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# INCLUSION THEOREMS FOR SOME SETS OF SEQUENCES DEFINED BY $\varphi$-FUNCTIONS

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ABSTRACT. For a sequence space $\lambda$ and a sequence of $\varphi$-functions $F = (f_k)$ let $\lambda^\rho(F) = \{x = (x_k) : F(x/\rho) \in \lambda\}$ $(\rho > 0)$, $\lambda^\exists(F) = \bigcup_{\rho>0} \lambda^\rho(F)$ and $\lambda^\forall(F) = \bigcap_{\rho>0} \lambda^\rho(F)$, where $F(x) = (f_k(|x_k|))$. We give necessary and sufficient conditions for the inclusions of the type $\lambda \subset \mu^\rho(F)$, $\lambda \subset \mu^\forall(F)$, $\lambda^\rho(F) \subset \mu$ and $\lambda^\exists(F) \subset \mu$, where $\lambda, \mu \in \{m, c_0, \ell_p\}$. Some special cases are also considered.

## 1. Introduction

By the term *sequence space* we shall mean, as usual, any linear subspace of the vector space $s$ of all (real or complex) sequences $x = (x_k) = (x_k)_{k\in\mathbb{N}}$, where $\mathbb{N} = \{1, 2, \dots\}$. A sequence space $\lambda$ is called *solid* if $(x_k) \in \lambda$ and $|y_k| \le |x_k|$ $(k \in \mathbb{N})$ yield $(y_k) \in \lambda$. Well-known solid sequence spaces are the space $m$ of all bounded sequences, the space $c_0$ of all convergent to zero sequences and the spaces

$$\ell_p = \Big\{x = (x_k) : \sum_{k=1}^{\infty} |x_k|^p < \infty\Big\} \qquad (1 \le p < \infty).$$

For $p = 1$ we write $\ell$ instead of $\ell_1$.

A function $f\colon [0,\infty) \to [0,\infty)$ is called a *modulus function* (or simply a *modulus*) if (see, for example, [22; p. 975])

(i) $f(t) = 0$ if and only if $t = 0$,
(ii) $f$ is non-decreasing,
(iii) $f(t + u) \le f(t) + f(u)$ $(t, u \ge 0)$,
(iv) $f$ is continuous.

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It is interesting to remark that the moduli are the same as the moduli of continuity: a function $f\colon [0,\infty) \to [0,\infty)$ is a modulus of continuity of a continuous function if and only if the conditions (i)–(iv) are satisfied (see [4; p. 866]).

If in the definition of a modulus the condition (iii) is replaced by the condition of convexity

(v) $f\bigl(\alpha t + (1-\alpha)u\bigr) \le \alpha f(t) + (1-\alpha) f(u)$ $(t, u \ge 0,\ 0 \le \alpha \le 1)$,

$f$ is called an *Orlicz function.*

Provided a modulus $f$, Ruckle [22] defined and studied the space

$$\ell(f) = \Bigl\{x = (x_k):\ \sum_{k=1}^{\infty} f\bigl(|x_k|\bigr) < \infty\Bigr\} = \bigl\{x = (x_k):\ \bigl(f(|x_k|)\bigr) \in \ell\bigr\}.$$

For an Orlicz function $f$, the *Orlicz sequence space* is determined by (see [14; p. 137])

$$\ell^{\exists}(f) = \Bigl\{x = (x_k):\ (\exists \rho{>}0)\Bigl(\sum_{k=1}^{\infty} f\bigl(|x_k|/\rho\bigr) < \infty\Bigr)\Bigr\}.$$

If $F = (f_k)$ is a sequence of Orlicz functions, the space

$$\ell^{\exists}(F) = \Bigl\{x = (x_k):\ (\exists \rho{>}0)\Bigl(\sum_{k=1}^{\infty} f_k\bigl(|x_k|/\rho\bigr) < \infty\Bigr)\Bigr\}$$

is called a *modular* or *Musielak-Orlicz sequence space* (see [18; p. 173]). Together with $\ell^{\exists}(f)$ and $\ell^{\exists}(F)$ there are examined also the sets

$$\ell^{\forall}(f) = \Bigl\{x = (x_k):\ (\forall \rho{>}0)\Bigl(\sum_{k=1}^{\infty} f\bigl(|x_k|/\rho\bigr) < \infty\Bigr)\Bigr\},$$

$$\ell^{\forall}(F) = \Bigl\{x = (x_k):\ (\forall \rho{>}0)\Bigl(\sum_{k=1}^{\infty} f_k\bigl(|x_k|/\rho\bigr) < \infty\Bigr)\Bigr\}.$$

In the mathematical literature there exist various modifications of these definitions, where $\ell$ is replaced by another solid sequence space (see, for example, [1], [2], [5]–[7], [10]–[13], [15], [17], [19]–[21], [23]). To investigate all such spaces from a more general point of view, we use the following notion.

**DEFINITION 1.** A function $f\colon [0,\infty) \to [0,\infty)$ is called a *$\varphi$-function* if the conditions (i) and (ii) are satisfied.

It should be noted that by our definition, a $\varphi$-function is not necessarily continuous and unbounded (cf. [18; p. 4]).

Let $F = (f_k)$ be a sequence of $\varphi$-functions and let $F(x) = \bigl(f_k(|x_k|)\bigr)$. For a sequence space $\lambda$ we define the sets

$$\lambda^{\rho}(F) = \bigl\{x = (x_k):\ F(x/\rho) \in \lambda\bigr\} \qquad (\rho > 0),$$

$$\lambda^{\exists}(F) = \bigl\{x = (x_k):\ (\exists \rho{>}0)\bigl(F(x/\rho) \in \lambda\bigr)\bigr\} = \bigcup_{\rho>0} \lambda^{\rho}(F),$$

$$\lambda^{\forall}(F) = \bigl\{x = (x_k):\ (\forall \rho{>}0)\bigl(F(x/\rho) \in \lambda\bigr)\bigr\} = \bigcap_{\rho>0} \lambda^{\rho}(F).$$

We write $\lambda(F)$ instead of $\lambda^1(F)$. If $f$ is a $\varphi$-function and $f_k = f$ $(k \in \mathbb{N})$, we write $\lambda^\rho(f)$, $\lambda^\exists(f)$ and $\lambda^\forall(f)$ instead of $\lambda^\rho(F)$, $\lambda^\exists(F)$ and $\lambda^\forall(F)$, respectively.

For an arbitrary sequence of $\varphi$-functions $F = (f_k)$ the sets $\lambda^\rho(F)$, $\lambda^\exists(F)$ and $\lambda^\forall(F)$ are different in general, and

$$\lambda^\forall(F) \subset \lambda^\rho(F) \subset \lambda^\exists(F). \tag{1}$$

At the same time, the sets $\lambda^\rho(F)$ $(\rho > 0)$ may not be linear, i.e., they may not be sequence spaces. However, a routine verification shows that, provided $\lambda$ is a solid sequence space, the sets $\lambda^\rho(F)$, $\lambda^\exists(F)$ and $\lambda^\forall(F)$ are solid sequence spaces whenever all $f_k$ satisfy either (iii) or (v). Moreover, the equalities

$$\lambda^\forall(F) = \lambda^\rho(F) = \lambda^\exists(F) \tag{2}$$

hold if the sequence of $\varphi$-functions $F$ satisfies so-called *uniform $\Delta_2$-condition*: there exists a constant $K > 0$ such that $f_k(2t) \le K f_k(t)$ $(k \in \mathbb{N},\ t > 0)$ (cf. [14; p. 167]).

In particular, for a solid sequence space $\lambda$, the sets $\lambda^\rho(F)$, $\lambda^\exists(F)$ and $\lambda^\forall(F)$ are sequence spaces whenever $f_k$ $(k \in \mathbb{N})$ are either moduli or Orlicz functions. Since uniform $\Delta_2$-condition holds (with $K = 2$) for every sequence of moduli $F = (f_k)$, we also conclude that (2) is true whenever all $f_k$ are either moduli or Orlicz functions such that $F$ satisfies uniform $\Delta_2$-condition.

The aim of this paper is to give necessary and sufficient conditions for the inclusions of the type $\lambda \subset \mu^\rho(F)$, $\lambda^\rho(F) \subset \mu$, $\lambda \subset \mu^\forall(F)$ and $\lambda^\exists(F) \subset \mu$, where $F = (f_k)$ is a sequence of $\varphi$-functions and $\lambda, \mu \in \{m, c_0, \ell_p\}$. Some simple special cases are also considered.

Our theorems generalize the results of [12], where the inclusions $\lambda \subset \mu(F)$ and $\lambda(F) \subset \mu$ have been characterized for a sequence of moduli $F = (f_k)$ and $\lambda, \mu \in \{m, c_0\}$. Our investigations are also motivated by the work of Grinnell [8] which is devoted to the study of the inclusions $\lambda \subset \mu_f$ for various sequence spaces $\lambda$ and $\mu$, by the assumptions that $f\colon \mathbb{R} \to \mathbb{R}$ and $\mu_f = \{x = (x_k) : (f(x_k)) \in \mu\}$.

Throughout this paper, by an *index sequence*, we mean any strictly increasing sequence of natural numbers.

## 2. Inclusions $\mu \subset \lambda(F)$

Let $F = (f_k)$ be a sequence of $\varphi$-functions, $1 \le p, q < \infty$ and

$$\ell_p^+ = \{x = (x_k) \in \ell_p :\ (\forall k \in \mathbb{N})(x_k \ge 0)\}.$$

Necessary and sufficient conditions for the inclusions $\mu \subset \lambda(F)$ in the case $\lambda, \mu \in \{m, c_0, \ell_p\}$ we derive from the results on superposition operators given by Dedagich and Zabreĭko [3].

Recall that every sequence $G = (g_k)$ of functions $g_k\colon \mathbb{R} \to \mathbb{R}$ $(k \in \mathbb{N})$ defines a *superposition operator* $P_G\colon s \to s$ by $P_G(x) = \big(g_k(x_k)\big)$. It is clear that $P_G\colon \mu \to \lambda$ if and only if $\mu \subset \lambda_G$, where $\lambda_G = \big\{x = (x_k) : \big(g_k(x_k)\big) \in \lambda\big\}$.

Now, if $\bar{f}_k$ $(k \in \mathbb{N})$ are even extensions of our $\varphi$-functions $f_k$, i.e.,

$$\bar{f}_k(t) = f_k\big(|t|\big) \qquad (t \in \mathbb{R}),$$

and $\bar{F} = (\bar{f}_k)$, then we have

$$\mu \subset \lambda(F) \iff P_{\bar{F}}\colon \mu \to \lambda$$

because of $\lambda_{\bar{F}} = \lambda(F)$. So by [3; Theorems 1, 7, 8] we may characterize the inclusions $\ell_q \subset \ell_p(F)$, $\ell_p \subset c_0(F)$, $c_0 \subset \ell_p(F)$, $c_0 \subset c_0(F)$, $m \subset \ell_p(F)$, $m \subset c_0(F)$ and $m \subset m(F)$, using the following classes of $\varphi$-function sequences:

$$\Phi_0 = \Big\{F = (f_k) : \ \big(\exists (a_k)\in\ell_p^+\big)\big(\exists b\geq 0\big)\big(\exists k_0\in\mathbb{N}\big)\big(\exists \delta>0\big) \big(\forall k\geq k_0\big)\big(\forall t\in[0,\delta]\big)\Big(f_k(t) \leq a_k + bt^{q/p}\Big)\Big\},$$

$$\Phi_1 = \Big\{F = (f_k) : \ \big(\exists t_0>0\big)\Big(\sum_{k=1}^{\infty} \big(f_k(t_0)\big)^p < \infty\Big)\Big\},$$

$$\Phi_2 = \Big\{F = (f_k) : \ \big(\forall t>0\big)\Big(\sum_{k=1}^{\infty} \big(f_k(t)\big)^p < \infty\Big)\Big\},$$

$$\Phi_3 = \Big\{F = (f_k) : \ \big(\exists k_0\in\mathbb{N}\big)\Big(\lim_{t\to 0+} \sup_{k\geq k_0} f_k(t) = 0\Big)\Big\},$$

$$\Phi_4 = \Big\{F = (f_k) : \ \big(\forall t>0\big)\Big(\lim_{k\to\infty} f_k(t) = 0\Big)\Big\},$$

$$\Phi_5 = \Big\{F = (f_k) : \ \big(\forall t>0\big)\Big(\sup_{k\in\mathbb{N}} f_k(t) < \infty\Big)\Big\},$$

$$\Phi_6 = \Big\{F = (f_k) : \ \big(\exists t_0>0\big)\Big(\sup_{k\in\mathbb{N}} f_k(t_0) < \infty\Big)\Big\}.$$

**THEOREM 1.** *The following equivalences are true:*

(1) $\ell_q \subset \ell_p(F) \iff F \in \Phi_0$;
(2) $c_0 \subset \ell_p(F) \iff F \in \Phi_1$;
(3) $m \subset \ell_p(F) \iff F \in \Phi_2$;
(4) $c_0 \subset c_0(F) \iff \ell_p \subset c_0(F) \iff F \in \Phi_3$;
(5) $m \subset c_0(F) \iff F \in \Phi_4$;
(6) $m \subset m(F) \iff F \in \Phi_5$.

[3; Theorem 7] asserts that a superposition operator $P_G$ maps $\ell_p$ into $m$ if and only if

$$\lim_{k\to\infty,\, t\to 0} |g_k(t)| < \infty. \tag{3}$$

It seems that this is not true in general. Defining, for example, $g_k(0) = 0$ and $g_k(t) = 1 - (-1)^k$ if $t \neq 0$, we clearly have $P_G\colon \ell_p \to m$ but the limit (3) does not exist.

Nevertheless, by [3; Theorem 8],

$$c_0 \subset m(F) \iff F \in \Phi_6 . \tag{4}$$

We show that the condition $F \in \Phi_6$ is necessary and sufficient also for $\ell_p \subset m(F)$.

**THEOREM 2.** *The following statements are equivalent:*

(a) $c_0 \subset m(F)$;
(b) $\ell_p \subset m(F)$;
(c) $F \in \Phi_6$.

Proof. Since (a) $\Longrightarrow$ (b) is obvious, then by (4) it suffices to prove that (b) $\Longrightarrow$ (c). Let $\ell_p \subset m(F)$. If $F \notin \Phi_6$, then $\sup_{k\in\mathbb{N}} f_k(t) = \infty$ for any $t > 0$. We thus can find an index sequence $(k_i)$ such that

$$f_{k_i}\big(2^{-i/p}\big) > i \qquad (i \in \mathbb{N}). \tag{5}$$

Define

$$x_k = \begin{cases} 2^{-i/p} & \text{for } k = k_i \ \ (i \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence $x = (x_k)$ belongs to $\ell_p$, but by (5) we get $x \notin m(F)$, contrary to $\ell_p \subset m(F)$. Therefore $F$ must be in $\Phi_6$. □

## 3. Inclusions $\lambda(F) \subset \mu$

Let, as in Section 2, $F = (f_k)$ be a sequence of $\varphi$-functions and $1 \le p < \infty$. In this section we study the inclusions $\lambda(F) \subset \mu$, where $\lambda \in \{m, c_0, \ell_p\}$ and $\mu \in \{m, c_0\}$. At it the following classes of $\varphi$-function sequences are important:

$$\Phi_7 = \Big\{F = (f_k):\ \big(\exists k_0 \in \mathbb{N}\big)\Big(\lim_{t\to\infty} \sup_{n\ge k_0} \inf_{k\ge n} f_k(t) = \infty\Big)\Big\},$$

$$\Phi_8 = \Big\{F = (f_k):\ \big(\exists t_0 > 0\big)\Big(\inf_{k\in\mathbb{N}} f_k(t_0) > 0\Big)\Big\},$$

$$\Phi_9 = \Big\{F = (f_k):\ \big(\forall t > 0\big)\Big(\lim_{k\to\infty} f_k(t) = \infty\Big)\Big\},$$

$$\Phi_{10} = \Big\{F = (f_k):\ \big(\forall t > 0\big)\Big(\inf_{k\in\mathbb{N}} f_k(t) > 0\Big)\Big\}.$$

**THEOREM 3.** *The inclusion $m(F) \subset m$ holds if and only if $F \in \Phi_7$.*

Proof.

*Necessity.* Let $m(F) \subset m$. Suppose that $F \notin \Phi_7$. Since the functions

$$\psi(t) = \sup_{n \geq k_0} \inf_{k \geq n} f_k(t)$$

are non-decreasing for every $k_0 \in \mathbb{N}$, there exists a number $H > 0$ such that $\inf_{k \in \mathbb{N}} f_k(t) \leq H$ for all $t > 0$. Thus, given $\varepsilon > 0$, we can choose an index sequence $(k_i)$ such that

$$f_{k_i}(i) \leq H + \varepsilon \qquad (i \in \mathbb{N}).$$

So, taking

$$x_k = \begin{cases} i & \text{if } k = k_i \ (i \in \mathbb{N}), \\ 0 & \text{otherwise,} \end{cases}$$

we get $(x_k) \in m(F)$. But $(x_k) \notin m$, contrary to $m(F) \subset m$. Therefore $F$ must be in $\Phi_7$.

*Sufficiency.* Let $x \in m(F)$, i.e., $f_k(|x_k|) \leq M$ $(k \in \mathbb{N})$ for some $M > 0$. If $F \in \Phi_7$, then there exists a number $T > 0$ such that $t \geq T$ implies

$$\inf_{k \geq n} f_k(t) \geq M \qquad (n \geq k_0).$$

This yields

$$f_n(t) \geq M \qquad (n \geq k_0, \ t \geq T). \tag{6}$$

Assuming $x \notin m$, we can choose indices $k_i \geq k_0$ $(i \in \mathbb{N})$ such that $|x_{k_i}| \geq T$, but

$$f_{k_i}(|x_{k_i}|) \leq M \qquad (i \in \mathbb{N}),$$

contrary to (6). Hence $x \in m$ and, consequently, $m(F) \subset m$. □

**THEOREM 4.** *The following statements are equivalent:*

(a) $c_0(F) \subset m$;
(b) $\ell_p(F) \subset m$;
(c) $F \in \Phi_8$.

Proof.

(a) $\Longrightarrow$ (b) follows immediately.

(b) $\Longrightarrow$ (c). Let $\ell_p(F) \subset m$. If $F \notin \Phi_8$, then $\inf_{k \in \mathbb{N}} f_k(t) = 0$ for all $t > 0$. Thus we can choose an index sequence $(k_i)$ with

$$f_{k_i}(i) \leq 2^{-i/p} \qquad (i \in \mathbb{N}).$$

So, if

$$x_k = \begin{cases} i & \text{for } k = k_i \ (i \in \mathbb{N}), \\ 0 & \text{otherwise}, \end{cases}$$

we have $x \in \ell_p(F)$. But $x \notin m$, contrary to $\ell_p(F) \subset m$. Hence $F \in \Phi_8$.

(c) $\Longrightarrow$ (a). Suppose that $F \in \Phi_8$ and $x = (x_k)$ belongs to $c_0(F)$. If we assume $x \notin m$, there exists an index sequence $(k_i)$ with $|x_{k_i}| \geq t_0$ $(i \in \mathbb{N})$. This gives

$$f_{k_i}(t_0) \leq f_{k_i}(|x_{k_i}|) \qquad (i \in \mathbb{N}),$$

which by $x \in c_0(F)$ shows that $\lim_{i\to\infty} f_{k_i}(t_0) = 0$, contrary to $F \in \Phi_8$. Consequently, $x \in m$ and the inclusion $c_0(F) \subset m$ holds. □

**THEOREM 5.** *The inclusion $m(F) \subset c_0$ holds if and only if $F \in \Phi_9$.*

Proof.

*Necessity.* Let $m(F) \subset c_0$. Assuming that $F \notin \Phi_9$, we can find numbers $t_0 > 0$, $M > 0$ and an index sequence $(k_i)$ such that $f_{k_i}(t_0) \leq M$ $(i \in \mathbb{N})$. So the sequence $x = (x_k)$, where

$$x_k = \begin{cases} t_0 & \text{for } k = k_i \ (i \in \mathbb{N}), \\ 0 & \text{otherwise}, \end{cases}$$

belongs to $m(F)$. But $x \notin c_0$. Consequently, $F \in \Phi_9$ is necessary for $m(F) \subset c_0$.

*Sufficiency.* Let $F \in \Phi_9$ and let $x = (x_k)$ belongs to $m(F)$. If $x \notin c_0$, there exist a number $\varepsilon_0 > 0$ and an index sequence $(k_i)$ such that $|x_{k_i}| \geq \varepsilon_0$ $(i \in \mathbb{N})$. Now, since the $\varphi$-functions are non-decreasing, by $x \in m(F)$ we have, for some $M > 0$,

$$f_{k_i}(\varepsilon_0) \leq f_{k_i}(|x_{k_i}|) \leq M \qquad (i \in \mathbb{N}),$$

contrary to $F \in \Phi_9$. Hence $x \in c_0$, proving $m(F) \subset c_0$. □

**THEOREM 6.** *The following statements are equivalent:*

(a) $c_0(F) \subset c_0$;
(b) $\ell_p(F) \subset c_0$;
(c) $F \in \Phi_{10}$.

Proof.

(a) $\Longrightarrow$ (b) is clear.

(b) $\Longrightarrow$ (c). Let $\ell_p(F) \subset c_0$. If $F \notin \Phi_{10}$, there exists a number $t_0 > 0$ such that $\inf_{k\in\mathbb{N}} f_k(t) = 0$ for all $t \leq t_0$. Thus, letting $t_i = t_0 i/(i+1)$, by induction we can choose an index sequence $(k_i)$ such that

$$f_{k_i}(t_i) \leq 2^{-i/p} \qquad (i \in \mathbb{N}).$$

Now, if $x = (x_k)$, where

$$x_k = \begin{cases} t_i & \text{for } k = k_i \ (i \in \mathbb{N}), \\ 0 & \text{otherwise,} \end{cases}$$

then $x \in \ell_p(F)$. But by $\lim\limits_{i\to\infty} x_{k_i} = \lim\limits_{i\to\infty} t_i = t_0 > 0$ we have $x \notin c_0$, which contradicts $\ell_p(F) \subset c_0$. So $F$ must be in $\Phi_{10}$.

(c) $\Longrightarrow$ (a). Let $F \in \Phi_{10}$ and let $x = (x_k)$ belongs to $c_0(F)$. If we suppose, that $x \notin c_0$, then there exist a number $\varepsilon_0 > 0$ and an index sequence $(k_i)$ such that $|x_{k_i}| \geq \varepsilon_0$ $(i \in \mathbb{N})$. This yields

$$0 < f_{k_i}(\varepsilon_0) \leq f_{k_i}\big(|x_{k_i}|\big) \qquad (i \in \mathbb{N}),$$

and by $x \in c_0(F)$ we have $\lim\limits_{i\to\infty} f_{k_i}(\varepsilon_0) = 0$, contrary to $F \in \Phi_{10}$. Hence $x$ must belong to $c_0$. Consequently, $c_0(F) \subset c_0$. □

## 4. The sets $\lambda^\rho(F)$, $\lambda^\exists(F)$ and $\lambda^\forall(F)$

Let $F = (f_k)$ be a sequence of $\varphi$-functions and $\lambda, \mu \in \{m, c_0, \ell_p\}$. For a fixed number $\rho > 0$ we consider a new sequence of $\varphi$-functions $F^\rho = (f_k^\rho)$, where

$$f_k^\rho(t) = f_k(t/\rho) \qquad (k \in \mathbb{N}).$$

It is not difficult to see that $\lambda^\rho(F) = \lambda(F^\rho)$ and

$$F^\rho \in \Phi_i \iff F \in \Phi_i \qquad (i = 0, 1, 2, \dots, 10).$$

Thus

$$\mu \subset \lambda(F) \iff \mu \subset \lambda^\rho(F), \qquad \lambda(F) \subset \mu \iff \lambda^\rho(F) \subset \mu \tag{7}$$

and, therefore, all our Theorems 1–6 remain true if there $\lambda(F)$ is replaced by $\lambda^\rho(F)$.

Further, because of (1) it is clear that for a sequence of $\varphi$-functions $F = (f_k)$ we have

$$\lambda \subset \mu^\forall(F) \implies \lambda \subset \mu(F), \qquad \lambda^\exists(F) \subset \mu \implies \lambda(F) \subset \mu.$$

It turns out that these implications are reversible.

**THEOREM 7.** *For a sequence of $\varphi$-functions $F = (f_k)$ and a pair of sequence spaces $\lambda, \mu$ we have*

$$\lambda \subset \mu^\forall(F) \iff \lambda \subset \mu(F), \qquad \lambda^\exists(F) \subset \mu \iff \lambda(F) \subset \mu.$$

Proof. It suffices to prove that

$$\mu \subset \lambda(F) \implies \mu \subset \lambda^{\forall}(F)\,, \qquad \lambda(F) \subset \mu \implies \lambda^{\exists}(F) \subset \mu\,.$$

But these implications follow immediately from the equalities $\lambda^{\forall}(F) = \bigcap_{\rho>0} \lambda^{\rho}(F)$, $\lambda^{\exists}(F) = \bigcup_{\rho>0} \lambda^{\rho}(F)$ because of the fact that $\lambda$ and $\mu$ as vector spaces contain together with an element $x$ also the element $x/\rho$, and conversely. □

The equivalences (7) and Theorem 7 show that we can give extended versions of all Theorems 1–6, replacing there $\lambda(F)$ by $\lambda^{\rho}(F)$ and adding to each statement of the type $\mu \subset \lambda^{\rho}(F)$ or $\lambda^{\rho}(F) \subset \mu$ the equivalent statement $\mu \subset \lambda^{\forall}(F)$ or $\lambda^{\exists}(F) \subset \mu$, respectively. Here we formulate extended versions of Theorems 2 and 6 only.

**THEOREM 8.** *Let $1 \le p < \infty$ and $\rho > 0$. The following statements are equivalent:*

(a) $c_0 \subset m^{\rho}(F)$;
(b) $c_0 \subset m^{\forall}(F)$;
(c) $\ell_p \subset m^{\rho}(F)$;
(d) $\ell_p \subset m^{\forall}(F)$;
(e) $F \in \Phi_6$.

**THEOREM 9.** *Let $1 \le p < \infty$ and $\rho > 0$. The following statements are equivalent:*

(a) $c_0^{\exists}(F) \subset c_0$;
(b) $c_0^{\rho}(F) \subset c_0$;
(c) $\ell_p^{\exists}(F) \subset c_0$;
(d) $\ell_p^{\rho}(F) \subset c_0$;
(e) $F \in \Phi_{10}$.

## 5. Some consequences

First let $F = (f_k)$ be a constant sequence of $\varphi$-functions, i.e., $f_k = f$ ($k \in \mathbb{N}$). In this case we write $\lambda(f)$ instead of $\lambda(F)$, and $f \in \Phi_i$ instead of $F \in \Phi_i$ for $i = 0, 1, 2, \dots, 10$. It is clear that for an arbitrary $\varphi$-function $f$ we have

$$f \notin \Phi_i \quad (i = 1, 2, 4, 9) \qquad \text{and} \qquad f \in \Phi_i \quad (i = 5, 6, 8, 10).$$

Moreover,

$$f \in \Phi_0 \iff (\exists\alpha>0)(\exists\delta>0)(\forall t\in[0,\delta])\big(f(t) \le \alpha t^{q/p}\big)\,,$$
$$f \in \Phi_3 \iff \lim_{t\to 0+} f(t) = 0\,,$$
$$f \in \Phi_7 \iff \lim_{t\to\infty} f(t) = \infty\,.$$

Thus our results permit to formulate:

**COROLLARY 1.** *Let $f$ be a $\varphi$-function, $1 \le p,q < \infty$ and $\rho > 0$. The following statements are true:*

(1) $\ell_q \subset \ell_p^{\forall}(f) \iff \ell_q \subset \ell_p^{\rho}(f)$

$$\iff (\exists\alpha>0)(\exists\delta>0)(\forall t\in[0,\delta])\big(f(t) \le \alpha t^{q/p}\big)\,;$$

(2) $c_0^{\exists}(f) \subset c_0$;

(3) $c_0 \subset c_0^{\forall}(f) \iff c_0 = c_0^{\forall}(f) = c_0^{\rho}(f) = c_0^{\exists}(f) \iff \lim\limits_{t\to 0+} f(t) = 0$;

(4) $m \subset m^{\forall}(f)$;

(5) $m^{\exists}(f) \subset m \iff m^{\forall}(f) = m^{\rho}(f) = m^{\exists}(f) = m \iff \lim\limits_{t\to\infty} f(t) = \infty$.

It should be noted that the inclusion $m \subset m(f)$ and the equivalences

$$\ell_q \subset \ell_p(f) \iff (\exists\alpha > 0)(\exists\delta > 0)(\forall t\in[0,\delta])\big(f(t) \le \alpha t^{q/p}\big)\,,$$
$$c_0 \subset c_0(f) \iff \lim_{t\to 0+} f(t) = 0$$

follow also from the corresponding results of Grinnell [8] because of $\lambda(f) = \lambda_{\bar f}$.

As an example of non-constant sequence of $\varphi$-functions we consider the sequence $F^{(r)} = \big(f_k^{(r)}\big)$ of $\varphi$-functions $f_k^{(r)}(t) = t^{r_k}$, where $r = (r_k)$ is a bounded sequence of positive numbers, i.e.,

$$0 < r_k \le \sup_{k\in\mathbb{N}} r_k = R < \infty\,.$$

For $F = F^{(r)}$ the sequence spaces $m(F)$, $c_0(F)$ and $\ell(F)$ are the sequence spaces of Maddox (see, for example, [9])

$$m(r) = \Big\{x = (x_k):\ \sup_{k\in\mathbb{N}} |x_k|^{r_k} < \infty\Big\}\,,$$
$$c_0(r) = \Big\{x = (x_k):\ \lim_{k\to\infty} |x_k|^{r_k} = 0\Big\}\,,$$
$$\ell(r) = \Big\{x = (x_k):\ \sum_{k=1}^{\infty} |x_k|^{r_k} < \infty\Big\}\,,$$

respectively. Since the functions $f_k^{(r/s)}(t) = t^{r_k/s}$ $(k \in \mathbb{N})$ with $s = \max\{1, R\}$ are moduli, and for $\rho > 0$ we have

$$m^{\rho}\big(F^{(r)}\big) = m^{\rho}\big(F^{(r/s)}\big)\,,\quad c_0^{\rho}\big(F^{(r)}\big) = c_0^{\rho}\big(F^{(r/s)}\big)\,,\quad \ell\big(F^{(r)}\big) = \ell_s^{\rho}\big(F^{(r/s)}\big)\,,$$

the equalities (2) hold if $F = F^{(r)}$ and $\lambda \in \{m, c_0, \ell\}$.

To apply our theorems for sequence spaces of Maddox, we must describe the classes of sequences $r = (r_k)$ with $F^{(r/s)} \in \Phi_0$ (for $p = s$) and $F^{(r)} \in \Phi_i$ for $i = 1, 2, \dots, 10$. By

$$\min\{1, t^R\} \le t^{r_k} \le \max\{1, t^R\}$$

it is easy to see that for any $r = (r_k)$ we have

$$F^{(r)} \in \Phi_i \quad (i = 5, 6, 8, 10) \qquad \text{and} \qquad F^{(r)} \notin \Phi_i \quad (i = 1, 2, 4, 9).$$

Further, from the definitions of the sets $\Phi_0$ and $\Phi_3$ it follows that

$$F^{(r/s)} \in \Phi_0 \iff r \in \mathcal{R}_0^q \qquad \text{and} \qquad F^{(r)} \in \Phi_3 \iff r \in \mathcal{R}_1,$$

where

$$\mathcal{R}_0^q = \Big\{ r = (r_k) : \ (\exists (a_k) \in \ell^+)(\exists k_0 \in \mathbb{N})(\exists b \ge 0)(\exists \delta > 0) \\ (\forall k \ge k_0)(\forall t \in [0, \delta])(t^{r_k} \le a_k + bt^q) \Big\},$$

$$\mathcal{R}_1 = \Big\{ r = (r_k) : \ \inf_{k \in \mathbb{N}} r_k > 0 \Big\}.$$

We claim that the $\varphi$-function sequences $F^{(r)}$ from $\Phi_7$ are also characterized by $r \in \mathcal{R}_1$. Indeed, for $t \ge 1$ and $k_0 \in \mathbb{N}$ we have

$$\sup_{n \ge k_0} \inf_{k \ge n} t^{r_k} = t^{\sup\limits_{n \ge k_0} \inf\limits_{k \ge n} r_k},$$

which gives that $F^{(r)} \in \Phi_7$ if and only if

$$(\exists k_0 \in \mathbb{N}) \Big( \sup_{n \ge k_0} \inf_{k \ge n} r_k > 0 \Big). \tag{8}$$

It is clear that $\inf_{k \in \mathbb{N}} r_k > 0$ yield (8). Conversely, let (8) be true. If $r \notin \mathcal{R}_1$, then for some index sequence $(k_i)$ we have $\lim_{i \to \infty} r_{k_i} = 0$, contrary to (8).

Consequently, from Theorems 1, 3 and 6 we get:

**Corollary 2.** *Let $1 \le q \le \infty$ and let $r = (r_k)$ be a bounded sequence of positive numbers. Then*

(1) $\ell_q \subset \ell(r) \iff r \in \mathcal{R}_0^q$;
(2) $\ell_q \subset c_0(r) \iff r \in \mathcal{R}_1$;
(3) $c_0(r) \subset c_0$ & $m \subset m(r)$;
(4) $c_0(r) = c_0 \iff m(r) = m \iff r \in \mathcal{R}_1$.

Corollary 2 shows that $\ell \subset \ell(r)$ if and only if $r \in \mathcal{R}_0^1$. A different necessary and sufficient condition for the inclusion $\ell \subset \ell(r)$ is contained in a (more general) result of Maddox (see [16; Theorem 1]).

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