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CONVERGENCES ON LATTICE ORDERED GROUPS WITH A FINITE NUMBER OF DISJOINT ELEMENTS

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ABSTRACT. For a lattice ordered group G we denote by $\operatorname{Conv} G$ the system of all sequential convergences on G satisfying the Urysohn's axiom. Let $\mathcal{F}$ be the class of all lattice ordered groups with a finite number of disjoint elements. In this paper we prove that if $G \in \mathcal{F}$, then $\operatorname{Conv} G$ is a finite Boolean algebra.

Introduction

In the papers [3]–[11] there has been investigated the system $\operatorname{Conv} G$ of all sequential convergences on a lattice ordered group G satisfying Urysohn's axiom. In some of these papers it was assumed that G is abelian.

The case when Urysohn's axiom was omitted has been dealt with in [12], [13], [14]; the corresponding system was denoted by $\operatorname{conv} G$. In [12] and [13], the commutativity of the group operation was assumed.

The system $\operatorname{Conv} G$ is partially ordered in a natural way. In general, $\operatorname{Conv} G$ fails to be a lattice. Namely, if α and β are elements of $\operatorname{Conv} G$, then $\alpha \vee \beta$ need not exist in $\operatorname{Conv} G$.

The class of all lattice ordered groups with a finite number of disjoint elements will be denoted by $\mathcal{F}$. Such lattice ordered groups have been studied in [1].

Some results concerning the system $\operatorname{conv} G$ for G belonging to $\mathcal{F}$ have been proved in [12].

In the present paper we show that if $G \in \mathcal{F}$, then $\operatorname{Conv} G$ is a finite Boolean algebra.

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In the particular case of abelian lattice ordered groups we prove the following stronger result:

(A) Let G be an abelian lattice ordered group. Then the following conditions are equivalent:

- (i) $\operatorname{Conv} G$ is a generalized Boolean algebra.
- (ii) $\operatorname{Conv} G$ is a Boolean algebra.
- (iii) $\operatorname{Conv} G$ is a finite Boolean algebra.
- (iv) $G \in \mathcal{F}$.

1. Preliminaries

The group operation in a lattice ordered group will be denoted additively, though it is not assumed to be commutative.

We start by recalling some definitions (cf. [12] and [14]). Let G be a lattice ordered group. Let $g \in G$ and $(g_n) \in G^{\mathbb{N}}$. If $g_n = g$ for each $n \in \mathbb{N}$, then we write $(g_n) = \operatorname{const} g$. For $(h_n) \in (G^{\mathbb{N}})^+$ we set $(h_n) \sim (g_n)$ if there is $m \in \mathbb{N}$ such that $h_n = g_n$ for each $n \in \mathbb{N}$ with $n \geqq m$.

Let α be a subset of the lattice ordered semigroup $(G^{\mathbb{N}})^+$. Consider the following conditions for the set α:

(I) If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.

(II) Let $(g_n) \in (G^{\mathbb{N}})^+$. If each subsequence of (g_n) has a subsequence belonging to α, then (g_n) belongs to α.

(II') Let $(g_n) \in \alpha$ and $(h_n) \in (G^{\mathbb{N}})^+$. If $(h_n) \sim (g_n)$, then $(h_n) \in \alpha$.

(III) Let $g \in G$. Then $\operatorname{const} g$ belongs to α if and only if $g = 0$.

The set α is called G-*normal* if for each $(x_n) \in \alpha$ and each $g \in G$ the relation $(-g + x_n + g) \in \alpha$ is valid. (Cf. [6].)

The system of all G-normal convex subsemigroups of the lattice ordered semigroup $(G^{\mathbb{N}})^+$ which satisfy the conditions (I), (II) and (III) (or the conditions (I), (II') and (III)) will be denoted by $\operatorname{Conv} G$ (or by $\operatorname{conv} G$, respectively).

Both $\operatorname{Conv} G$ and $\operatorname{conv} G$ are partially ordered by the set-theoretical inclusion.

For $(g_n) \in G^{\mathbb{N}}$, $g \in G$ and $\alpha \in \operatorname{Conv} G$ we put $g_n \to_\alpha g$ if and only if $(|g_n - g|) \in \alpha$.

Let $\alpha(d)$ be the set of all $(g_n) \in (G^{\mathbb{N}})^+$ such that $(g_n) \sim \operatorname{const} 0$. Then $\alpha(d)$ is the least element of both $\operatorname{Conv} G$ and $\operatorname{conv} G$.

Further, let $\alpha(o)$ be the set of all $(g_n) \in (G^{\mathbb{N}})^+$ having the property that there exists $(h_n) \in (G^{\mathbb{N}})^+$ such that

(i) $h_{n+1} \leqq h_n$ is valid for each $n \in \mathbb{N}$,
(ii) $\bigwedge_{n\in\mathbb{N}} h_n = 0$,
(iii) there is $m \in \mathbb{N}$ such that $h_n \geqq g_n$ for each $n \in \mathbb{N}$ with $n \geqq m$.

The set $\alpha(o)$ will be called the *o-convergence* in G. We have $\alpha(o) \in \operatorname{conv} G$; but, in general, $\alpha(o)$ need not belong to $\operatorname{Conv} G$. If G is linearly ordered, then $\alpha(o) \in \operatorname{Conv} G$.

2. The case $G \in \mathcal{F}$

A lattice ordered group G is said to be a *lexico extension* of its ℓ-subgroup H if, whenever $0 < g \in G \setminus H$, then $g > h$ for each $h \in H$.

It is well known that if G belongs to $\mathcal{F}$, then it can be built up from a finite number of linearly ordered groups by forming direct products and lexico extensions. Moreover, by each step applying the construction of lexicographic extension, the corresponding ℓ-subgroup H fails to be linearly ordered. (Cf. [1], [2].)

For a lattice ordered group G we denote by $\mathcal{L}(G)$ the system of all convex ℓ-subgroups X of G such that

(i) X is linearly ordered and $X \neq \{0\}$;
(ii) whenever Y is a convex linearly ordered subgroup of G with $X \subseteq Y$, then $X = Y$.

2.1. Lemma. (Cf. [15].) *Let $X_1, X_2 \in \mathcal{L}(G)$, $X_1 \neq X_2$. Then $X_1 \cap X_2 = \{0\}$.*

From the definition of $\mathcal{L}(G)$ we immediately obtain:

2.2. Lemma. *Let $X \in \mathcal{L}(G)$ and $g \in G$. Then $-g + X + g \in \mathcal{L}(G)$.*

2.3. Lemma. (Cf. [6].) *Let G be a linearly ordered group and $\alpha \in \operatorname{Conv} G$. Then either $\alpha = \alpha(d)$ or $\alpha = \alpha(o)$.*

The following assertion is an easy consequence of the definition of $\operatorname{Conv} G$.

2.4. Lemma. *Let H be a convex ℓ-subgroup of G and let $\alpha \in \operatorname{Conv} G$. Let β be the set of all $(h_n) \in (H^{\mathbb{N}})^+$ such that there exists $(g_n) \in \alpha$ with $(h_n) \sim (g_n)$. Then $\beta \in \operatorname{Conv} H$.*

Now suppose that G belongs to $\mathcal{F}$. Then from the structure of G mentioned at the beginning of the present section we conclude that the set $\mathcal{L}(G)$ is nonempty and finite; namely, $\mathcal{L}(G)$ is the system of all linearly ordered groups by means of which G is constructed in the above described way. Thus we can put

$$\mathcal{L}(G) = \{X_1, X_2, \dots, X_m\}.$$

For $X_i \in \mathcal{L}(G)$ we denote by $\alpha_i(d)$ the least element of $\operatorname{Conv} X_i$ and by $\alpha_i(o)$ the corresponding o-convergence on X_i. We put

$$\mathcal{L}_0(G) = \{X_i \in \mathcal{L}(G) : \ \alpha_i(d) \neq \alpha_i(o)\}.$$

Of course, it may happen that $\mathcal{L}_0(G)$ is an empty set.

We denote by $B(G)$ the Boolean algebra of all subsets of $\mathcal{L}_0(G)$. Further, let $B_0(G)$ be the collection of all $S \in B(G)$ such that

(i) $S \subseteq \mathcal{L}_0(G)$,

(ii) if $X_i \in S$ and $g \in G$, then $-g + X_i + g \in S$.

Then we clearly have:

2.5. LEMMA. *$B_0(G)$ is a subalgebra of the Boolean algebra $B(G)$.*

Our aim is to show that the partially ordered set $\operatorname{Conv} G$ is isomorphic to $B_0(G)$. We need some auxiliary results.

Let $\alpha \in \operatorname{Conv} G$ and $X_i \in \mathcal{L}_0(G)$. Hence X_i is a convex ℓ-subgroup of G. Thus we can apply 2.4 with X_i instead of H. We write α_i instead of β, where β is as in 2.4. We put

$$f_1(\alpha) = \{X_i \in \mathcal{L}_0(G) : \ \alpha_i(o) \subseteq \alpha\}.$$

2.6. LEMMA. *For each $\alpha \in \operatorname{Conv} G$, $f_1(\alpha)$ belongs to $B_0(G)$.*

Proof. This is the consequence of the fact that α is a normal subset of $(G^{\mathbb{N}})^+$. □

Next, let S be an element of $B_0(G)$. If $S = \emptyset$, then we put $f_2(S) = \alpha(d)$. Further, assume that S is nonempty; for fixing the notation let us set

$$S = \{X_1, X_2, \dots, X_k\}.$$

We denote by S^1 the set of all elements $x \in G^+$ which can be represented in the form

$$x = x_1 + x_2 + \dots + x_k$$

with $x_i \in X_i$ $(i = 1, 2, \dots, k)$.

In view of 2.1, whenever $i(1)$ and $i(2)$ are distinct elements of the set $\{1, 2, \dots, k\}$, then $x_{i(1)} \wedge x_{i(2)} = 0$; therefore

$$x_{i(1)} + x_{i(2)} = x_{i(1)} \vee x_{i(2)} = x_{i(2)} + x_{i(1)}.$$

Thus we have

$$x = x_1 \vee x_2 \vee \dots \vee x_k.$$

This easily yields that if y is another element of S^1 with

$$y = y_1 + y_2 + \dots + y_k$$

(under analogous notation as above), then

$$x + y = (x_1 + y_1) + (x_2 + y_2) + \dots + (x_k + y_k),$$
$$x \circ y = (x_1 \circ y_1) + (x_2 \circ y_2) + \dots + (x_k \circ y_k) \qquad \text{for} \quad \circ \in \{\wedge, \vee\}.$$

Moreover, if $x \in S^1$, $0 \leqq g \in G$, and $g \leqq x$, then $g \in S^1$. Hence we have:

2.7. Lemma. *S^1 is a subsemigroup of the semigroup G^+. Further, S^1 is a sublattice of the lattice G^+.*

In view of 2.2 and of the definition of the set S, the relation

$$-g + S^1 + g = S^1 \tag{1}$$

is valid for each $g \in G$.

We define $f_{20}(S)$ to be the set of all $(g_n) \in (G^{\mathbb{N}})^+$ such that

(i) $g_n \in S^1$ for each $n \in \mathbb{N}$ (hence, under analogous notation as above, g_n has a representation

$$g_n = g_{n1} + g_{n2} + \dots + g_{nk};$$

(ii) if $i \in \{1, 2, \dots, k\}$, then $(g_{ni}) \in \alpha_i(o)$.

Further, let $f_2(S)$ be the set of all $(h_n) \in (G^{\mathbb{N}})^+$ such that there exists $(g_n) \in f_{20}(S)$ with $(h_n) \sim (g_n)$.

In view of 2.7 and (1) we conclude that $f_2(S)$ is a B-normal convex subsemigroup of the lattice ordered group $(G^{\mathbb{N}})^+$ satisfying the conditions (I), (II) and (III). In other words, we have:

2.8. Lemma. $f_2(S) \in \operatorname{Conv} G$.

The definition of $f_2(S)$ immediately yields:

2.9. Lemma. *Let S and S' be elements of $B_0(G)$ such that $S \subseteq S'$. Then $f_2(S) \subseteq f_2(S')$.*

2.10. Lemma. *Let S and S' be elements of $B_0(G)$ such that $S \subset S'$. Then $f_2(S) \subset f_2(S')$.*

Proof. There exists $X_i \in S' \setminus S$. Further, there exists $(g_n) \in (X_i^{\mathbb{N}})^+$ such that $(g_n) \in \alpha_i(o)$ and $(g_n) \notin \alpha_i(d)$. Then we have

$$(g_n) \in f_2(S'), \qquad (g_n) \notin f_2(S).$$

Thus in view of 2.9, the relation $f_2(S) \subset f_2(S')$ holds. □

2.11. Lemma. *Let $\alpha \in \operatorname{Conv} G$ and let $S = f_1(\alpha)$. Then $f_2(S) \subseteq \alpha$.*

Proof. Let $(h_n) \in f_2(S)$. Then there exists $(g_n) \in f_{20}(S)$ such that $(g_n) \sim (h_n)$. For elements g_n we apply the same notation as above. For each $i \in \{1, 2, \dots, k\}$ we have $(g_{ni}) \in \alpha_i(o)$. Thus in view of the definition of $f_1(\alpha)$ we get $(g_{ni}) \in \alpha$. Since α is a subsemigroup of $(G^{\mathbb{N}})^+$ we infer that (g_n) belongs to α. Therefore (h_n) belongs to (α) as well. □

2.12. Lemma. *Let $0 \leqq g \in G$, $i \in \{1, 2, \dots, m\}$ and suppose that g does not exceed all elements of X_i. Then the set $\{t \in X_i : t \leqq g\}$ has the greatest element.*

Proof. Cf. [15; p. 56]. □

Under the assumptions as in 2.12, the greatest element of the set under consideration will be denoted by g^i.

2.13. Lemma. *Let $0 \leqq g \in G$ and suppose that for each $i \in \{1, 2, \dots, m\}$ there exists $x^i \in X_i$ such that $x^i \nleqq g$. Then*

$$g = g^1 + g^2 + \dots + g^m .$$

Proof. In view of 2.1 we conclude that the set $\{g^1, g^2, \dots, g^m\}$ is disjoint, whence

$$g^1 + g^2 + \dots + g^m = g^1 \vee g^2 \vee \dots \vee g^m .$$

Denote $g^1 \vee g^2 \vee \dots \vee g^m = g'$. Clearly $g' \leqq g$.

By way of contradiction, assume that $g' < g$. Hence there is $0 < h \in G$ with $g' + h = g$.

From the structure of G we conclude that $\{x^1, x^2, \dots, x^m\}$ is a maximal disjoint subset of G. Hence there is $i \in \{1, 2, \dots, m\}$ such that $h \wedge x^i > 0$. We get $h \wedge x^i \in X_i$, thus $g^i + (h \wedge x^i) \in X_i$ and

$$g^i < g^i + (h \wedge x^i) \leqq g^i + h = g ;$$

in view of the definition of g^i we arrived at a contradiction. □

Again, let $\alpha \in \operatorname{Conv} G$ and let $(g_n) \in \alpha$. Let $\mathcal{L}(G)$ be as above and let $i \in \{1, 2, \dots, m\}$. There exists $x^i \in X_i$ with $x^i > 0$. If $x^i \leqq g_n$ for infinitely many n, then in view of (I) and according to the convexity of α we would have $\operatorname{const} x^i \in \alpha$, which is a contradiction. Therefore there exists $(g_n^1) \in \alpha$ such that $(g_n^1) \sim (g_n)$ and no g_n^1 exceeds x^i. If we choose $x^i > 0$ for each $i \in \{1, 2, \dots, m\}$, then by induction we conclude that there exists $(g_n^2) \in \alpha$ with $(g_n^2) \sim (g_n)$ such that for each $n \in \mathbb{N}$ and each $i \in \{1, 2, \dots, m\}$ we have

$$x^i \nleqq g_n^2 .$$

Hence in view of 2.13, each g_n^2 can be represented in the form

$$g_n^2 = g_n^{21} + g_n^{22} + \cdots + g_n^{2m}$$

with $0 \leqq g_n^{2i} \in X_i$ for $i \in \{1, 2, \ldots, m\}$.

As above, put $f_1(\alpha) = S$.

2.14. LEMMA. *Suppose that i is an element of the set $\{1, 2, \ldots, m\}$ such that X_i does not belong to S. Then the set*

$$N_1 = \{n \in \mathbb{N} : \ g_n^{2i} \neq 0\}$$

is finite.

Proof. By way of contradiction, assume that the set N_1 is infinite. Hence there exists a subsequence (g_n^3) of (g_n^2) such that, under an analogous notation as above, we have

$$g_n^{3i} > 0 \qquad \text{for each} \quad n \in \mathbb{N}.$$

Since $(g_n^3) \in \alpha$ and $0 < g_n^{3i} \leqq g_n^3$ for each $n \in \mathbb{N}$, we infer that $(g_n^{3i}) \in \alpha$.

Consider the element β of $\operatorname{Conv} X_i$ which is constructed by means of α and by applying 2.4 with X_i instead of H. Then

$$(g_n^{3i}) \in \beta, \qquad (g_n^{3i}) \notin \alpha_i(d),$$

therefore $\beta \neq \alpha_i(d)$, hence in view of 2.3 we obtain $\beta = \alpha_i(o)$. Thus $\alpha_i(o) \subseteq \alpha$, yielding that $X_i \in S$, which is a contradiction. □

By applying 2.14 and the induction we conclude:

2.15. LEMMA. *There exists $(g_n^4) \in (G^{\mathbb{N}})^+$ such that*

(i) $(g_n^4) \sim (g_n^2)$,

(ii) $g_n^{4i} = 0$ *whenever* $X_i \notin S$.

Therefore for each $n \in \mathbb{N}$, g_n^4 has a representation

$$g_n^4 = g_n^{41} + g_n^{42} + \cdots + g_n^{4m}.$$

Hence $(g_n^4) \in f_2(S)$. Since $(g_n) \sim (g_n^4)$, we have also $(g_n) \in f_2(S)$. Summarizing, we obtain $\alpha \subseteq f_2(S)$.

Thus according to 2.11 we get:

2.16. LEMMA. *Let $\alpha \in \operatorname{Conv} G$, $f_1(\alpha) = S$. Then $f_2(S) = \alpha$.*

From the definition of f_2 we easily obtain:

2.17. LEMMA. *Let $S \in B_0(G)$, $f_2(S) = \alpha$. Then $f_1(\alpha) = S$.*

It is obvious that the mapping f_1 is monotone. Hence from 2.9, 2.16 and 2.17 we conclude:

2.18. THEOREM. *The mapping f_1 is an isomorphism of the Boolean algebra $B_0(G)$ onto the partially ordered set $\operatorname{Conv} G$; further, $f_2 = f_1^{-1}$.*

3. Proof of (A)

In the present section we assume that G is an abelian lattice ordered group.

A lattice L is said to be a *generalized Boolean algebra* if it has the least element and if each interval of L is a Boolean algebra.

A subset M of $(G^{\mathbb{N}})^+$ is called *regular* if there is $\alpha \in \operatorname{Conv} G$ such that $M \subseteq \alpha$.

Each interval of the partially ordered set $\operatorname{Conv} G$ is a complete distributive lattice (cf. [3]). Hence if M is regular, then there exists $\beta \in \operatorname{Conv} G$ such that

(i) $M \subseteq \beta$,

(ii) whenever $\beta_1 \in \operatorname{Conv} G$, $M \subseteq \beta_1$, then $\beta \subseteq \beta_1$.

We say that the *convergence β is generated by the set M*.

Let $(x_n) \in (G^{\mathbb{N}})^+$ such that $x_n > 0$ for each $n \in \mathbb{N}$ and $x_{n(1)} \wedge x_{n(2)} = 0$ whenever $n(1)$ and $n(2)$ are distinct elements of $\mathbb{N}$.

The following lemma is a consequence of [4; Theorem 7.3].

3.1. Lemma. *The one-element set $\{(x_n)\}$ is regular.*

Let us denote by α the convergence on G which is generated by the set $\{(x_n)\}$.

For each $n \in \mathbb{N}$ we put $y_n = nx_n$. Then $y_{n(1)} \wedge y_{n(2)} = 0$ if $n(1)$ and $n(2)$ are distinct elements of $\mathbb{N}$. Thus in view of 3.1, the set $\{(y_n)\}$ is regular. The convergence generated by this set will be denoted by β.

3.2. Lemma. *Let M be a regular subset of $(G^{\mathbb{N}})^+$ and let $(z_n) \in (G^{\mathbb{N}})^+$. Let γ be convergence generated by the set M. Then the following conditions are equivalent:*

(i) $(z_n) \in \gamma$.

(ii) *For each subsequence (z'_n) of (z_n) there exist a subsequence (z''_n) of (z'_n), positive integers k and m, sequences $(a^i_n) \in M$ and subsequences (b^i_n) of (a^i_n) $(i = 1, 2, \dots, m)$ such that for each $n \in \mathbb{N}$ the relation*

$$z''_n \leqq k(b^1_n + b^2_n + \dots + b^m_n)$$

is valid.

Proof. This is a consequence of [5; Theorem 2.2]. □

From 3.2 we immediately obtain:

3.3. Lemma. *We have $\alpha \leqq \beta$.*

3.4. Lemma. *(y_n) does not belong to α.*

Proof. By way of contradiction, assume that (y_n) belongs to α. Thus in view of 3.2 there exists a subsequence (y'_n) of (y_n) and there are positive integers k and m, and subsequences (b^i_n) of (x_i) $(i = 1, 2, \dots, m)$ such that the relation

$$y'_n \leqq k\big(b^1_n + b^2_n + \dots + b^m_n\big) \tag{1}$$

is valid for each $n \in \mathbb{N}$.

Then for each $n \in \mathbb{N}$ we have

(i) there is a positive integer $t(n) \geqq n$ such that $y'_n = t(n)x_{t(n)}$;

(ii) there are positive integers $s(n,i)$ such that

$$b^i_n = x_{s(n,i)} \qquad (i = 1, 2, \dots, m).$$

If $t(n) \neq s(n,i)$, then $y'_n \wedge kb^i_n = 0$; thus such elements b^i_n can be omitted in (1). Hence in view of (1) we have

$$y'_n = t(n)x_{t(n)} \leqq kmx_{t(n)} \tag{2}$$

for each $n \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that $t(n) > km$ and then the relation (2) cannot hold. □

From 3.3 and 3.4 we conclude:

3.5. Corollary. *We have $\alpha < \beta$.*

It is obvious that $\alpha(d) < \alpha$.

3.6. Lemma. *The element α has no relative complement in the interval $[\alpha(d), \beta]$.*

Proof. By way of contradiction, assume that γ is a complement of α in the interval $[\alpha(d), \beta]$. Thus

$$\alpha \wedge \gamma = \alpha(d)\,, \tag{3a}$$

$$\alpha \vee \gamma = \beta\,. \tag{3b}$$

In view of (3b) and 3.5 we have $\alpha(d) < \gamma$. Hence there exists $(z_n) \in \gamma$ such that the relation $(z_n) \sim \operatorname{const} 0$ fails to be valid. Then there is a subsequence (z'_n) of (z_n) such that $z'_n > 0$ for each $n \in \mathbb{N}$.

Since $(z'_n) \in \gamma \subseteq \beta$, in view of 3.2 there exist a subsequence (z''_n) of (z'_n), positive integers k, m and sequences (b^i_n) $(i = 1, 2, \dots, m)$ such that each (b^i_n) is a subsequence of (y_n) and

$$z''_n \leqq k\big(b^1_n + b^2_n + \dots + b^m_n\big) \qquad \text{for each} \quad n \in \mathbb{N}.$$

For each $i \in \{1, 2, \ldots, m\}$ there is a subsequence $\big(t(n,i)\big)$ of the sequence (n) such that

$$b_n^i = y_{t(n,i)} = t(n,i)x_{t(ni)}\,.$$

Denote

$$v_n = k\big(x_{t(n,1)} + x_{t(n,2)} + \cdots + x_{t(n,m)}\big)\,. \tag{4}$$

Then $0 < v_n \leqq z_n''$ for each $n \in \mathbb{N}$. Hence $(v_n) \notin \alpha(d)$. Further, $(z_n'') \in \gamma$ and thus $(v_n) \in \gamma$. But, at the same time, (in view of (4)) we have $(v_n) \in \alpha$. This yields that the relation (3a) fails to be valid and we arrived at a contradiction. □

Summarizing the above results of the present section we obtain:

3.7. Lemma. *Let G be an abelian lattice ordered group which does not belong to the class $\mathcal{F}$. Then* $\operatorname{Conv} G$ *fails to be a generalized Boolean algebra.*

Let (A) be as in Introduction.

P r o o f o f (A) . Let (i)–(iv) be the conditions in the assertion (A). Then we have (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). In view of 3.7, (i) $\Longrightarrow$ (iv). Further, according to 2.18, (iv) $\Longrightarrow$ (iii). □

REFERENCES

[1] CONRAD, P.: *The structure of a lattice-ordered group with a finite number of disjoint elements*, Michigan Math. J. **7** (1960), 171–180.

[2] FUCHS, L.: *Partially Ordered Algebraic Systems*, Pergamon Press, Oxford, 1963.

[3] HARMINC, M.: *Sequential convergence on abelian lattice-ordered groups.* In: Convergence Structures, 1984. Math. Res. 24, Akademie Verlag, Berlin, 1985, pp. 153–158.

[4] HARMINC, M.: *The cardinality of the system of all convergences on an abelian lattice ordered group*, Czechoslovak Math. J. **37** (1987), 533–546.

[5] HARMINC, M.: *Sequential convergences on lattice ordered groups*, Czechoslovak Math. J. **39** (1989), 232–238.

[6] HARMINC, M.: *Convergences on Lattices Ordered Groups.* Dissertation, Math. Inst. Slovak Acad. Sci., 1986. (Slovak)

[7] HARMINC, M.—JAKUBÍK, J.: *Maximal convergences and minimal proper convergences in ℓ-groups*, Czechoslovak Math. J. **39** (1989), 631–640.

[8] JAKUBÍK, J.: *Convergences and complete distributivity of lattice ordered groups*, Math. Slovaca **38** (1988), 269–272.

[9] JAKUBÍK, J.: *On some types of kernels of a convergence ℓ-group*, Czechoslovak Math. J. **39** (1989), 239–247.

[10] JAKUBÍK, J.: *Lattice ordered groups having a largest convergence*, Czechoslovak Math. J. **39** (1989), 717–729.

[11] JAKUBÍK, J.: *Convergences and higher degrees of distributivity of lattice ordered groups and of Boolean algebras*, Czechoslovak Math. J. **40** (1990), 453–458.

[12] JAKUBÍK, J.: *Sequential convergences in ℓ-groups without Urysohn's axiom*, Czechoslovak Math. J. **42** (1992), 101–116.
[13] JAKUBÍK, J.: *Nearly disjoint sequences in convergence ℓ-groups*, Math. Bohem. **125** (2000), 139–144.
[14] JAKUBÍK, J.: *On iterated limits of subsets of a convergence ℓ-groups*, Math. Bohem. **126** (2001), 53–61.
[15] JAKUBÍK, J.: *Konvexe Ketten in ℓ-Gruppen*, Časopis Pěst. Mat. **84** (1959), 53–63.

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