Mathematica Slovaca

Zbigniew Walczak

Error estimates and the Voronovskaja theorem for modified Szász-Mirakyan operators

Mathematica Slovaca, Vol. 55 (2005), No. 4, 465--476

Persistent URL: http://dml.cz/dmlcz/136921

Math. Slovaca, **55** (2005), No. 4, 465–476

Mathematica Slovaca

ERROR ESTIMATES AND THE VORONOVSKAJA THEOREM FOR MODIFIED SZÁSZ-MIRAKYAN OPERATORS

Zbigniew Walczak

(*Communicated by Michal Zajac*)

ABSTRACT. In this paper we introduce certain positive linear operators. We give theorem on the degree of approximation of functions from polynomial weighted spaces by introduced operators, using norms of these spaces. This note was inspired by the results given in author's previous papers.

1. Introduction

M. Becker in his paper [1] studied approximation problems for functions $f \in C_p$, $p \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, and Szász-Mirakyan operators

$$S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad x \in \mathbb{R}_0 = [0, +\infty), \quad n \in \mathbb{N} := \{1, 2, \dots\}, \tag{1}$$

where

$$C_p := \{f \in C(\mathbb{R}_0) : \ w_p f \ \text{ is uniformly continuous and bounded on } \mathbb{R}_0\}$$

and

$$w_0(x) := 1, \qquad w_p(x) := (1 + x^p)^{-1} \quad \text{if } p \geq 1. \tag{2}$$

The norm on C_p is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in \mathbb{R}_0} w_p(x)|f(x)|. \tag{3}$$

In [1] it was proved that S_n gives a positive linear operator $C_p \to C_p$. For $f \in C_p$, $p \in \mathbb{N}_0$, it was proved that

$$w_p(x)|S_n(f; x) - f(x)| \leq M_1(p)\omega_2\left(f; C_p; \sqrt{\tfrac{x}{n}}\right), \qquad x \in \mathbb{R}_0, \ \ n \in \mathbb{N},$$

2000 Mathematics Subject Classification: Primary 41A36.
Keywords: Szász-Mirakyan operator, degree of approximation, Voronovskaja type theorem.

where $\omega_2(f;\cdot)$ is the modulus of smoothness of the order 2 and $M_1(p)$ is a positive constant.

Thus it follows that if $f \in C_p^1 := \{f \in C_p : f' \in C_p\}$ and $p \in \mathbb{N}$, then

$$w_p(x)|S_n(f;x) - f(x)| \le M_2\sqrt{\frac{x}{n}}$$

for every fixed $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ ($M_2 = \text{const} > 0$). From these theorems it was deduced that

$$\lim_{n\to\infty} S_n(f;x) = f(x) \tag{4}$$

for every $f \in C_p$, $p \in \mathbb{N}_0$ and $x \in \mathbb{R}_0$. Moreover the convergence (4) is uniform on every interval $[x_1, x_2]$, $x_2 > x_1 \ge 0$.

Recently in many papers various modification of operators S_n were introduced. We cite works of P. Gupta and V. Gupta [3], V. Gupta and U. Abel [4], V. Gupta and R. P. Pant [5], V. Gupta, V. Vasishtha and M. K. Gupta [6], M. Herzog [7], H. G. Lehnhoff [8], M. Leśniewicz and L. Rempulska [9], and S. Xiehua [18]. These operators are very interesting approximation processes. In particular, the authors [3]–[6] studied the rate of convergence of introduced operators. Their results improve other related results in the literature.

Approximation properties of modified Szász-Mirakyan operators

$$S_n^q(f;x) := \mathrm{e}^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\Big(\frac{k}{n+q}\Big)$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $q > 0$ in exponential weighted spaces were examined in [10]. Their extensions can be found in [11]–[17]. For example, in the paper [16], there were considered certain linear positive operators

$$B_n^r(f;x) := \frac{1}{g\big((nx+1)^2;r\big)}\sum_{k=0}^{\infty}\frac{(nx+1)^{2k}}{(k+r)!}f\left(\frac{k+r}{n(nx+1)}\right),$$
$$q > 0, \quad r \in \mathbb{N}, \quad x \in \mathbb{R}_0,$$

where

$$g(t;r) := \sum_{k=0}^{\infty}\frac{t^k}{(k+r)!}, \qquad t \in \mathbb{R}_0,$$

i.e.

$$g(0;r) = \frac{1}{r!}, \qquad g(t;r) = \frac{1}{t^r}\left(\mathrm{e}^t - \sum_{j=0}^{r-1}\frac{t^j}{j!}\right) \quad \text{if } t > 0.$$

In [16], there were proved theorems on the degree of approximation of $f \in C_p$ by operators B_n.

Similar theorems for the operators of the Szász-Mirakyan type are given in [11], [13]–[15] and [17].

Theorems of many papers (e.g. [1], [7], [9], [10], [12]) concern point nice approximation. We give theorems on the degree of approximation of functions from polynomial weighted spaces by introduced operators, using norms of these spaces. Moreover, we shall prove that the operators A_n (defined by (8)) give better degree of approximation of functions $f \in C_p$ and $f \in C_p^1$ than that proved for S_n. This together with the simple form of the operator makes results given in the present paper more helpful.

For $f \in C_p$ we define the modulus of continuity $\omega_1(f;\cdot)$ as usual ([2]) by the formula

$$\omega_1(f;C_p;t) := \sup_{0\le h\le t} \|\Delta_h f(\cdot)\|_p\,, \qquad t \in \mathbb{R}_0\,, \tag{5}$$

where $\Delta_h f(x) := f(x+h) - f(x)$, for $x, h \in \mathbb{R}_0$. From the above it follows that

$$\lim_{t\to 0+} \omega_1(f;C_p;t) = 0 \tag{6}$$

for every $f \in C_p$. Moreover, if $f \in C_p^1$, then there exists $M_3 = \text{const} > 0$ such that

$$\omega_1(f;C_p;t) \le M_3 \cdot t \qquad \text{for} \quad t \in \mathbb{R}_0\,. \tag{7}$$

In this note we introduce in the space C_p, $p \in \mathbb{N}_0$, a new modification of the Szász-Mirakyan operators as follows

$$A_n(f;r,q;x) := \mathrm{e}^{-(n^q x+1)^r} \sum_{k=0}^{\infty} \frac{(n^q x+1)^{rk}}{k!} f\left(\frac{k}{n^q (n^q x+1)^{r-1}}\right) \tag{8}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $r \in \mathbb{R}_2 := [2, +\infty)$ and $q > 0$.

Similarly as S_n, the operator A_n is linear and positive.

Using technique found in [1]–[2], [7]–[17], we shall obtain error estimates and the Voronovskaja type asymptotic formula for A_n. It turns out that the order of approximation by the operators examined in [11], [13]–[14] and [16] is $O(1/n)$ for $f \in C_p$. Thus, to improve the order of approximation, we consider the operators (8).

2. Auxiliary results

In this section we shall give some properties of the above operators, which we shall apply in the proofs of the main theorems.

From (8) we easily derive the following formulas

$$\begin{aligned}
A_n(1;r,q;x) &= 1\,,\\
A_n(t;r,q;x) &= x+\frac{1}{n^q}\,,\\
A_n(t^2;r,q;x) &= \Big(x+\frac{1}{n^q}\Big)^2\left[1+\frac{1}{(n^qx+1)^r}\right],\\
A_n(t^3;r,q;x) &= \Big(x+\frac{1}{n^q}\Big)^3\left[1+\frac{3}{(n^qx+1)^r}+\frac{1}{(n^qx+1)^{2r}}\right]
\end{aligned}\tag{9}$$

for every fixed $r\in\mathbb{R}_2$, $q>0$ and for all $n\in\mathbb{N}$ and $x\in\mathbb{R}_0$.

From formulas (8), (9) and $A_n(t^k;r,q;x)$, $1\le k\le 3$, given above we obtain:

Lemma 1. *Fix $r\in\mathbb{R}_2$, $q>0$. Then for all $x\in\mathbb{R}_0$ and $n\in\mathbb{N}$ we have*

$$\begin{aligned}
A_n(t-x;r,q;x) &= \frac{1}{n^q}\,,\\
A_n\big((t-x)^2;r,q;x\big) &= \frac{1}{n^{2q}}\left[1+\frac{1}{(n^qx+1)^{r-2}}\right],\\
A_n\big((t-x)^3;r,q;x\big) &= \frac{1}{n^{3q}}\left[1+\frac{3}{(n^qx+1)^{r-2}}+\frac{1}{(n^qx+1)^{2r-3}}\right].
\end{aligned}$$

Next we shall prove:

Lemma 2. *Let $s\in\mathbb{N}$, $q>0$ and $r\in\mathbb{R}_2$ be fixed numbers. Then there exist positive numbers $\lambda_{s,j}$, $1\le j\le s$, depending only on j and s such that*

$$A_n(t^s;r,q;x)=\Big(x+\frac{1}{n^q}\Big)^s\sum_{j=1}^{s}\frac{\lambda_{s,j}}{(n^qx+1)^{(j-1)r}}\tag{10}$$

for all $n\in\mathbb{N}$ and $x\in\mathbb{R}_0$. Moreover, $\lambda_{s,1}=\lambda_{s,s}=1$.

P r o o f. We shall use the mathematical induction for s.

Formula (10), for $s=1,2,3$, is given above.

Let (10) holds for $f(x)=x^j$, $1\le j\le s$, with fixed $s\in\mathbb{N}$. We shall prove (10) for $f(x) = x^{s+1}$. From (8) it follows that

$$
\begin{aligned}
&A_n(t^{s+1}; r, q; x) = \\
&= \mathrm{e}^{-(n^q x+1)^r} \sum_{k=1}^{\infty} \frac{(n^q x + 1)^{rk}}{(k-1)!} \frac{k^s}{\left(n^q (n^q x + 1)^{r-1}\right)^{s+1}} \\
&= \frac{(n^q x + 1)^r}{\left(n^q (n^q x + 1)^{r-1}\right)^{s+1}} \mathrm{e}^{-(n^q x+1)^r} \sum_{k=0}^{\infty} \frac{(n^q x + 1)^{rk}}{k!} (k+1)^s \\
&= \frac{(n^q x + 1)^r}{\left(n^q (n^q x + 1)^{r-1}\right)^{s+1}} \mathrm{e}^{-(n^q x+1)^r} \sum_{k=0}^{\infty} \frac{(n^q x + 1)^{rk}}{k!} \sum_{\mu=0}^{s} \binom{s}{\mu} k^\mu \\
&= \frac{(n^q x + 1)^r}{\left(n^q (n^q x + 1)^{r-1}\right)^{s+1}} \sum_{\mu=0}^{s} \binom{s}{\mu} \left(n^q (n^q x + 1)^{r-1}\right)^\mu A_n(t^\mu; r, q; x) .
\end{aligned}
$$

By our assumption we get

$$
\begin{aligned}
&A_n(t^{s+1}; r, q; x) = \\
&= \frac{(n^q x + 1)^r}{\left(n^q (n^q x + 1)^{r-1}\right)^{s+1}} \left\{ 1 + \sum_{\mu=1}^{s} \binom{s}{\mu} (n^q x + 1)^{r\mu} \sum_{j=1}^{\mu} \frac{\lambda_{\mu,j}}{(n^q x + 1)^{(j-1)r}} \right\} \\
&= \left(x + \frac{1}{n^q}\right)^{s+1} \left\{ \frac{1}{(n^q x + 1)^{rs}} + \sum_{j=1}^{s} \sum_{\mu=j}^{s} \binom{s}{\mu} \frac{\lambda_{\mu,j}}{(n^q x + 1)^{(s+j-\mu-1)r}} \right\} \\
&= \left(x + \frac{1}{n^q}\right)^{s+1} \left\{ \frac{1}{(n^q x + 1)^{rs}} + \sum_{j=1}^{s} \frac{1}{(n^q x + 1)^{(j-1)r}} \sum_{\mu=s-j+1}^{s} \binom{s}{\mu} \lambda_{\mu,\mu+j-s} \right\} \\
&= \left(x + \frac{1}{n^q}\right)^{s+1} \sum_{j=1}^{s+1} \frac{\lambda_{s+1,j}}{(n^q x + 1)^{(j-1)r}}
\end{aligned}
$$

and $\lambda_{s+1,1} = \lambda_{s+1,s+1} = 1$, which proves (10) for $f(x) = x^{s+1}$. Hence the proof of (11) is completed. □

Lemma 3. *Fix $p \in \mathbb{N}_0$, $r \in \mathbb{R}_2$, $q > 0$. Then there exists a positive constant $M_4 \equiv M_4(p, r)$ depending only on the parameters p and r such that*

$$\left\| A_n\big(1/w_p(t); r, q; \cdot\big) \right\|_p \leq M_4 , \qquad n \in \mathbb{N}. \tag{11}$$

Moreover, for every $f \in C_p$ we have

$$\| A_n(f; r, q; \cdot) \|_p \leq M_4 \| f \|_p , \qquad n \in \mathbb{N}. \tag{12}$$

Formula (8) and inequality (12) show that A_n, $n \in \mathbb{N}$, is a positive linear operator from the space C_p into C_p for every $p \in \mathbb{N}_0$.

P r o o f . Inequality (11) is obvious for $p=0$ by (2), (3) and (9). Let $p \in \mathbb{N}$. Then by (2) and (8)–(10) we have

$$\begin{aligned} w_p(x)A_n\big(1/w_p(t);r,q;x\big) &= w_p(x)\big\{1+A_n(t^p;r,q;x)\big\} \\ &= \frac{1}{1+x^p} + \frac{(x+1/n^q)^p}{1+x^p}\sum_{j=1}^{p}\frac{\lambda_{p,j}}{(n^q x+1)^{(j-1)r}} \\ &\le 1+\sum_{\mu=0}^{p}\binom{p}{\mu}\frac{x^\mu}{1+x^p}\sum_{j=1}^{p}\frac{\lambda_{p,j}}{(n^q x+1)^{(j-1)r}} \le M_4(p,r) \end{aligned}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $q>0$ and $r \in \mathbb{R}_2$, where $M_4(p,r)$ is a positive constant depending only on p and r. From this (11) follows.

Formula (8) and (3) imply

$$\big\|A_n\big(f(t);r,q;\cdot\big)\big\|_p \le \|f\|_p\big\|A_n\big(1/w_p(t);r,q;\cdot\big)\big\|_p, \qquad n \in \mathbb{N}, \quad r \in \mathbb{R}_2,$$

for every $f \in C_p$. Applying (11), we obtain (12). This completes the proof of Lemma 3. □

Lemma 4. *Fix $p \in \mathbb{N}_0$, $q>0$ and $r \in \mathbb{R}_2$. Then there exists a positive constant $M_5 \equiv M_5(p,r)$ such that*

$$\Big\|A_n\Big(\tfrac{(t-\cdot)^2}{w_p(t)};r,q;\cdot\Big)\Big\|_p \le \frac{M_5}{n^{2q}} \qquad \textit{for all} \quad n \in \mathbb{N}. \tag{13}$$

P r o o f . The formulas given in Lemma 1 and (2), (3) imply (13) for $p=0$. By (2) and (9) we have

$$A_n\big((t-x)^2/w_p(t);r,q;x\big) = A_n\big((t-x)^2;r,q;x\big) + A_n\big(t^p(t-x)^2;r,q;x\big)$$

for $p,n \in \mathbb{N}$, $q>0$ and $r \in \mathbb{R}_2$. If $p=1$, then by the equality we get

$$\begin{aligned} A_n\big((t-x)^2/w_1(t);r,q;x\big) &= A_n\big((t-x)^2;r,q;x\big) + A_n\big(t(t-x)^2;r,q;x\big) \\ &= A_n\big((t-x)^3;r,q;x\big) + (1+x)A_n\big((t-x)^2;r,q;x\big), \end{aligned}$$

which by (2), (3) and Lemma 1 yields (13) for $p=1$.

Let $p \geq 2$. Applying (10), we get

$$
\begin{aligned}
&w_p(x)A_n\big(t^p(t-x)^2;r,q;x\big)\\
&= w_p(x)\{A_n(t^{p+2};r,q;x) - 2xA_n(t^{p+1};r,q;x) + x^2A_n(t^p;r,q;x)\}\\
&= w_p(x)\Bigg\{\Big(x+\frac{1}{n^q}\Big)^{p+2}\sum_{j=1}^{p+2}\frac{\lambda_{p+2,j}}{(n^qx+1)^{(j-1)r}}\\
&\qquad - 2x\Big(x+\frac{1}{n^q}\Big)^{p+1}\sum_{j=1}^{p+1}\frac{\lambda_{p+1,j}}{(n^qx+1)^{(j-1)r}} + x^2\Big(x+\frac{1}{n^q}\Big)^{p}\sum_{j=1}^{p}\frac{\lambda_{p,j}}{(n^qx+1)^{(j-1)r}}\Bigg\}\\
&= w_p(x)\Big(x+\frac{1}{n^q}\Big)^{p}\Bigg\{\frac{1}{n^{2q}} + \Big(x+\frac{1}{n^q}\Big)^{2}\sum_{j=2}^{p+2}\frac{\lambda_{p+2,j}}{(n^qx+1)^{(j-1)r}}\\
&\qquad - 2x\Big(x+\frac{1}{n^q}\Big)\sum_{j=2}^{p+1}\frac{\lambda_{p+1,j}}{(n^qx+1)^{(j-1)r}} + x^2\sum_{j=2}^{p}\frac{\lambda_{p,j}}{(n^qx+1)^{(j-1)r}}\Bigg\},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&w_p(x)A_n\big(t^p(t-x)^2;r,q;x\big)\\
&\qquad \leq \frac{1}{n^{2q}}\frac{(1+x)^p}{1+x^p}\Bigg\{1+\frac{1}{(n^qx+1)^{r-2}}\Bigg(\sum_{j=2}^{p+2}\lambda_{p+2,j} + 2\sum_{j=2}^{p+1}\lambda_{p+1,j} + \sum_{j=2}^{p}\lambda_{p,j}\Bigg)\Bigg\}\\
&\qquad \leq \frac{M_5(p,r)}{n^{2q}}
\end{aligned}
$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $q > 0$ and $r \in \mathbb{R}_2$. This ends the proof of (13). □

3. Error estimates

In this part we shall give some estimates of the rate of convergence of the operators A_n.

We shall apply the method used in [1], [7], [9]–[17].

THEOREM 1. *Let $p \in \mathbb{N}_0$, $q > 0$ and $r \in \mathbb{R}_2$ be fixed numbers. Then there exists a positive constant $M_6 \equiv M_6(p,r)$ such that for every $f \in C_p^1$ we have*

$$\|A_n(f;r,q;\cdot) - f(\cdot)\|_p \leq \frac{M_6}{n^q}\|f'\|_p, \qquad n \in \mathbb{N}. \tag{14}$$

P r o o f. Fix $x \in \mathbb{R}_0$. Since we have

$$f(t) - f(x) = \int_x^t f'(u)\,\mathrm{d}u, \qquad t \in \mathbb{R}_0,$$

for $f \in C_p^1$ and the fixed x, we have by (8) and (9) that

$$A_n\big(f(t);r,q;x\big) - f(x) = A_n\Big(\int\limits_x^t f'(u)\,\mathrm{d}u;r,q;x\Big), \qquad n \in \mathbb{N}.$$

But by (2) and (3) we have

$$\left|\int\limits_x^t f'(u)\,\mathrm{d}u\right| \le \|f'\|_p \left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right)|t-x|, \qquad t \in \mathbb{R}_0 .$$

Consequently

$$\begin{aligned} &w_p(x)|A_n(f;r,q;x) - f(x)| \\ &\qquad \le \|f'\|_p \left\{A_n\big(|t-x|;r,q;x\big) + w_p(x)A_n\Big(\tfrac{|t-x|}{w_p(t)};r,q;x\Big)\right\} \end{aligned} \tag{15}$$

for $n \in \mathbb{N}$. By the Hölder inequality and by (9) and Lemmas 1, 3, 4, it follows that

$$A_n\big(|t-x|;r,q;x\big) \le \left\{A_n\big((t-x)^2;r,q;x\big)A_n(1;r,q;x)\right\}^{1/2} \le \frac{\sqrt{2}}{n^q},$$

$$\begin{aligned} w_p(x)A_n\Big(\tfrac{|t-x|}{w_p(t)};r,q;x\Big) &\le w_p(x)\left\{A_n\Big(\tfrac{(t-x)^2}{w_p(t)};r,q;x\Big)\right\}^{1/2}\left\{A_n\Big(\tfrac{1}{w_p(t)};r,q;x\Big)\right\}^{1/2} \\ &\le \frac{M_6}{n^q} \end{aligned}$$

for $n \in \mathbb{N}$. From this and by (15) we immediately obtain (14). □

THEOREM 2. *Let $p \in \mathbb{N}_0$, $q > 0$ and $r \in \mathbb{R}_2$ be fixed numbers. Then there exists $M_7 \equiv M_7(p,r)$ such that for every $f \in C_p$ and $n \in \mathbb{N}$ we have*

$$\|A_n(f;r,q;\cdot) - f(\cdot)\|_p \le M_7\omega_1\Big(f;C_p;\tfrac{1}{n^q}\Big). \tag{16}$$

P r o o f . We use Steklov function f_h of $f \in C_p$

$$f_h(x) := \frac{1}{h}\int\limits_0^h f(x+t)\,\mathrm{d}t, \qquad x \in \mathbb{R}_0,\ \ h > 0. \tag{17}$$

From (17) we get

$$f_h(x) - f(x) = \frac{1}{h}\int\limits_0^h \Delta_t f(x)\,\mathrm{d}t,$$

$$f_h'(x) = \frac{1}{h}\Delta_h f(x), \qquad x \in \mathbb{R}_0,\ \ h > 0,$$

and

$$\|f_h - f\|_p \le \omega_1(f; C_p; h), \tag{18}$$
$$\|f'_h\|_p \le h^{-1}\omega(f; C_p; h) \tag{19}$$

for $h > 0$. From this we deduce that $f_h \in C_p^1$ if $f \in C_p$ and $h > 0$. Observe that

$$\begin{aligned} &w_p(x)|A_n(f; r, q; x) - f(x)| \\ &\le w_p(x)\{|A_n(f - f_h; r, q; x)| + |A_n(f_h; r, q; x) - f_h(x)| + |f_h(x) - f(x)|\} \\ &:= L_1(x) + L_2(x) + L_3(x) \end{aligned}$$

for $n \in \mathbb{N}$, $h > 0$ and $x \in \mathbb{R}_0$. From (12) and (18) we get

$$\begin{aligned} &\|L_1\|_p \le M_4\|f_h - f\|_p \le M_4\omega_1(f; C_p; h), \\ &\|L_3\|_p \le \omega_1(f; C_p; h). \end{aligned}$$

By Theorem 1 and (19) it follows that

$$\|L_2\|_p \le \frac{M_6}{n^q}\|f'_h\|_p \le \frac{M_6}{n^q h}\omega_1(f; C_p; h).$$

Consequently

$$\|A_n(f; r, q; \cdot) - f(\cdot)\|_p \le \left(1 + M_4 + \frac{M_6}{n^q h}\right)\omega_1(f; C_p; h).$$

Setting $h = \frac{1}{n^q}$ we obtain the assertion of Theorem 2. □

From Theorem 1 and Theorem 2, applying (6)–(7), classical theorems of mathematical analysis and elementary calculations, we can prove following two corollaries:

Corollary 1. *For every fixed $r \in \mathbb{R}_2$, $q > 0$ and $f \in C_p$, $p \in N_0$, we have*

$$\lim_{n\to\infty} \|A_n(f; r, q; \cdot) - f(\cdot)\|_p = 0.$$

Corollary 2. *If $f \in C_p^1$, $p \in \mathbb{N}_0$ and $r \in \mathbb{R}_2$, $q > 0$, then*

$$\|A_n(f; r, q; \cdot) - f(\cdot)\|_p = O(1/n^q).$$

4. The Voronovskaja type theorem

THEOREM 3. *Let $f \in C_p^1$ and let $r \in \mathbb{R}_2$, $q > 0$ be fixed numbers. Then,*

$$\lim_{n\to\infty} n^q \{A_n(f; r, q; x) - f(x)\} = f'(x) \tag{20}$$

for every $x > 0$.

Proof. Let $x > 0$ be a fixed point. Then by the Taylor formula we have

$$f(t) = f(x) + f'(x)(t - x) + \varepsilon(t; x)(t - x)$$

for $t \in \mathbb{R}_0$, where $\varepsilon(t) \equiv \varepsilon(t; x)$ is a function belonging to C_p and $\varepsilon(x) = 0$. Hence by (8) and (9) we get

$$A_n(f; r, q; x) = f(x) + f'(x) A_n(t - x; r, q; x) + A_n\big(\varepsilon(t)(t - x); r, q; x\big), \qquad n \in \mathbb{N}, \tag{21}$$

and by the Hölder inequality

$$\big|A_n\big(\varepsilon(t)(t - x); r, q; x\big)\big| \le \big\{A_n\big(\varepsilon^2(t); r, q; x\big)\big\}^{1/2} \big\{A_n\big((t - x)^2; r, q; x\big)\big\}^{1/2}.$$

By Corollary 1 we deduce that

$$\lim_{n\to\infty} A_n\big(\varepsilon^2(t); r, q; x\big) = \varepsilon^2(x) = 0.$$

From this and by Lemma 1 we get

$$\lim_{n\to\infty} n^q A_n\big(\varepsilon(t)(t - x); r, q; x\big) = 0. \tag{22}$$

Applying (22) and Lemma 1 to (21), we obtain the desired assertion (20). □

Remark. It is easy to verify that analogous approximation properties in the space C_p have the operators

$$A_n^{[1]}(f; r, q; x) := \mathrm{e}^{-(n^q x + 1)^r} \sum_{k=0}^{\infty} \frac{(n^q x + 1)^{rk}}{k!} n^q (n^q x + 1)^{r-1} \int_{(k+r)/(n^q(n^q x+1)^{r-1})}^{(k+1+r)/(n^q(n^q x+1)^{r-1})} f(t)\, \mathrm{d}t$$

and

$$A_n^{[2]}(f; r, q; x) :=$$

$$:= \mathrm{e}^{-(n^q x + 1)^r} \sum_{k=0}^{\infty} \frac{(n^q x + 1)^{rk}}{k!} n^q (n^q x + 1)^{r-1} \int_0^{\infty} \mathrm{e}^{-(n^q t + 1)^r} \frac{(n^q t + 1)^{rk}}{k!} f(t)\, \mathrm{d}t,$$

for $f \in C_p$, $p \in \mathbb{N}_0$, $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $q > 0$ and $r \in \mathbb{R}_2$.

Acknowledgment

The author would like to thank the referee for his valuable comments.

REFERENCES

[1] BECKER, M.: *Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J. **27** (1978), 127–142.

[2] DE VORE, R. A.—LORENTZ, G. G.: *Constructive Approximation*, Springer-Verlag, Berlin, 1993.

[3] GUPTA, P.—GUPTA, V.: *Rate of convergence on Baskakov-Szász type operators*, Fasc. Math. **31** (2001), 37–44.

[4] GUPTA, V.—ABEL, U.: *On the rate of convergence of Bezier variant of Szász-Durrmeyer operators*, Anal. Theory Appl. **19** (2003), 81–88.

[5] GUPTA, V.—PANT, R. P.: *Rate of convergence of the modified Szász-Mirakyan operators on functions of bounded variation*, J. Math. Anal. Appl. **233** (1999), 476–483.

[6] GUPTA, V.—VASISHTHA, V.—GUPTA, M. K.: *Rate of convergence of the Szász-Kantorovitch-Bezier operators for bounded variation functions*, Publ. Inst. Math. (Beograd) (N.S.) **72**(**86**) (2002), 137–143.

[7] HERZOG, M.: *Approximation theorems for modified Szász-Mirakjan operators in polynomial weight spaces*, Matematiche (Catania) **54** (1999), 77–90.

[8] LEHNHOFF, H. G.: *On a modified Szász-Mirakjan operator*, J. Approx. Theory **42** (1984), 278–282.

[9] LEŚNIEWICZ, M.—REMPULSKA, L.: *Approximation by some operators of the Szász-Mirakjan type in exponential weight spaces*, Glas. Mat. Ser. III **32** (1997), 57–69.

[10] REMPULSKA, L.—WALCZAK, Z.: *Approximation properties of certain modified Szász-Mirakyan operators*, Matematiche (Catania) **55** (2000), 121–132.

[11] WALCZAK, Z.: *On certain linear positive operators in exponential weighted spaces*, Math. J. Toyama Univ. **25** (2002), 109–118.

[12] WALCZAK, Z.: *On approximation by modified Szász-Mirakyan operators*, Glas. Mat. Ser. III **37**(**57**) (2002), 303–319.

[13] WALCZAK, Z.: *On certain modified Szász-Mirakyan operators in polynomial weighted spaces*, Note Mat. **22** (2003/2004), 17–25.

[14] WALCZAK, Z.: *Certain modification of Szász-Mirakyan operators*, Fasc. Math. **34** (2004), 141–148.

[15] WALCZAK, Z.: *Approximation by some linear positive operators in polynomial weighted spaces*, Publ. Math. Debrecen **64** (2004), 353–367.

[16] WALCZAK, Z.: *On certain positive linear operators in polynomial weight spaces*, Acta Math. Hungar. **101** (2003), 179–191.

[17] WALCZAK, Z.: *On some linear positive operators in polynomial weighted spaces*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **49** (2003), 289–300.

[18] XIEHUA, S.: *On the convergence of Szász-Mirakyan operator*, Approx. Theory Appl. **10** (1994), 20–25.

Received June 2, 2003
Revised June 30, 2004

Institute of Mathematics
Poznań University of Technology
Piotrowo 3A
PL–60-965 Poznań
POLAND
E-mail: zwalczak@math.put.poznan.pl