorical equivalence between σ-MV-algebras and Dedekind σ-complete ℓ-groups and theory of compressible groups recently developed by Foulis in [12], [13], [14], [15], enables us to prove the following statement ([29]).

THEOREM 1.4. *Every regular spectral representation of a σ-MV-algebra coincides with the canonical spectral representation.*

To illustrate the proof, let M be a σ-MV-algebra, and let (G, u) be a Dedekind σ-complete ℓ-group such that $\Gamma(G, u) = M$. Recall that an element $p \in G$ is characteristic, if $0 \leq p \leq u$, and $p \wedge (u - p) = 0$ ([18]). The set $\mathcal{P}$ of characteristic elements in G is isomorphic with the Boolean σ-algebra $\mathcal{B}(M)$ of M.

A generalization of the Loomis-Sikorski theorem for Dedekind σ-complete ℓ-groups with strong unit was proved in [7]. In this theorem, a generalization of a tribe, so-called g-tribe is used, where a g-tribe is a set $\mathcal{T}_g$ of bounded functions on a (nonempty) set X such that

(i) $0_x, 1_x \in \mathcal{T}_g$,
(ii) $f \pm g \in \mathcal{T}_g$ whenever $f, g \in \mathcal{T}_g$,
(iii) if $(f_n)_n$ is a sequence of elements from $\mathcal{T}_g$ for which there is $f \in \mathcal{T}_g$ with $f_n \leq f$ (pointwise) for all n, then $\sup\limits_n f_n \in \mathcal{T}_g$.

The analogue of the Loomis-Sikorski theorem then claims the following.

THEOREM 1.5. *For every Dedekind σ-complete ℓ-group G with a strong unit u there is a g-tribe $\mathcal{T}_g$ of bounded functions on a compact Hausdorff space X and a σ-lattice group homomorphism η from $\mathcal{T}_g$ onto G with $\eta(1_X) = u$.*

Notice that a g-tribe is Dedekind σ-complete ℓ-group with unit 1_X, its unit interval is a tribe, and the restriction of η to $\Gamma\big(\mathcal{T}_g, 1_X\big)$ yields a Loomis-Sikorski representation of the σ-MV-algebra $\Gamma(G, u)$.

An extension of the Butnariu and Klement theorem can also be proved, that is, every $f \in \mathcal{T}_g$ is measurable with respect to the σ-algebra of sets $\mathcal{B}(\mathcal{T}_g) := \{A \subset X : \chi_A \in \mathcal{T}_g\}$. Moreover, every σ-additive state m (positive functional with $m(1_X) = 1$)) on $\mathcal{T}_g$ admits an integral representation $m(f) = \int_{\mathbb{R}} f(x)\, \mathrm{d}\mu(x)$, where μ is the probability measure on $\mathcal{B}(\mathcal{T}_g)$ obtained by the restriction of m ([10]).

The notion of a regular representation can be extended to a Dedekind-σ-complete unital ℓ-group (G,u) as follows: a representation of (G,u) is a triple $(X,\mathcal{T}_g,\eta)$, where $\mathcal{T}_g$ is a g-tribe of functions on a set X and η is a σ-ℓ-group homomorphism of $\mathcal{T}_g$ onto G with $\eta(1_x) = u$. A representation is regular if $\eta(f) = 0$ iff $\eta(\chi_{N(f)}) = 0$, where $N(f) = \{x \in X : f(x) \neq 0\}$ is the support of f.

In [29], the following theorem was proved.

THEOREM 1.6. *Let G be a Dedekind σ-complete ℓ-group with strong unit u. Let $(X,\mathcal{T}_g,\eta)$ be a regular representation of (G,u). To every $g \in G$, there is a σ-homomorphism $\Lambda_g \colon \mathcal{B}(\mathbb{R}) \to \mathcal{P}$ such that $g \mapsto \Lambda_g$ is one-to-one, and for every σ-additive state m on (G,u) we have $m(g) = \int_{\mathbb{R}} \lambda\, m(\Lambda_g(\mathrm{d}\lambda))$.*

Notice that in the above theorem, for $g \in G$ and $E \in \mathcal{B}(\mathbb{R})$, $\Lambda_g(E) = \eta \circ f^{-1}(E)$, where $f \in \mathcal{T}_g$ is any element such that $\eta(f) = g$.

Let (G,u) be a Dedekind σ-complete ℓ-group. For $e \in G$, $0 \le e \le u$, and $p \in \mathcal{P}$, put $J_p(e) = e \wedge p$. This mapping can be extended to a positive group homomorphism $J_p \colon G \to G$, with the properties: $J_p(u) = p$, $0 \le e \le p \implies J_p(e) = e$, and $J_p(e) = e$ iff $J_{u-p}(e) = 0$, $J_p(e) = 0$ iff $J_{u-p}(e) = e$. That is, J_p is a *compression* in the sense of Foulis, and since also $J_p = J_q$ iff $p = q$, G is a *compressible group* ([12], [13]).

In addition, for every $p \in \mathcal{P}$, and every $g \in G$, $g = J_p(g) + J_{u-p}(g)$. Further, for every $g \in G$ there is at least one $p \in \mathcal{B}(G)$ such that $J_{u-p}(g) \le 0 \le J_p(g)$, and the elements $J_p(g)$, $J_{u-p}(g)$ do not depend on the choice of a suitable p. Therefore, there is a uniquely defined decomposition into positive and negative parts, $g^+ = J_p(g)$, $g^- = -J_{u-p}(g)$ for every $g \in G$. This means that G has the *general comparability property* in the sense of [15]. Finally, G admits a *Rickart mapping*, that is, a mapping ${}^\sim \colon G \to \mathcal{P}$ such that $p \in \mathcal{P}$, $J_p(g) = 0 \iff p \le g^\sim$ ([14]).

Let $q \in \mathbb{Q}$ be any rational number that can be written as $q = m/n$, where m,n are integers and $n > 0$. In [14], it is proved that for any $g \in G$, the element $p_q := \big((ng - mu)^+\big)^\sim$ does not depend on the choice of m, n in the expression for q, and the family of elements $\big(p_q := \big((ng-mu)^+\big)^\sim\big)_{q\in\mathbb{Q}}$ is called a *rational* *spectral resolution* of G. In addition, every element $g \in G$ is uniquely defined by its rational spectral resolution.

In [29], it was proved that for any regular spectral representation $g \mapsto \Lambda_g$, we have $\Lambda_g((-\infty, q]) = \big((ng - mu)^+\big)^\sim$. Since the spectral measure Λ_g is uniquely defined by the family $\big(\Lambda_g(-\infty, q]\big)_{q \in \mathbb{Q}}$, it follows that Λ_g is uniquely defined by the rational spectral resolution of g, and does not depend on the choice of a particular regular representation of (G, u). Restricting these considerations to $\Gamma(G, u)$, we obtain analogous result for regular representations of σ-MV-algebras, and the proof of Theorem 1.4 follows.

Let M be a σ-MV-algebra. According to Varadarajan ([35]), every spectral measure $\Lambda_a : \mathcal{B}([0,1]) \to \mathcal{B}(M)$ can be considered as a *real observable* associated with the Boolean σ-algebra $\mathcal{B}(M)$. Notice that, since $\Lambda_a([0,1]) = 1$, we can consider Λ_a as an observable from $\mathcal{B}(\mathbb{R})$ to $\mathcal{B}(M)$ by putting $\Lambda_a(E) = \Lambda_a(E \cap [0,1])$.

[35; Theorem 1.6] enables us to introduce a functional calculus for real observables associated with a Boolean σ-algebra $\mathcal{L}$, which assigns, to any real observables ξ_i, $i = 1, 2, \dots, n$, associated with $\mathcal{L}$ and any Borel function $\phi \colon \mathbb{R}^n \to \mathbb{R}$, a well-defined real observable associated with $\mathcal{L}$, which is called the *ϕ-function* of the observables ξ_i, $i = 1, \dots, n$, and is denoted by $\phi(\xi_1, \xi_2, \dots, \xi_n)$.

Moreover, let ξ_n, ξ be real observables associated with $\mathcal{L}$. According to [31; Definition 6.1.2], we say that the sequence $(\xi_n)_n$ converges to ξ *everywhere* if for every $\varepsilon > 0$, $\liminf\limits_{n \to \infty} (\xi_n - \xi)(-\varepsilon, \varepsilon) = 1$. [31; Theorem 6.1.3] gives a characterization of everywhere convergence.

The following theorem is proved in [28; Theorem 5.3].

THEOREM 1.7. *Let M be a σ-MV-algebra. For $a \in M$, let Λ_a be the corresponding canonical spectral measure. Then for every $a, b \in M$, the following holds:*

(i) $\Lambda_{a \oplus b} = \Lambda_a \oplus \Lambda_b$,
(ii) $\Lambda_{a \vee b} = \Lambda_a \vee \Lambda_b$,
(iii) $\Lambda_{a \wedge b} = \Lambda_a \wedge \Lambda_b$,
(iv) $\Lambda_{a^*} = (\Lambda_a)^*$,

where the observables on the right-hand side are defined by the functional calculus. Moreover,

(v) *if $(a_i)_i$ is a sequence of elements of M, then $a_i \nearrow a$ implies $\Lambda_{a_i} \to \Lambda_a$ everywhere.*

Notice that if $(\Omega, \mathcal{T}, h)$ is a regular representation of M, then for $a \in M$, $\Lambda_a = h \circ \widehat{a}^{-1}$, where $\widehat{a} \in \mathcal{T}$ is such that $h(\widehat{a}) = a$. Then, according the Varadarajan functional calculus, $\Lambda_a \oplus \Lambda_b = h \circ \big(\min(\widehat{a} + \widehat{b}, 1)\big)^{-1} = \Lambda_{a \oplus b}$, the last equality holds by Theorem 1.7. Similarly, $\Lambda_{a\vee b} = h\circ\max(\widehat{a},\widehat{b})^{-1}$, $\Lambda_{a\wedge b} = h\circ\min(\widehat{a},\widehat{b})^{-1}$, $\Lambda_{a^*} = h\circ(1-\widehat{a})^{-1}$. Moreover, if $a_i \nearrow a$, then $\widehat{a}_i(\omega)\to\widehat{a}(\omega)$ for all ω except on a set N with $h(N)=0$. By [31; Theorem 6.1.3], the latter convergence is equivalent with the everywhere convergence of the observables Λ_{a_i} to the observable Λ_a.

Now assume that $a\mapsto\Lambda_a$ is a one-to-one mapping on a σ-MV-algebra M, where for every $a\in M$, Λ_a is a real observable associated with the Boolean σ-algebra $\mathcal{B}(M)$. Then we can use [35; Theorem 1.4] to find a couple (X,Σ), where X is a nonempty set and Σ is a σ-algebra of subsets of X, and a σ-homomorphism (of Boolean σ-algebras) u from Σ onto $\mathcal{B}(M)$, such that to every Λ_a there is a Σ-measurable function $f_a\colon X\to[0,1]$ with $\Lambda_a = u\circ f_a^{-1}$. The function f_a is unique in the sense that if $g_a\colon X\to[0,1]$ is another measurable function such that $\Lambda_a = u\circ g_a^{-1}$, then $u\big(\{x\in X:\ f_a(x)\neq g_a(x)\}\big)=0$, i.e., $f_a = g_a$ a.e. $[u]$. Denote $\mathcal{T} := \{f_a:\ a\in M\}$, and assume that conditions (i)–(v) of Theorem 1.7 are satisfied. We will prove that $\mathcal{T}$ is a tribe. If $f_a, f_b\in\mathcal{T}$, then $\Lambda_a = u\circ f_a^{-1}$, $\Lambda_b = u\circ f_b^{-1}$. According to our assumptions, $\Lambda_{a\oplus b} = u\circ\big(\min(f_a+f_b,1)\big)^{-1}$, and hence $\min(f_a+f_b,1)\in\mathcal{T}$. For every $a\in M$, $a\oplus 1 = 1$ implies $f_1 = \min(f_1+f_a,1)$, and this entails $f_1 = 1_X$ a.e. $[u]$. Thus we have $1_X\in\mathcal{T}$, and for every $a\in M$, $\Lambda_{a^*} = u\circ(1_X - f_a)^{-1}$, so that $1_X - f_a\in\mathcal{T}$. If $\{f_{a_i}:\ i\in\mathbb{N}\}\subset\mathcal{T}$, then $g_n := \min\Big(\sum\limits_{i=1}^{n} f_{a_i},1\Big)\in\mathcal{T}$, and $\Lambda_{\bigoplus\limits_{i=1}^{n} a_i} = u\circ g_n^{-1}$ for all n. Then $(g_n)_n$ is an increasing sequence with $g_n\to g$ pointwise, where $g = \sup g_n = \min\Big(\sum\limits_{i=1}^{\infty} f_{a_i},1\Big)$. By [31; Theorem 6.1.3], $u\circ g_n^{-1}\to u\circ g^{-1}$ everywhere, which entails $\Lambda_{\bigoplus\limits_i a_i} = u\circ g^{-1}$, whence $g\in\mathcal{T}$. Let $h\colon\mathcal{T}\to M$ be a mapping defined by $h(f)=a$, where a is such that $\Lambda_a = u\circ f^{-1}$. Since $a\mapsto\Lambda_a$ is one-to-one, this mapping is well defined σ-homomorphism from $\mathcal{T}$ onto M. Therefore the triple $(X,\mathcal{T},h)$ is a representation of M. If $f,g\in\mathcal{T}$ are such that $h(f)=h(g)=a$, then $\Lambda_a = u\circ f^{-1} = u\circ g^{-1}$, which implies that $f=g$ a.e. $[u]$. In particular, $h(f)=0$ implies $f=0_X$ a.e. $[u]$. Let $h\big(\chi_{N(f)}\big)=a$, $a\in M$. Then $\Lambda_a = u\circ\chi_{N(f)}^{-1}$, $\Lambda_a\{0\} = u\big(N(f)^c\big)=1$, so that $a=0$, and hence $h\big(\chi_{N(f)}\big)=0$. Inverting the direction of the reasonings, we obtain that $h\big(\chi_{N(f)}\big)=0$ iff $h(f)=0$, hence the representation is regular. As a consequence, we obtain the following statement.

THEOREM 1.8. *Let M be a σ-MV-algebra, and let $a\mapsto\Lambda_a$ be a spectral representation of M, satisfying conditions* (i)–(v) *of Theorem 1.7. Then $a\to\Lambda_a$ is induced by a regular representation of M.*

2. σ-Complete lattice effect algebras

In this section, we apply the theory of spectral representations for σ-MV-algebras to σ-complete lattice ordered effect algebras. The main tool used here is the result, that every lattice effect algebra can be covered by blocks, that is, maximal pairwise compatible sets, which form MV-(effect) algebras [33].

According to [24], the compatibility relation in a lattice effect algebra E satisfies the following property: If $a_i \in E$, $i = 1, 2, \dots$, are such that $a_i \leftrightarrow b$ for all i, and $a = \bigvee_i a_i$ exists in E, then $a \leftrightarrow b$, and $a \wedge b = \bigvee_i a_i \wedge b$.

Using the above property, we can prove that if E is a σ-complete lattice effect algebra, then every block B of E is a σ-MV-algebra. Moreover, the set of all sharp elements $\mathrm{Sh}(E)$ is an orthomodular σ-lattice. Indeed, let $a_i \nearrow a$, where a_i, $i \in \mathbb{N}$, are sharp. Then $a \wedge a' = \left(\bigvee_j a_j\right) \wedge \left(\bigwedge_i a_i'\right) = \bigvee_j \left(\bigwedge_i a_j \wedge a_i'\right) = 0$, hence a is sharp.

An *observable* on an effect algebra E is defined as a mapping $\xi \colon \Sigma \to E$, where X is a nonempty set and Σ is a σ-algebra of subsets of X, such that

(O1) $\xi(X) = 1$,
(O2) for every sequence $(A_i)_{i\in\mathbb{N}}$, of disjoint elements of Σ, $\xi\left(\bigcup A_i\right) = \bigoplus_i \xi(A_i)$.

If $(X, \Sigma) \subseteq \left(\mathbb{R}, \mathcal{B}(\mathbb{R})\right)$, we say that the observable ξ is a *real* observable. Denote by $\mathcal{R}(\xi) := \left\{\xi(A) : A \in \Sigma\right\}$ the range of an observable ξ. We say that the observable ξ is *sharp* if its range consists of sharp elements, i.e., $\mathcal{R}(\xi) \subseteq \mathrm{Sh}(E)$. Evidently, a sharp observable on E can be considered as an observable on the orthomodular σ-lattice $\mathrm{Sh}(E)$, and all results about observables on orthomodular lattices (see e.g. [31], [35]) can be applied. In particular, the range of a sharp observable is a Boolean σ-algebra and the observable is a σ-homomorphism of Boolean σ-algebras. Moreover, if $(X, \Sigma) \subseteq \left(\mathbb{R}, \mathcal{B}(\mathbb{R})\right)$, $\xi \colon \Sigma \to E$ is a sharp real observable and m is a σ-additive state on E, the mapping $m \circ \xi$ is a probability measure on $\Sigma \subseteq \mathcal{B}(\mathbb{R})$ and we may define the *expectation* $m(\xi)$ of the observable ξ in the state m as follows:

$$m(\xi) := \int_X \lambda\, m \circ \xi(\mathrm{d}\lambda) \tag{3}$$

if the integral exists.

Using the results from [28], the following statement can be proved.

THEOREM 2.1. *Let E be a σ-complete lattice effect algebra. Then for every $a \in E$ there is a σ-homomorphism $\Lambda_a : \mathcal{B}([0,1]) \to \mathrm{Sh}(E)$ such that for every σ-additive state m on E,*

$$m(a) = \int_0^1 \lambda\, m(\Lambda_a(\mathrm{d}\lambda)) . \tag{4}$$

Proof. Every $a \in E$ is contained in a block B of E, which is a σ-MV-algebra, and the Boolean σ-algebra $\mathcal{B}(B)$ of idempotents in B equals $B \cap \mathrm{Sh}(E) \subset \mathrm{Sh}(E)$. By Theorem 1.2, there is a σ-homomorphism $\Lambda_a^B : \mathcal{B}([0,1]) \to \mathrm{Sh}(E) \cap B$, which satisfies (4). □

The result of the next theorem can be interpreted as follows: every σ-additive state m on E is obtained by "smearing" of a σ-additive state on $\mathrm{Sh}(E)$. This extends the results of [34].

THEOREM 2.2. *Let E be a σ-complete lattice effect algebra. Then every σ-additive state on the orthomodular σ-lattice $\mathrm{Sh}(E)$ of all sharp elements in E can be uniquely extended to a σ-additive state on the whole E. In addition, a pure state extends to a pure state.*

Proof. Let m be a σ-additive state on $\mathrm{Sh}(E)$. For every block B, the restriction of m to $\mathrm{Sh}(E) \cap B$ is a σ-additive state on the Boolean σ-algebra of all idempotents in B. Using Theorem 1.3, the map $\tilde{m}(a) = \int_0^1 \lambda\, m(\Lambda_a^B(\mathrm{d}\lambda))$, where $\Lambda^B : a \mapsto \Lambda_a^B$ is the canonical spectral representation of B, uniquely extends m to the block B. Let $a \in E$ belong to two blocks B_1 and B_2 of E, and let $\tilde{m}_1$ and $\tilde{m}_2$ be the extensions of m to B_1 and B_2, respectively. Let $B = B_1 \cap B_2$. The restriction of m to $\mathrm{Sh}(E) \cap B$ is a σ-additive state on $\mathcal{B}(B)$, and by Theorem 1.3, m can be uniquely extended to a σ-additive state on B, hence $\tilde{m}_1(a) = \tilde{m}_2(a)$.

Therefore the map $a \mapsto \tilde{m}(a)$ is well defined. Moreover, the restriction of $\tilde{m}$ to every block is a σ-additive mapping. Hence $\tilde{m}$ defines a σ-additive state on E which extends m.

Let m be a pure σ-additive state on $\mathrm{Sh}(E)$. Assume that its extension on E is of the form $\tilde{m} = \alpha m_1 + (1-\alpha) m_2$, where $0 < \alpha < 1$ and m_1, m_1 are states on E. Then $m = \tilde{m}/\mathrm{Sh}(E) = \alpha m_1/\mathrm{Sh}(E) + (1-\alpha) m_2/\mathrm{Sh}(E)$ implies $m_1/\mathrm{Sh}(E) = m_2/\mathrm{Sh}(E) = m$. Since the extensions from $\mathrm{Sh}(E)$ to E are unique, we have $m_1 = m_2 = \tilde{m}$. Hence $\tilde{m}$ is pure on E. □

In the next theorem we show that canonical spectral measures of an element of a lattice σ-effect algebra are the same in all blocks to which this element belongs. First we prove a lemma.

LEMMA 2.3. *Let M be a σ-MV-algebra, and N be a sub-σ-MV-algebra of M. Let $(X, \mathcal{T}, h)$ be regular representation of M. Then $(X, \mathcal{T}_N, h_N)$, where $\mathcal{T}_N = \{f \in \mathcal{T} : h(f) \in N\}$ and $h_N = h|_{\mathcal{T}_N}$, is a regular representation of N.*

Proof. Clearly, $\mathcal{T}_N$ is a subtribe of the tribe $\mathcal{T}$, and h_N maps $\mathcal{T}_N$ onto N. By the Butnariu and Klement theorem, each $f \in \mathcal{T}_N$ is measurable with respect to the σ-algebra $\mathcal{B}(\mathcal{T}_N) = \mathcal{B}(\mathcal{T}) \cap \mathcal{T}_N$. If $f \in \mathcal{T}_N$, then its support $N(f) \in \mathcal{B}(\mathcal{T}_N)$, and regularity of $(X, \mathcal{T}, h)$ implies that $h_N(f) = h(f) = 0$ iff $h_N(N(f)) = h(N(f)) = 0$, hence $(X, \mathcal{T}_N, h_N)$ is a regular representation of N. □

THEOREM 2.4. *Let E be a σ-complete lattice effect algebra. Let $a \in E$ belong to two blocks M_1 and M_2 of E. Denote Λ_a^1 and Λ_a^2 the canonical spectral measure of a taken in M_1 and M_2, respectively. Then $\Lambda_a^1 = \Lambda_a^2$.*

Proof. Denote $M := M_1 \cap M_2$. Then $a \in M$, and M is a σ-MV-algebra. Let Λ_a denote the canonical spectral measure of a in M. Let $(\Omega_1, \mathcal{T}_1, h_1)$ be the canonical representation of M_1 and $(\Omega_2, \mathcal{T}_2, h_2)$ be the canonical representation of M_2. Applying Lemma 2.3, we construct regular representations of M as subrepresentations of $(\Omega_1, \mathcal{T}_1, h_1)$ and $(\Omega_2, \mathcal{T}_2, h_2)$, and from this we can derive, using Theorem 1.4 that the spectral measures of Λ_a^1 and Λ_a^2 are both equal to the canonical spectral measure of a in M. □

Denote, for every $a \in E$,

$$a^\sim := \Lambda_a(\{0\}).$$

Then $a^\sim \in \mathrm{Sh}(E)$, and $a^\sim$ belongs to every block B of E to which a belongs.

THEOREM 2.5. *For every a in E, $a^\sim$ satisfies the following conditions:*

(R) $p \in \mathrm{Sh}(E),\ p \leftrightarrow a,\ p \wedge a = 0 \iff p \le a^\sim$.

Proof. We recall that every block B of E is a subalgebra and a sublogic of E, therefore lattice operations taken in B coincide with those in E. If the left-hand side of (R) is satisfied, there is a block B of E such that $a \in B$ and $p \in \mathcal{B}(B) = \mathrm{Sh}(E) \cap B$. Let $(X, \mathcal{T}, h)$ be a regular representation of B. Let $f \in \mathcal{T}$, $\chi_A \in \mathcal{T}$ be such that $h(f) = a$ and $h(\chi_A) = p$. Then $0 = a \wedge p = h(f \wedge \chi_A)$ implies, by regularity, that $h(N(f \wedge \chi_A)) = 0$, hence

$$\begin{aligned} 0 &= h\big(\{x \in X : f(x) \cdot \chi_A(x) \neq 0\}\big) = h\big(\{x \in X : f(x) \neq 0\} \cap A\big) \\ &= h\big(N(f) \cap A\big). \end{aligned}$$

Now $A = N(f) \cap A \cup N(f)^c \cap A$ implies that

$$p = h(A) = h\big((N(f)^c \cap A\big) = h \circ f^{-1}\big(\{0\}\big) \wedge h(\chi_A) = a^\sim \wedge p\,,$$

i.e., $p \le a$.

On the other hand, observe that $a^\sim \wedge a = h\big(f \cap f^{-1}(\{0\})\big)$. But $N\big(f \cap f^{-1}(\{0\})\big) = \big\{x \in X : f(x) \cdot \chi_{N(f)^c} \neq 0\big\} = \emptyset$, and by regularity, $a^\sim \wedge a = 0$. Therefore $a \wedge p = 0$ whenever $p \le a^\sim$. □

That is, $a^\sim$ is the greatest sharp element p compatible with a (possibly zero) such that $p \wedge a = 0$. In analogy with [14], we will call the mapping $a \mapsto a^\sim$ from E to $\mathrm{Sh}(E)$ the *Rickart mapping*. Notice that if $p \in \mathrm{Sh}(E)$, then $p^\sim = p'$.

THEOREM 2.6. *For every $a \in E$, $a^{\sim\sim}$ is the smallest sharp element that dominates a. Moreover, $a^{\sim\sim}$ is contained in every block that contains a.*

Proof. Let $p \in \mathrm{Sh}(E)$ be such that $a \le p$. Then $a \leftrightarrow p$ and $a \wedge p' \le p \wedge p' = 0$, hence $p' \le a^\sim$. This yields that $a^{\sim\prime} = a^{\sim\sim} \le p$. Hence $a^{\sim\sim} = \bigwedge\big\{p \in \mathrm{Sh}(E) : a \le p\big\}$, which yields that $a \le a^{\sim\sim}$. The rest of the proof follows from $a^\sim = \Lambda_a\big(\{0\}\big)$ and $a^{\sim\sim} = a^{\sim\prime}$. □

Notice that a similar result for complete lattice effect algebras was recently obtained in [21], using different methods.

According to [19], an effect algebra E is *sharply dominating* if for every $a \in E$ there exists the smallest sharp element p such that $a \le p$. If, moreover, $p \wedge a$ exist for every $a \in E$ and every sharp element $p \in E$, then E is called *S-dominating*. As a consequence of Theorem 2.6 we have:

COROLLARY 2.7. *Every σ-complete lattice effect algebra is S-dominating.*

In what follows, we will investigate relations between commutators of elements a, b in E and commutators of the corresponding observables Λ_a, Λ_b in the orthomodular σ-lattice $\mathrm{Sh}(E)$.

Let L be an orthomodular lattice. Then $a, b \in L$ are compatible ($a \leftrightarrow b$) if $a = (a \wedge b) \vee (a \wedge b')$, equivalently, if it holds $\mathrm{com}(a,b) = 0$, where

$$\mathrm{com}(a,b) := (a \vee b) \wedge (a' \vee b) \wedge (a \vee b') \wedge (a' \vee b')\,.$$

The element $\mathrm{com}(a,b)$ is called the *commutator* of the elements a, b. The commutator $c := \mathrm{com}(a,b)$ has the following properties:

(i) $a \leftrightarrow c$, $b \leftrightarrow c$,
(ii) $a \wedge c' \leftrightarrow b \wedge c'$,
(iii) c is the smallest element in L with properties (i) and (ii) ([31; §5.1]).

If $\xi\colon \mathcal{B}(\mathbb{R}) \to L$, $\eta\colon \mathcal{B}(\mathbb{R}) \to L$ are real observables on an orthomodular σ-lattice L (that is, σ-homomorphisms from Borel subsets of the real line to L), then the element

$$\operatorname{com}(\xi,\eta) := \bigvee \big\{ \operatorname{com}\big(x(E), y(F)\big) : \ E, F \in \mathcal{B}(\mathbb{R}) \big\}$$

exists and is called the *commutator* of the observables ξ, η. Put $c = \operatorname{com}(\xi,\eta)$, then we have

(i) $c \leftrightarrow \xi(E), \eta(E)$ for all $E \in \mathcal{B}(\mathbb{R})$,
(ii) $\xi(E) \wedge c' \leftrightarrow \eta(F) \wedge c'$ for all $E, F \in \mathcal{B}(\mathbb{R})$,
(iii) c is the smallest element with properties (i) and (ii) ([31; §5.1]).

Now let E be a (σ-complete) lattice effect algebra. An analogue of the commutator was introduced in [22]: for $a, b \in E$, define

$$\operatorname{com}_e(a,b) := \big((a \vee b) \mathbin{\dot -} b\big)\Delta\big(a \mathbin{\dot -} (a \wedge b)\big)\,,$$

where Δ is the symmetric difference, $e\Delta f = (e \vee f) \mathbin{\dot -} (e \wedge f)$, $e, f \in E$. Put $d := \operatorname{com}_e(a,b)$. Then we have

(i) $a \leftrightarrow b$ iff $d = 0$,
(ii) $a \leftrightarrow d$, $b \leftrightarrow d$,
(iii) for any $e \in E$, $e \leftrightarrow a$, $e \leftrightarrow b$ implies $e \leftrightarrow d$ ([23], [30]).

Moreover, if $a, b \in \operatorname{Sh}(E)$, then $\operatorname{com}_e(a,b) = \operatorname{com}(a,b)$ ([22]). Consequently, two sharp elements a, b are compatible in E iff a, b are compatible in $\operatorname{Sh}(E)$ (as elements of an orthomodular lattice).

THEOREM 2.8. *For any $a, b \in E$, if $d = \operatorname{com}_e(a,b)$ and $c = \operatorname{com}(\Lambda_a, \Lambda_b)$, then $d^{\sim} = c'$.*

P r o o f. Fix elements $a, b \in E$, and let Λ_a, Λ_b be the corresponding canonical observables. Put $c = \operatorname{com}(\Lambda_a, \Lambda_b)$, $d = \operatorname{com}_e(a,b)$. Then $c \in \operatorname{Sh}(E)$, and there is a block B_1 which contains the range $\mathcal{R}(\Lambda_a) = \{\Lambda_a(E) : \ E \in \mathcal{B}(\mathbb{R})\}$, and the element c. Similarly, there is a block B_2 that contains $\mathcal{R}(\Lambda_b)$ and c. It then follows that $a \in B_1$, $b \in B_2$. Assume that $(X, \mathcal{T}, h)$ is a regular representation of B_1. Let $f, \chi_C \in \mathcal{T}$ be such that $a = h(f)$, $c' = h(\chi_C)$. Then $\Lambda_{a\wedge c'}(E) = h \circ (f \wedge \chi_C)^{-1}(E)$. Since $(f \wedge \chi_C)^{-1}(E) = \{x : \ f(x) \in E\} \cap C$ if $0 \notin E$, and $(f \wedge \chi_C)^{-1}(E) = \{x : \ f(x) \in E\} \cap C \cup C^c$ if $0 \in E$, we have $\Lambda_{a\wedge c'}(E) = \Lambda_a(E) \wedge c'$ if $0 \notin E$, and $\Lambda_{a\wedge c'}(E) = \big(\Lambda_a(E) \wedge c'\big) \vee c$ if $0 \in E$. Analogous relations between $\Lambda_{b\wedge c'}(F)$ and $\big(\Lambda_b(F)\big) \wedge c'$, $F \in \mathcal{B}(\mathbb{R})$, are obtained, taking a regular representation of B_2.

From the properties of the commutator c we have that for all $E, F \in \mathcal{B}(\mathbb{R})$, $\Lambda_a(E) \wedge c' \leftrightarrow \Lambda_b(F) \wedge c'$. By the relations proved in the previous paragraph, this implies that $\Lambda_{a\wedge c'}(E) \leftrightarrow \Lambda_{b\wedge c'}(F)$ for all E, F, hence $\Lambda_{a\wedge c'} \leftrightarrow \Lambda_{b\wedge c'}$, and this in turn implies that $a \wedge c' \leftrightarrow b \wedge c'$. Therefore $\operatorname{com}_e(a \wedge c', b \wedge c') = 0$.

Define $C(c) := \{e \in E: \ e \leftrightarrow c\}$. From the properties of compatibility ([24], [9; Theorem 1.10.17, Theorem 1.10.15])

(i) $0, 1 \in C(c)$,
(ii) $e \in C(c) \implies e' \in C(c)$,
(iii) $e, f \in C(c) \implies e \vee f, e \wedge f \in C(c)$,
(iv) $e, f \in C(c)$, $e \perp f \implies e \dot{+} f \in C(c)$,
(v) if M is an at most countable subset of $C(c)$, then $\bigvee M \in C(c)$, and moreover, $(\bigvee M) \wedge c = \bigvee\{m \wedge c: \ m \in M\}$.

It follows that $C(c)$ is a sub-σ-effect algebra and sub-σ-lattice of E. The element c is sharp and is compatible with all elements in $C(c)$, hence c is central in $C(c)$. This entails that for $e, f \in C(c)$, $e \perp f$, we have $(e \dot{+} f) \wedge c = (e \wedge c) \dot{+} (f \wedge c)$ ([24]).

Now let us return to the fixed elements $a, b \in E$. We have $a, b \in C(c)$, and since c' is central in $C(c)$, we obtain, similarly as in [30], that $\mathrm{com}_e(a, b) \wedge c' = \mathrm{com}_e(a \wedge c', b \wedge c')$ in $C(c)$, but since the operations in $C(c)$ are inherited from those in E, the latter equality holds in E. So we obtained that $\mathrm{com}_e(a, b) \wedge c' = d \wedge c' = 0$. From $a, b \in C(c)$ we obtain $d \in C(c)$. So we have $c' \in \mathrm{Sh}(E)$, $c' \leftrightarrow d$ and $d \wedge c' = 0$, which implies that $c' \leq d^{\sim}$.

To obtain the converse inequality, recall that $d^{\sim}$ is compatible with all with which d is compatible, and hence $d^{\sim} \leftrightarrow a, b$. This implies that $d^{\sim} \leftrightarrow \Lambda_a(E), \Lambda_b(F)$ for all $E, F \in \mathcal{B}(\mathbb{R})$. From $d^{\sim} \in \mathrm{Sh}(E)$, and $a, b \in C(d^{\sim})$ we obtain that $0 = d \wedge d^{\sim} = \mathrm{com}_e(a \wedge d^{\sim}, b \wedge d^{\sim})$, which implies that $a \wedge d^{\sim} \leftrightarrow b \wedge d^{\sim}$, and consequently, $\Lambda_{a \wedge d^{\sim}}$ and $\Lambda_{b \wedge d^{\sim}}$ are compatible observables. This yields that $\Lambda(E) \wedge d^{\sim} \leftrightarrow \Lambda_b(F) \wedge d^{\sim}$ for all $E, F \in \mathcal{B}(\mathbb{R})$, where we used again the relation between $\Lambda_a(E) \wedge d^{\sim}$ and $\Lambda_{a \wedge d^{\sim}}(E)$. Properties of the commutator c then imply that $d^{\sim} \leq c'$.

This concludes the proof that $d^{\sim} = c'$. $\square$

Notice that in the proof of Theorem 2.8 we can replace $\mathrm{com}_e(a, b)$ by any function $d(a, b)\colon E \times E \to E$ such that $d(a, b) = 0$ iff $a \leftrightarrow b$ (see examples in [30]), and we obtain that $d(a, b)^{\sim} = c'$.

We note that in [19], compressible effect algebras were introduced and studied. In a σ-complete lattice effect algebra E, every block with the compressions $J_p(a) = p \wedge a$, $p \in \mathrm{Sh}(E) \cap B$ is a compressible effect algebra. The author does not know if compressions can be defined on the whole E.

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