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ON 7- AND 8-DECOMPOSABLE FINITE GROUPS

ALI REZA ASHRAFI* — WUJIE SHI**

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ABSTRACT. Let G be a finite group and $\mathcal{N}_G$ denote the set of all non-trivial proper normal subgroups of G. An element K of $\mathcal{N}_G$ is said to be n-decomposable if K is a union of n distinct conjugacy classes of G. G is called n-decomposable, if $\mathcal{N}_G \neq \emptyset$ and every element of $\mathcal{N}_G$ is n-decomposable. In this paper, we will completely describe all 7- and 8-decomposable finite groups.

1. Introduction and preliminaries

Let G be a finite group and let $\mathcal{N}_G$ be the set of all non-trivial proper normal subgroups of G. An element K of $\mathcal{N}_G$ is said to be *n-decomposable* if K is a union of n distinct conjugacy classes of G. If $\mathcal{N}_G \neq \emptyset$ and every element of $\mathcal{N}_G$ is n-decomposable, then we say that G is *n-decomposable.*

In [1], [2] and [3], the first author characterize the structure of n-decomposable finite groups for $n \leq 6$. In this paper we continue this problem and characterize the non-perfect 7- and 8-decomposable finite groups. To this end some deeper results in the field of the quantitative structure of finite groups are needed. For the motivation of this problem and background material the reader is encouraged to consult [5] [6], [10], [18] [21] and their references.

Let G be a group. Denote by $\pi_e(G)$ the set of all orders of elements in G. Following Wujie Shi [21], a finite group G is called *EPO-group* if every non-identity element of G has prime order. In [20], Wujie Shi and Wenze Yang discussed finite EPO-groups and got an interesting result:

THEOREM 1.1. (Wujie Shi and Wenze Yang, [20]) *The characteristic property of A_5 is:*

(1) *the order of the group contains at least three different prime factors,*
(2) *the order of every non-identity element in the group is a prime.*

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COROLLARY 1.2. *If G is a non-abelian finite simple group and the order of every non-identity element of G is prime, then G is isomorphic to A_5.*

Let G be a finite simple group and set $\pi(G) = \{p : p \text{ is a prime and } p \mid |G|\}$. Following D. Gorenstein, a finite simple group G is called a K_3*-group* if $|\pi(G)| = 3$. For the sake of completeness we mention below the following theorem of Herzog on the structure of simple K_3-groups.

THEOREM 1.3. (Herzog, [13]) *If G is a simple K_3-group, then G is isomorphic to one of the simple groups A_5, A_6, $U_3(3)$, $U_4(2)$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$ and $\mathrm{PSL}(3,3)$.*

Throughout this paper, as usual, G' denotes the derived subgroup of G. $Z(G)$ is the center of G, x^G, $x \in G$, denotes the conjugacy class of G with the representative x, and G is called non-perfect if $G' \neq G$. Also, $\psi(G)$ denotes the number of composite integers of $\pi_e(G)$. All groups considered are assumed to be finite. Our notation is standard and taken mainly from [8] and [14].

2. Main results

Suppose n is a positive integer such that there are non-abelian simple groups A and B, not necessarily different, with exactly n conjugacy classes and $G = A \times B$. Then G is a perfect n-decomposable finite group. Thus, there are n-decomposable perfect finite groups. However, the investigation of such finite groups does not seem to be simple. Hence, in this paper we restrict our attention to the non-perfect finite group.

LEMMA 2.1. *Let G be a 7- or 8-decomposable non-solvable non-perfect finite group. Then G' is simple.*

Proof. Since G' is a maximal normal subgroup of G, $|G : G'| = p$, p is prime, and G' is a minimal normal subgroup of G, which is not abelian. So G' is a direct product of k isomorphic non-abelian simple groups, say $H_1, \dots, H_k$. If $k \geq 2$ and H_1 is not a K_3-group, then $|\pi_e(H_1 \times H_2)| \geq 11$, which is a contradiction. Thus, G' is simple or H_1 is a K_3-group. Suppose G' is not simple. Then, by Theorem 1.3, H_1 is isomorphic to A_5, A_6, $U_3(3)$, $U_4(2)$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$ or $\mathrm{PSL}(3,3)$. Using a simple calculation with GAP on element orders of these groups, we can see that $H_1 \cong A_5$. Our main proof will consider a number of cases:

Case 1. G is 7-decomposable.

It is an easy fact that $|\pi_e((A_5)^r)| = 8$ for $r \geq 3$. Thus G' is simple or $G' \cong A_5 \times A_5$. Suppose $G' \cong A_5 \times A_5$. Since $\pi_e(G') = \{1, 2, 3, 5, 6, 10, 15\}$, elements of the same order of G' must be conjugate in G. On the other hand,

$|\operatorname{Aut}(G')| = 28800$, which implies that $p = 2$ and $|G| = 7200$. But G' has exactly three conjugacy classes of elements of order 2 with lengths 15, 15 and 225, respectively. This shows that 7200 must be divisible by 255, which is a contradiction.

Case 2. G is 8-decomposable.

Choose elements x, y and z of G' such that $\operatorname{ord}(x) = 2$, $\operatorname{ord}(y) = 3$ and $\operatorname{ord}(z) = 5$. We first assume that $G' \cong A_5 \times A_5$. Then there is at most one conjugacy class of G containing all of elements of G' with a prime order. Now a similar argument as in Case 1 leads to a contradiction. Thus $G' \cong (A_5)^r$ for $r \geq 3$. Since $\left|\pi_e\big((A_5)^r\big)\right| = 8$, the elements of order 2, as well as the elements of order 3, in G' must be conjugate in G. By a well-known result in character theory, since $G' = A_5 \times A_5 \times \cdots \times A_5$, every conjugacy class of G' is a direct product of the conjugacy classes of the group A_5. But A_5 has a unique conjugacy class of elements of order 2 with length 15, so G' has exactly $\binom{r}{1} = r$ conjugacy classes of elements of order 2 with length 15, $\binom{r}{2}$ conjugacy classes of elements of order 2 with length 15^2, $\binom{r}{3}$ conjugacy classes of elements of order 2 with length 15^3, $\ldots$ and, $\binom{r}{r} = 1$ conjugacy classes of elements of order 2 with length 15^r. Therefore,

$$|x^G| = 15\binom{r}{1} + 15^2\binom{r}{2} + \cdots + 15^r\binom{r}{r} = 16^r - 1 = 2^{4r} - 1\,.$$

A similar argument shows that $|y^G| = 21^r$. This implies that for some integers u and v, we have:

$$\begin{aligned}(2^{4r} - 1)u &= 2^{2r} \cdot 3^r \cdot 5^r \cdot p\,,\\ (21^r - 1)v &= 2^{2r} \cdot 3^r \cdot 5^r \cdot p\,.\end{aligned}$$

If $p \notin \{2,3,5\}$, then $G \cong (A_5)^r \times \mathbb{Z}_p$, which is contradiction. For $p = 2,3$, the first and second equation does not have an integer solution, respectively. Thus $p = 5$. Since $3^r \mid v$, $y = 3^r y_1$. If $y_1 \geq 5$, then $(21^r - 1)v > 2^{2r} \cdot 3^r \cdot 5^{r+1}$. Also, if $y_1 = 2$, then $2 \cdot (21^r - 1) > 5 \cdot 20^r$ for $r \geq 19$, and if $y_1 = 4$, then $4 \cdot (21^r - 1) > 5 \cdot 20^r$ for $r \geq 5$. For other values of r, there is no solution for the second equation. This completes the proof. □

Lemma 2.2. *Let G be a n-decomposable non-solvable non-perfect finite group and $|\mathcal{N}_G| \geq 2$. Then $|\mathcal{N}_G| = 2$, n is a prime number and $G \cong \mathbb{Z}_n \times B$, where B is a non-abelian simple group with exactly n conjugacy classes.*

Proof. Let A and B be elements of $\mathcal{N}_G$. Then by [1; Theorem 2], $G \cong A \times B$. It is easy to see that A and B are simple groups. By [18; p. 88], A and B are the only proper non-trivial normal subgroups of G. So $|\mathcal{N}_G| = 2$. If A and B are non-abelian simple groups, then $G' = G$, which is a contradiction.

Therefore, one of A or B, say A, is abelian. Since A is simple, n is a prime number and $A \cong \mathbb{Z}_n$, proving the lemma.

Suppose $\omega(G')$ denotes the number of orbits of G' under the action of $\mathrm{Aut}(G')$. In the following lemma, we show that n is an upper bound for ωG in the case that G is n-decomposable. In fact, we have:

LEMMA 2.3. *Let G be a n-decomposable non-solvable non-perfect finite group with the unique normal subgroup G'. Then G is isomorphic to a subgroup of $\mathrm{Aut}(G')$. Moreover, if G' is simple, then $n \geq \omega(G')$.*

Proof. Define $\alpha\colon G \to \mathrm{Aut}(G')$ by $\alpha(g) = T_g\colon G' \to G'$, where $T_g(a$ gag^{-1} for all $a \in G'$. It is obvious that α is well defined. We show that α is one-to-one. Suppose $\alpha(g) = I_{G'}$, where g is a non-identity element of G Then $G' \subseteq C_G(g)$ and so $C_G(g) \trianglelefteq G$. If $G' = C_G(g)$, then $g \in Z(C_G(g)) = Z(G'$. But G' is the unique normal subgroup of G, so $Z(G') = G'$. Hence G' is abelian and G is solvable, a contradiction. Thus $g \in Z(G)$. Since G' is unique and G is non-abelian, $G' = Z(G)$. This leads to a contradiction Therefore α is one-to-one and G is isomorphic to a subgroup of $\mathrm{Aut}(G')$. Now it is easy to see that for every elements $a, b \in G'$, $a^G = b^G$ if and only if a and b lie in the same orbit under the action of G, proving the lemma. □

Suppose T is the set of all groups $L_2(q)$, where $q = p^m$, p, m are primes and S is the set of all groups $L_2(p)$, where p is prime. In the following lemma, we investigate the 7- and 8-decomposable finite groups with $G' \in T \cup S$.

LEMMA 2.4. *Suppose G is a 7- or 8-decomposable finite group with $G' \in T \cup S$. Then $G \cong \mathrm{PSL}(2,27):3$, $\mathrm{Aut}(\mathrm{PSL}(2,11))$ or $\mathrm{Aut}(\mathrm{PSL}(2,13))$.*

Proof. Let G be a 7- or 8-decomposable finite group with $G' \in T$. If $2 \mid q$, then by Lemma 2.3 and a theorem of Kohl, [17; Theorem 2.5], $n > \omega(G') \geq \omega(G') = 3 + \frac{2^m-2}{m}$. This shows that $m = 2, 3$ and so $G' \cong A_5$ or $\mathrm{PSL}(2,8)$, which contradicts Table I. Next we assume that q is an odd integer. In this case, by the previously mentioned theorem of Kohl

$$\omega(G') = \begin{cases} 1 + \frac{(p+1)^2}{4} & \text{if } m - 2\,, \\ \frac{p^m+(m-1)p+3m}{2m} & \text{if } m \neq 2\,, \end{cases}$$

and so, by Lemma 2.3 and Table I, $G' \cong \mathrm{PSL}(2,27)$. Finally, we assume that $G' \in S$. Then by the Kohl's results, p is odd and $\omega(G') = \frac{p+3}{2}$. This shows that $p = 11, 13$ and $G \cong \mathrm{Aut}(\mathrm{PSL}(2,11))$ or $\mathrm{Aut}(\mathrm{PSL}(2,13))$, which concludes the lemma. □

THEOREM 2.5. *Let G be a non-perfect 7-decomposable finite group. Then G is isomorphic to an abelian group of order 49, $\mathrm{Aut}(\mathrm{PSL}(2,11))$, $\mathbb{Z}_7 \times A_6$, $\mathrm{Aut}(\mathrm{Sz}(8))$ or a Frobenius group of order $\frac{1}{6}p^r(p^r-1)$, $p \geq 5$ is prime, and r is a positive integer, such that the kernel of G is elementary abelian of order p^r and its complement is cyclic.*

P r o o f. We first assume that G is solvable. If G is abelian, then it is clear that G is an abelian group of order 49, as desired. Suppose G is non-abelian. Then $|G : G'| = q$, where q is prime. Since G' is a minimal normal subgroup of G, G' is an elementary abelian subgroup of order, say p^r. Thus, $|G| = p^r q$. Since G is not abelian, $q \neq p$ and $C_G(x) = G'$ for any $x \in G'$, $x \neq 1$. Therefore, G is a Frobenius group with kernel G'. Since G' is abelian, by [15; p. 1160, Theorem 5.1], $n-1 = \frac{|G'|-1}{q}$. This implies that $p^r - 1 = 6q$, as desired.

Next we assume that G is non-solvable. If $|\mathcal{N}_G| = 2$, then by Lemma 2.2, $G \cong \mathbb{Z}_7 \times A_6$. So, we can restrict our investigation to the case that G' is the unique normal subgroup of G, which is simple by Lemma 2.1. It is clear that $|\pi(G')| \leq 6$. If $|\pi(G')| = 6$, then G' is an EPO-group and by Corollary 1.2, $G' \cong A_5$, which is a contradiction. Suppose $|\pi(G')| = 3$. Then by Theorem 1.3, G' is isomorphic to A_5, A_6, $U_3(3)$, $U_4(2)$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$ or $\mathrm{PSL}(3,3)$ and by Lemma 2.3, G is isomorphic to a subgroup of $\mathrm{Aut}(G')$. But, G' cannot be isomorphic to the groups A_5 and $\mathrm{PSL}(2,7)$ since these groups have exactly five and six conjugacy classes, respectively. Suppose $G' \cong A_6$. Since $|\mathrm{Aut}(A_6) : A_6| = 4$ and G is a subgroup of $\mathrm{Aut}(A_6)$ with prime index, G is isomorphic to $S_6 = A_6.2_1, A_6.2_2$ or $A_6.2_3$, in ATLAS notation. [9]. But by Table I, such a group is 5- or 6-decomposable, which is a contradiction. On the other hand, by this table, $L_2(8)$ is a 5-decomposable subgroup of $\mathrm{Aut}(L_2(8))$, $L_2(17)$ is a 10-decomposable subgroup of $\mathrm{Aut}(L_2(17))$, $L_3(3)$ is a 9-decomposable subgroup of $\mathrm{Aut}(L_3(3))$, $U_3(3)$ is a 10-decomposable subgroup of $\mathrm{Aut}(U_3(3))$ and $U_4(2)$ is a 15-decomposable subgroup of $\mathrm{Aut}(U_4(2))$, also $|\mathrm{Aut}(G') : G'| = p$, $p = 2,3$, which are impossible. Thus $|\pi(G')| = 4,5$. In our main proof, we consider two separate cases:

Case 1. $|\pi(G')| = 5$.
In this case $\psi(G') = 1$ and by [22], G is isomorphic to $\mathrm{PSL}(2,q)$, $q = 5,7,8,9,11,13,16$, $\mathrm{PSL}(3,4)$, $\mathrm{Sz}(8)$, $\mathrm{PSL}(2,3^n)$, where $\frac{3^n-1}{2}$ and $\frac{3^n+1}{4}$ are primes, or $\mathrm{PSL}(2,2^n)$, where 2^n-1 and $\frac{2^n+1}{3}$ are primes. But, by a calculation, the orders of all of these groups have at most four prime divisors, which is a contradiction.

Case 2. $|\pi(G')| = 4$.
In this case $\psi(G') = 1,2$. We first assume that $\psi(G') = 1$. Apply the previously mentioned result of S h i and Y a n g. By Lemma 2.4, Table I and [9], $|\mathrm{Aut}(\mathrm{Sz}(8)) : \mathrm{Sz}(8)| = 3$ and $\mathrm{Aut}(\mathrm{Sz}(8))$ is 7-decomposable. Also, by Table I,

Aut(PSL(2, 11)) is another 7-decomposable group with $\psi(G') = 1$. Next we suppose that $\psi(G') = 2$. Applying [10; Theorem 2], [24; Theorem 2] and [7; Theorem 2], it is enough to investigate the simple groups PSL$(2,q)$. Suppose $G' \cong$ PSL$(2,q)$, then by Lemma 2.4 and Table I, G is not 7-decomposable. This completes the proof. □

THEOREM 2.6. *Let G be a non-perfect 8-decomposable finite group. Then G is isomorphic to* Aut(PSL(2, 13)), PSL(2, 27) : 3, PSL(3, 4) : 2 (*including* PSL$(3,4).2_1$, PSL$(3,4).2_2$ *and* PSL$(3,4).2_3$), PSL(3, 4) : 3, S_7 *or a Frobenius group of order* $\frac{1}{7}2^r(2^r-1)$, *r is a positive integer, such that the kernel of G is elementary abelian of order 2^r and its complement is cyclic.*

Proof. It is clear that such a group cannot be abelian. If G is a non-abelian solvable group, then using a similar argument as in Theorem 2.5, we can see that G is a Frobenius group of order $\frac{1}{7}p^r(p^r-1)$, p is odd prime and r is a positive integer. Suppose that G is non-solvable. Then by Lemmas 2.2 and 2.3, $|\mathcal{N}_G| = 1$ and G' is simple. Also, by Corollary 1.2, Theorem 1.3 and Table I, G' cannot be an EPO-group or a K_3-group. So, $4 \leq |\pi(G')| \leq 6$. If $|\pi(G')| = 6$, then $\psi(G') = 1$. But in this case, by [22] and [5], such a group has at most four prime divisors, which is a contradiction. In our main proof, we consider two separate cases:

Case 1. $|\pi(G')| = 5$.
Since G is not EPO-group, $\psi(G') = 1, 2$. Also by Lemma 2.4 and [22], there is no group G with $\psi(G') = 1$. Thus $\psi(G') = 2$. By Table I, PSL(3, 4) : 2, PSL(3, 4) : 3 and Aut(PSL(2, 13)) are solutions for our problem. So by [11; Theorem A], it is enough to investigate the cases that G' is isomorphic to the Suzuki group Sz(q) or a projective special linear group PSL$(2,q)$ for some special values of q. By Lemma 2.4, if $G' \cong$ PSL$(2,p^m)$, where p and m are primes, then $G' \cong$ PSL(2, 27), which is a contradiction. If $G' \cong$ PSL$(2,p)$, where p is a prime with $p > 13$, then by the previously mentioned theorem of Kohl, we obtain a contradiction. Finally, assume that $G' \cong$ Sz(q), where $q = 2^{2m+1}$ is such that each of $q-1$, $q-(2q)^{\frac{1}{2}}+1$ and $q+(2q)^{\frac{1}{2}}+1$ is either a prime or a product of two distinct primes. By [17; Theorem 3.4], $\omega(\mathrm{Sz}(q)) = \omega(\mathrm{PSL}(2,q)) + 2$ and by Lemma 2.3 and [17; Theorem 2.5], $8 \geq \omega(\mathrm{Sz}(q)) = 2 + \omega(\mathrm{PSL}(2,q)) = 5 + \frac{2^{2m+1}-2}{2m+1}$. This shows that $G' \cong$ Sz(8) and by Table I, we get our final contradiction.

Case 2. $|\pi(G')| = 4$.
Using a tedious calculation for applying the [24; Theorem 2], [7; Theorem 2], [17; Theorem 2.5], Lemma 2.4 and Table I, we can see that $G \cong$ Aut(PSL(2, 13)), PSL(2, 27) : 3, PSL(3, 4) : 2 or PSL(3, 4) : 3, which completes the proof. □

Table I: The fusion maps of some simple groups into their automorphism groups.

A_5-Classes	1a	2a	3a	5a	5b				
Fusion into S_5	1A	2A	3A	5A	5A				
A_6-Classes	1a	2a	3a	3b	4a	5a	5b		
Fusion into S_6	1A	2A	3A	3B	4A	5A	5A		
A_6-Classes	1a	2a	3a	3b	4a	5a	5b		
Fusion into $A_6.2_2$	1A	2A	3A	3A	4A	5A	5B		
A_6-Classes	1a	2a	3a	3b	4a	5a	5b		
Fusion into $A_6.2_3$	1A	2A	3A	3A	4A	5A	5A		
A_7-Classes	1a	2a	3a	3b	4a	5a	6a	7a	7b
Fusion into S_7	1A	2A	3A	3B	4A	5A	6A	7A	7B
PSL(2, 7)-Classes	1a	2a	3a	4a	7a	7b			
Fusion into Aut(PSL(2, 7))	1A	2A	3A	4A	7A	7A			
PSL(2, 8)-Classes	1a	2a	3a	7a	7b	7c	9a	9b	9c
Fusion into Aut(PSL(2, 8))	1A	2A	3A	7A	7A	7A	9A	9A	9A
PSL(2, 11)-Classes	1a	2a	3a	5a	5b	6a	11a	11b	
Fusion into Aut(PSL(2, 11))	1A	2A	3A	5A	5B	6A	11A	11A	
PSL(2, 13)-Classes	1a	2a	3a	6a	7a	7b	7c	13a	13b
Fusion into Aut(PSL(2, 13))	1A	2A	3A	6A	7A	7B	7C	13A	13A
PSL(2, 16)-Classes	1a	2a	3a	5a	5b	15a	15b	15c	15d
Fusion into PSL(2, 16).2	1A	2A	3A	5A	5B	15A	15B	15A	15B
PSL(2, 16)-Classes	17a	17b	17c	17d	17e	17f	17g	17h	
Fusion into PSL(2, 16).2	17B	17A	17D	17B	17D	17C	17C	17A	
PSL(2, 19)-Classes	1a	2a	3a	5a	5b	9a	9b	9c	10a
Fusion into Aut(PSL(2, 19))	1A	2A	3A	5A	5B	9A	9B	9C	10A
PSL(2, 19)-Classes	10b	19a	19b						
Fusion into Aut(PSL(2, 19))	10B	19A	19A						
PSL(2, 17)-Classes	1a	2a	3a	4a	8a	8b	9a	9b	9c
Fusion into Aut(PSL(2, 17))	1A	2A	3A	4A	8A	8B	9A	9B	9C
PSL(2, 17)-Classes	17a	17b							
Fusion into Aut(PSL(2, 17))	17A	17A							
PSL(2, 27)-Classes	1a	2a	3a	3b	7a	7b	7c	13a	13b
Fusion into PSL(2, 27) : 2	1A	2A	3A	3A	7A	7B	7C	13A	13B
PSL(2, 27)-Classes	13c	13d	13e	13f	14a	14b	14c		
Fusion into PSL(2, 27) : 2	13C	13D	13E	13F	14A	14B	14C		

Table I: (Continued).

PSL(2, 27)-Classes	1a	2a	3a	3b	7a	7b	7c	13a	13b
Fusion into PSL(2, 27) : 3	1A	2A	3A	3B	7A	7A	7A	13A	13A
PSL(2, 19)-Classes	13c	13d	13e	13f	14a	14b	14c		
Fusion into PSL(2, 27) : 3	13A	13B	13B	13B	14A	14A	14A		
PSL(3, 3)-Classes	1a	2a	3a	3b	4a	6a	8a	8b	13a
Fusion into Aut(PSL(3, 3))	1A	2A	3A	3B	4A	6A	8A	8A	13A
PSL(3, 3)-Classes	13b	13c	13d						
Fusion into Aut(PSL(3, 3))	13A	13B	13B						
PSL(3, 4)-Classes	1a	2a	3a	4a	4b	4c	5a	5b	7a
Fusion into PSL(3, 4).2	1A	2A	3A	4D	4A	4C	5A	5A	7A
PSL(3, 4)-Classes	7b								
Fusion into PSL(3, 4).2	7A								
PSL(3, 4)-Classes	1a	2a	3a	4a	4b	4c	5a	5b	7a
Fusion into PSL(3, 4).3	1A	2A	3C	4A	4A	4A	5A	5B	7A
PSL(3, 4)-Classes	7b								
Fusion into PSL(3, 4).3	7B								
$U_3(3)$-Classes	1a	2a	3a	3b	4a	4b	4c	6a	7a
Fusion into $U_3(3):2$	1A	2A	3A	3B	4A	4A	4B	6A	7A
$U_3(3)$-Classes	7b	8a	8b	12a	12b				
Fusion into $U_3(3):2$	7A	8A	8A	12A	12A				
$U_4(2)$-Classes	1a	2a	2b	3a	3b	3c	3d	4a	4b
Fusion into $U_4(2):2$	1A	2A	2B	3A	3A	3B	3C	4A	4B
$U_4(2)$-Classes	5a	6a	6b	6c	6d	6e	6f	9a	9b
Fusion into $U_4(2):2$	5A	6A	6A	6B	6B	6C	6D	9A	9A
$U_4(2)$-Classes	12a	12b							
Fusion into $U_4(2):2$	12A	12A							
Sz(8)-Classes	1a	2a	4a	4b	5a	7a	7b	7c	13a
Fusion into Aut(Sz(8))	1A	2A	4A	4B	5A	7A	7A	7A	13A
Sz(8)-Classes	13b	13c							
Fusion into Aut(Sz(8))	13A	13A							
M_{22}-Classes	1a	2a	3a	4a	4b	5a	6a	7a	7b
Fusion into Aut(M_{22})	1A	2A	3A	4A	4B	5A	6A	7A	7B
M_{22}-Classes	8a	11a	11b						
Fusion into Aut(M_{22})	8A	11A	11B						

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REFERENCES

[1] ASHRAFI, A. R. SAHRAEI, H.: *On finite groups whose every normal subgroup is a union of the same number of conjugacy classes*, Vietnam J. Math. **30** (2002), 289 294.

[2] ASHRAFI, A. R. SAHRAEI, H.: *Subgroups which are a union of a given number of conjugacy classes.* In: Groups St. Andrews 2001 in Oxford, Vol. I. London Math. Soc. Lecture Note Ser. 304, Cambridge Univ. Press, Cambridge, 2003, pp. 22 30.

[3] ASHRAFI, A. R.—ZHAO YAOQING: *On 5- and 6-decomposable finite groups*, Math. Slovaca **53** (2003), 373 383.

[4] BERKOVICH, YA. G. ZHMUD, E.: *Characters of Finite Groups. Part 2.* Transl. Math. Monogr. 181, Amer. Math. Soc., Providence, RI, 1998.

[5] BRANDL, R. WUJIE SHI: *The characterization of* PSL$(2, q)$ *by its element orders*, J. Algebra **163** (1994), 109 114.

[6] BRANDL, R. WUJIE SHI: *A characterization of finite simple groups with abelian Sylow 2-subgroups*, Ricerche Mat. **42** (1993), 193 198.

[7] BUGEAUD YANN—ZHENFU CAO MAURICE MIGNOTTE: *On simple K_4-group*, J. Algebra **241** (2001), 658 668.

[8] COLLINS, M. J.: *Representations and Characters of Finite Groups*, Cambridge University Press, Cambridge, 1990.

[9] CONWAY, J. H. CURTIS, R. T.—NORTON, S. P. PARKER, R. A. WILSON, R. A.: *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[10] HUIWEN DENG: *The number of composite numbers in the set of element orders of a finite groups*, J. Group Theory **1** (1998), 339 355.

[11] HUIWEN DENG WUJIE SHI: *A simplicity criterion for finite groups*, J. Algebra **191** (1997), 371 381.

[12] GORENSTEIN, D.: *Finite Simple Groups. An Introduction to Their Classification*, Plenum, New York-London, 1982.

[13] HERZOG, M.: *On finite simple groups of order divisible by three primes only*, J. Algebra **10** (1968), 383 388.

[14] HUPPERT, B.: *Endliche Gruppen*, Springer-Verlag, Berlin, 1967.

[15] KARPILOVSKY, G.: *Group Representations, Vol. I.* North-Holland Math. Stud. 175, North-Holland, Amesterdam, 1992.

[16] ISAACS, I. M.: *Character Theory of Finite Groups.* Pure Appl. Math. 69, Academic Press, New York-San Francisco-London, 1976.

[17] KOHL, S.: *Counting the orbits on finite simple groups under the action of the automorphism group Suzuki groups vs. linear groups*, Comm. Algebra **30** (2002), 3515 3532.

[18] ROBINSON, D. J. S.: *A Course in the Theory of Groups* (2nd ed.). Grad. Text in Math. 80, Springer-Verlag, New York, 1996.

[19] SCHONERT, M. et al.: *GAP, Groups, Algorithms and Programming.* Lehrstuhl D für Math., Rheinisch Westfalische Technische Hochschule, Aachen, 1993.

[20] WUJIE SHI—WENZE YANG: *A new characterization of A_5 and the finite groups in which every non-identity element has prime order*, J. Southwest Teachers College **9** (1984), 36–40. (Chinese)

[21] WUJIE SHI: *The quantitative structure of groups and related topics.* In: Group Theory in China (Zhe-Xian Wan, Sheng-Ming Shi et al., eds.), Kluwer Academic Publishers, Dordrecht, 1996, pp. 163–181.

[22] WUJIE SHI—CHEN YANG: *A class of special finite groups*, Chinese Sci. Bull. **37** (1992), 252–253.

[23] WUJIE SHI: *A characterization of Suzuki's simple groups*, Proc. Amer. Math. Soc. **114** (1992), 589–591.

[24] WUJIE SHI: *On simple K_4-groups*, Chinese Sci. Bull. **36** (1991), 1281–1283. (Chinese)

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