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BOOK REVIEWS

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Peter Lang, Europäischer Verlag der Wissenschaften, Frankfurt am Main 2005, xxii + 454 p.
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This is the first monograph aiming in a systematized description of details of distribution properties of one-dimensional or multi-dimensional sequences. The heart of the book is formed by more than 500 items of sequences, for instance those involving logarithmic functions, trigonometric functions, polynomials, sum-of-digits functions, number-theoretic functions, primes, normal numbers, the van der Corput sequence, pseudorandom number generators, etc.. Properties of sequences like low discrepancy sequences, good lattice points, nets, lattice rules, completely uniformly distributed sequences, block sequences, circle sequences, etc. are also covered. The reader finds here a complete mathematical description of the sequences, characterization of the currently known distribution properties (in terms of distribution functions, or various types of discrepancy, diaphony, dispersion, etc.), comments documenting the history, development, open problems (if exist) and the relevant literature for each of the referred to sequences. The cumulative updated bibliography at the end of the book contains more than 1000 items.

The book will be useful for the advanced students and mathematicians working in the area of the uniform distribution of sequences and in applications of quasi-Monte Carlo methods.

The main interest of the present monograph is focused to the set $G(x_n)$ of all distribution functions of a given sequence x_n of real numbers or vectors in unit cubes. The notion of the distribution of a sequence x_n will be identified with the set $G(x_n)$. Let us notice that only a relatively small number of sequences x_n are known with a completely described infinite set $G(x_n)$. On the other hand, the majority of sequences x_n for which $G(x_n)$ is completely known is formed by the set of uniformly distributed sequences, i.e., sequences x_n for which $G(x_n)$ is a singleton $\{g(x)\}$ with $g(x) = x$. That the set $G(x_n)$ is important is reflected on the fact that most properties of a sequence x_n expressed in terms of limiting processes may be characterized just using $G(x_n)$.

Let us emphasize that the description of $G(x_n)$ has high theoretical importance in the theory of uniform distributions and in the number theory generally. For instance, the detailed knowledge of $G(x_n)$ influences an application of the sequence x_n when calculating some series of arithmetical functions using the generalized Weyl's limit relation. Let us note that the sequences x_n for which $G(x_n)$ is a singleton (i.e., x_n has a limit law) have numerical applications through the so-called Quasi-Monte Carlo method

1. in numerical integration,
2. when approximating the solutions of differential equations,
3. or when approximating the global extremes of continuous functions,
4. in searching theory,
5. in cryptology,
6. or in financial applications

There are, of course, mentioned only some areas of applications.

For multi-dimensional sequences $\mathbf{x}_n$ the set $G(\mathbf{x}_n)$ can be used in the correlation analysis of co-ordinate sequences $\mathbf{x}_n$. This brings results different from those obtained by the statistical analysis.

Now we shall outline the body of conception of this book. The authors first list deterministic — mainly infinite — sequences, including block sequences. A finite sequence will be included if an estimation of its discrepancy is known. There are not covered

(1) metric aspects of the theory of distribution,
(2) integer sequences and sequences from generalized metric spaces,
(3) distribution problems in finite abstract sets,
(4) continuously uniform distributions.

In most cases the terms of the listed sequences will be supposed to lie in the unit interval $[0,1]$ or that they are reduced $\bmod 1$. In some special cases there are also included unbounded sequences with distribution functions defined on $(-\infty,\infty)$. Infinite sequences will be listed together with their distribution functions, the upper and lower distribution function, discrepancy, diaphony, dispersion, or with their known estimates, of course, depending on a present state of knowledge of all these quantities (often if even authors do not know anything about their density properties).

The sequences having limits are not listed for the obvious reasons, they have a one-jump asymptotic distribution function and so can be found in other sources. On the other hand, dense statistically convergent sequences which also possess one-jump asymptotic distribution functions are included.

The book itself is divided into four chapters. To make the book more self-contained there are recalled the basic definitions or listed the fundamental results in *Chapter 1*. This also will help to unify the exact meaning of the utilized notions which may be in use and to some extent hardly noticeable as to difference in their meaning. Simultaneously, it is hoped, this also will help the non-specialized reader to find the fundamental notions and results of the classical theory on the real line or in multi-dimensional real spaces in one source.

Additional theoretical results can be found in *Chapter 4*, *Appendix*.

Chapters 2 and *3* contain the promised lists of sequences, which are divided into two main categories:

(1) one-dimensional sequences (Chapter 2),
(2) multi-dimensional sequences (Chapter 3).

The sequences are grouped within these two categories according to a dominant (from authors' point of view) or characteristic feature mainly represented by

- distribution criteria,
- the distribution as a result of some operations on sequences,
- general functions involved in the definition of the sequence,
- some important special functions appearing in their definitions.

One can say that it is hard to find a unique classification scheme in the labyrinth of the various aspects. From the other classification attributes there are mentioned

sequences involving primes,
sequences of rational numbers or reduced rational numbers,
the van der Corput sequence and van der Corput-Halton sequence,
pseudorandom number generators,
circle sequences.

The so-called completely uniformly distributed sequences can be found in *Chapter 3*.

Not all of these classification attributes may be immediately clear. Moreover, they are neither uniquely determined nor even disjoint, therefore many cross-references should help the reader in orientation among other related sequences.

Open problems are also included, not only to complete the picture. These may provide the impetus for further possible research. Having the same aim in mind reader's attention is also directed to gaps in the presently known results in the theory of the distribution of sequences.

The sections of the book are numbered consecutively, their subsections, too. The numbering of the entries starts afresh in each section. The entries are then numbered indicating the chapter by the first number, then the section by the second one, and the final number gives the order within the section. The theorems have the additional fourth number giving their order within the entry number. The notes containing a brief survey of related results together with relevant bibliographies follow immediately the main body of the entry. Here the numbering of the notes corresponds to the numbering in the main part of the entry if any, otherwise the numbering only separates notes from each other (the numbering may also continue if there is no relation to the numbering within the main body of the entry).

The book ends with an extended bibliography with cross-references to the main text, followed by the index of names referred to in the text and the subject index.

There are listed monographs here which are recommended as the prerequisites for using this book, namely monographs which are usually recommended as standard references in the general theory of the uniform distribution.

It was the main aim of the authors to make the presented selection of results as complete as possible in order to reflect the current state of things.

It is a positive result that the present monograph covers material scattered throughout books and journals. The enormous wealth of information contained in the book may be of value not only to those working directly in this field, but also to those working in related branches of number theory, combinatorics, real or numerical analysis in the process of finding sequence possessing the required properties. Last, but not least, browsing throughout the book may provide the impulse for prospective further research. This is what may address a wide class of working mathematicians and applying mathematics in the fields mentioned in this review.

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