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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATORY BEHAVIOUR OF SOLUTIONS OF A FORCED NONLINEAR NEUTRAL EQUATION OF FIRST ORDER WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. In this paper necessary and sufficient conditions are obtained so that every nonoscillatory solution of

$$\big(y(t) - p(t)y(t-\tau)\big)' + Q(t)G\big(y(t-\sigma)\big) - R(t)G\big(y(t-\alpha)\big) = f(t)$$

tends to zero or to ∞ as $t \to \infty$, where $p, f \in C([0,\infty),\mathbb{R})$, $Q, R \in C([0,\infty),[0,\infty))$, $G \in C(\mathbb{R},\mathbb{R})$, $\tau, \sigma, \alpha \geq 0$. $p(t)$ is considered in various ranges and the nonlinear function G could be linear, sublinear, or super linear. This work indicates that the non-linearity of G depends on α and σ. The results also hold when $f(t) \equiv 0$. This paper improves and generalizes some recent results. (See [DAS, P.—Misra, N.: *A necessary and sufficient condition for the solution of a functional differential equation to be oscillatory or tend to zero*, J. Math. Anal. Appl. **204** (1997), 78–87], [Parhi, N.—Chand, S.: *On forced first order neutral differential equations with positive and negative coefficients*, Math. Slovaca **50** (2000), 81–94], [Parhi, N.—Rath, R. N.: *On oscillation criteria for a forced neutral differential equation*, Bull. Inst. Math. Acad. Sinica **28** (2000), 59–70], [Parhi, N.—Rath, R. N.: *Oscillation criteria for forced first order neutral differential equations with variable coefficients*, J. Math. Anal. Appl. **256** (2001), 525–541]).

1. Introduction

In the present work the first order forced nonlinear neutral delay differential equation (NDDE)

$$\big(y(t) - p(t)y(t-\tau)\big)' + Q(t)G\big(y(t-\sigma)\big) - R(t)G\big(y(t-\alpha)\big) = f(t) \tag{E}$$

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is considered, where τ, σ, α are positive real numbers, $p, f \in C([0,\infty),\mathbb{R})$, $Q, R \in C([0,\infty),[0,\infty))$, $G \in C(\mathbb{R},\mathbb{R})$, G is nondecreasing with $xG(x) > 0$ for $x \neq 0$ and there exists a bounded function $F \in C^{(1)}([0,\infty),\mathbb{R})$ such that $F'(t) = f(t)$. Then it is proved that every nonoscillatory solution of (E) tends either to zero or to ∞ if and only if the primary assumption

(H_1) $\int\limits_{t_0}^{\infty} Q(t)\,\mathrm{d}t = \infty$

is satisfied. Further, the following assumptions are needed in the sequel:

(H_2) $\int\limits_{t_0}^{\infty} R(t)\,\mathrm{d}t < \infty$,

(H_3) $Q(t) > R(t-\sigma+\alpha)$,

(H_4) $\lim\limits_{t\to\infty} F(t) = 0$,

(H_5) $\liminf\limits_{|u|\to\infty} \frac{G(u)}{u} \leq \beta$ where $\beta > 0$,

(H_6) G is Lipschitzian in every interval of the form $[a,b]$ where $0 < a < b$.

Here the following ranges of $p(t)$ are considered:

(A_1) $0 \leq p(t) \leq p < 1$,

(A_2) $-1 < -p \leq p(t) \leq 0$,

(A_3) $-p_2 \leq p(t) \leq -p_1 < -1$,

(A_4) $1 < p_1 \leq p(t) \leq p_2$,

(A_5) $1 \leq p(t) \leq p_2$,

(A_6) $-p_2 \leq p(t) \leq 0$,

(A_7) $0 \leq p(t) \leq p_2$,

where p, p_1, p_2 are positive real numbers.

During the last decade there has been extensive research about the oscillatory behaviour of solutions of neutral delay differential equations (briefly NDDEs) which are defined as differential equations in which the highest order derivative of the unknown function appears with and without delays. The study of oscillatory behaviour of solutions of NDDEs has also manifold importance in applications (see [3; p. 111]). The first order NDDE

$$\big(y(t) - p(t)y(t-\tau)\big)' + Q(t)G\big(y(t-\sigma)\big) = f(t) \tag{1}$$

with different symbols having their usual meaning is studied by many authors (see [5], [6]). If in the equation (1), $p(t)$ is differentiable and $p'(t) < 0$, then (1) reduces to a first order NDDE with positive and negative coefficients. Such equations are studied by many authors (see [1], [4], [7], [8]). But our interest in this paper centres around [4], where the authors have considered

$$\left(y(t)+\sum_{i=1}^{\ell} p_i(t)y(t-\tau_i)\right)' + \sum_{j=1}^{m} Q_j(t)y(t-\sigma_j) - \sum_{k=1}^{n} R_k(t)y(t-\alpha_k) = f(t) \quad (2)$$

and obtained the sufficient conditions for all nonoscillatory solutions of (2) to tend to zero as $t \to \infty$. Most of the results in [4] hold only for bounded solutions of (2). In literature, [4] seems to be the only paper with a forced term so far as study of first order NDDE with positive and negative coefficients are concerned (see [1]–[8]). The purpose of this paper is to improve all the results of [4] by removing not only the boundedness conditions assumed on the solution but also relaxing other conditions as well. Moreover, our motive is to have a systematic and complete study of oscillatory and asymptotic behaviour of solutions of the equation (E) and generalize [4] to nonlinear NDDEs with positive and negative coefficients, which is not yet studied. For simplicity and clarity we take $\ell = m = n = 1$ in the equation (2) and get the results for the equation (E). But our method and technique does not fail to deliver the results when applied to the equation (2). The nonlinear function G we used could be linear, sublinear or super linear. The results of this paper indicate that the nonlinearity of G depends on α and σ. But when G is linear we are able to remove the restriction $\sigma > \alpha$ which most authors assume in their work (see [1], [4], [7], [8]). Above all, our results hold when $f(t) \equiv 0$, and we show that our condition is both necessary and sufficient, which is rare in the literature, as rightly remarked by the authors in [2].

By a *solution of* (E) we mean a real valued continuous function y on $[t_y-\rho, \infty)$ such that $y(t) - p(t)y(t-\tau)$ is once continuously differentiable for $t \geq t_y$ and (E) is satisfied identically for $t \geq t_y$ where $\rho = \max\{\tau, \sigma, \alpha\}$. A solution of (E) is said to be *oscillatory* if and only if it has arbitrarily large zeros. Otherwise it is said to be nonoscillatory.

2. Main results

First we state a lemma found in [3; p. 19].

LEMMA 2.1. *Let $u, \nu, p\colon [0,\infty) \to \mathbb{R}$ be such that*

$$u(t) = \nu(t) - p(t)\nu(t-c), \qquad t \geq c,$$

where $c \geq 0$. Suppose that $p(t)$ is in one of the ranges (A_2), (A_3) or (A_7). If $\nu(t) > 0$ for $t \geq 0$ and $\liminf_{t\to\infty} \nu(t) = 0$ and $\lim_{t\to\infty} u(t) = L \in \mathbb{R}$ exists, then $L = 0$.

Remark 1. Since the ranges (A_1), (A_4) and (A_5) are contained in (A_7), the above lemma also holds when $p(t)$ lies in one of the ranges (A_1), (A_4), or (A_5).

THEOREM 2.2. *Suppose that $\sigma > \alpha$ and $p(t)$ lies in one of the ranges (A_1), (A_2), or (A_3). Let (H_1)–(H_5) hold. Then every solution of the equation* (E) *oscillates or tends to zero as $t \to \infty$.*

P r o o f . Let $y(t)$ be any nonoscillatory positive solution of the equation (E) on $[t_y, \infty)$. For $t \geq t_0 = t_y + \rho$, we set

$$z(t) = y(t) - p(t)y(t-\tau) \tag{3}$$

and

$$w(t) = z(t) - \int_{t-\sigma+\alpha}^{t} R(s)G\big(y(s-\alpha)\big)\,\mathrm{d}s - F(t)\,. \tag{4}$$

Thus for $t \geq t_0$,

$$w'(t) = -G\big(y(t-\sigma)\big)\big\{Q(t) - R(t-\sigma+\alpha)\big\} \leq 0\,. \tag{5}$$

Hence $w(t) > 0$ or $w(t) < 0$ for $t \geq t_1 \geq t_0$ and $\lim\limits_{t\to\infty} w(t) = \ell$, where $-\infty \leq \ell < \infty$. We claim that $y(t)$ is bounded. Otherwise, there exists a sequence $\{T_n\}_{n=1}^{\infty}$ such that

$$\begin{gathered} T_n \to \infty\,, \quad y(T_n) \to \infty \qquad \text{as} \quad n \to \infty \\ \text{and} \\ y(T_n) = \max\big\{y(s) :\ t_1 \leq s \leq T_n\big\}\,. \end{gathered} \tag{6}$$

We may choose n sufficiently large such that $T_n - \rho > t_1$. Suppose that $p(t)$ satisfies (A_1). Then using (H_2), (H_4) and (H_5) we obtain for $0 < \varepsilon < (1-p)/\beta$

$$\begin{aligned} w(T_n) &\geq y(T_n)(1-p) - \varepsilon G\big(y(T_n)\big) - \varepsilon \\ &\geq y(T_n)\{1 - p - \beta\varepsilon\} - \varepsilon\,. \end{aligned} \tag{7}$$

Taking limit $n \to \infty$, we find $\ell = \infty$, which is a contradiction. If $p(t)$ satisfies (A_2) or (A_3), then we choose $0 < \varepsilon < \frac{1}{\beta}$ and use (H_2), (H_4) and (H_5) to obtain the following inequality in place of (7).

$$w(T_n) \geq y(T_n)\{1 - \beta\varepsilon\} - \varepsilon\,. \tag{7'}$$

Taking $n \to \infty$ in (7′), we get $\ell = \infty$, a contradiction. Hence our claim holds, and $-\infty < \ell < \infty$. Consequently $\lim\limits_{t\to\infty} z(t) = \ell$ by (H_2) and (H_4). It may be noted that if (H_2) and (H_3) hold, then (H_1) is equivalent to the condition

$$\int_{t_0}^{\infty} \big\{Q(s) - R(s-\sigma+\alpha)\big\}\,\mathrm{d}s = \infty\,. \tag{8}$$

Next we claim $\liminf\limits_{t\to\infty} y(t) = 0$. Otherwise, for large $t \geq t_2$, $y(t) > m > 0$, and since G is nondecreasing,

$$\int_{t_2}^{\infty} G\big(y(t-\sigma)\big)\big\{Q(t) - R(t-\sigma+\alpha)\big\}\ dt = \infty$$

by (8). However, integrating (5) between t_2 and ∞ we obtain

$$\int_{t_2}^{\infty} G\big(y(t-\sigma)\big)\big\{Q(t) - R(t-\sigma+\alpha)\big\}\ dt < \infty\,,$$

which is a contradiction. Hence our claim holds. Application of Lemma 2.1 yields $\ell = 0$. If $p(t)$ satisfies (A_1), then

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}\big[y(t) - p(t)y(t-\tau)\big] \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\big[-py(t-\tau)\big] \\ &\geq (1-p)\limsup_{t\to\infty} y(t)\,, \end{aligned}$$

which implies $\limsup\limits_{t\to\infty} y(t) \leq 0$. Hence $\lim\limits_{t\to\infty} y(t) = 0$. If $p(t)$ satisfies (A_2) or (A_3), then, since $y(t) \leq z(t)$, we get $\limsup\limits_{t\to\infty} y(t) \leq 0$. Consequently, $\lim\limits_{t\to\infty} y(t) = 0$. The proof for the case $y(t) < 0$ for $t \geq t_y$ is similar. Thus the theorem is completely proved. □

From the proof of the above theorem we obtain the following result:

Corollary 2.3. *Suppose all the conditions of Theorem* 2.2 *hold except* (H_4). *Then every unbounded solution of* (E) *oscillates.*

Remark 2. Theorem 2.2 holds when G is linear and sublinear. We say that G is *sublinear* if G satisfies $\int_0^k \frac{du}{G(u)} < \infty$ or $\int_{-k}^{\infty} \frac{du}{G(u)} < \infty$ for $k > 0$. The prototype of G satisfying the hypothesis of Theorem 2.2 is $G(u) = |u|^r \operatorname{sgn} u$ for $r \leq 1$.

Remark 3. (H_4) is equivalent to the condition

$$\lim_{t\to\infty} F(t) = M \in \mathbb{R},$$

because one may replace F by F_1 where

$$F_1(t) = F(t) - M \qquad \text{so that} \qquad \lim_{t\to\infty} F_1(t) = 0 \quad \text{and} \quad F_1'(t) = f(t)\,.$$

Further, (H_4) is again equivalent to the condition

$$\left|\int_0^\infty f(t)\,\mathrm{d}t\right| < \infty .$$

Remark 4. If $Q(t)$ is monotonic increasing, then (H_1) holds.

Remark 5. In view of Remark 3 and 4, Theorem 2.2 improves and generalize [4; Theorems 2, 3, 5].

Remark 6. In Theorem 2.2, the assumption $\sigma > \alpha$ forces us to assume (H_5), which holds for linear and sublinear G. But (H_5) fails to hold for super linear G satisfying $\int_k^\infty \frac{\mathrm{d}u}{G(u)} < \infty$ or $\int_{-\infty}^{-k} \frac{\mathrm{d}u}{G(u)} < \infty$ for $k > 0$, that is for example when $G(u) = |u|^r \operatorname{sgn} u$ with $r > 1$.

The case $\sigma < \alpha$ is considered in the following theorem where (H_5) is not required. Thus the following theorem holds for all types of G.

THEOREM 2.4. *Suppose that $\sigma < \alpha$ and $p(t)$ satisfies one of the ranges (A_1), (A_2), or (A_3). Let (H_1)–(H_4) hold. Then every solution of the equation* (E) *oscillates or tends to zero as $t \to \infty$.*

Proof. We proceed as in the proof of Theorem 2.2 and obtain (3), (4), (5) and (6). Then using (H_2), (H_4) for the case when $p(t)$ is in the range (A_1), we obtain the following inequality in place of (7).

$$\begin{aligned} w(T_n) &\ge y(T_n)\{1-p\} - \int_{T_n-\sigma+\alpha}^{T_n} R(s)G\big(y(s-\alpha)\big)\,\mathrm{d}s - F(t_n) \\ &\ge y(T_n)\{1-p\} - \varepsilon . \end{aligned} \tag{7''}$$

Taking $n \to \infty$ in $(7'')$, we obtain $w(T_n) \to \infty$, hence $\ell = \infty$, which is a contradiction. If $p(t)$ is as in (A_2) or (A_3), then $w(T_n) \ge y(T_n) - \varepsilon$, which implies $\ell = \infty$, which is a contradiction. Hence $y(t)$ is bounded. The rest of the proof is similar to that of Theorem 2.2. □

Remark 7. In view of Remarks 3 and 4, Theorem 2.2 and Theorem 2.4 improve and generalize [4; Theorems 2, 3, 4]. Moreover Theorem 2.4 extends and generalizes the work of [2], [5], [6].

Remark 8. We assumed $\sigma > \alpha$ in Theorem 2.2 and $\sigma < \alpha$ in Theorem 2.4. If $\sigma = \alpha$, then the equation (E) reduces to the NDDE (1), which is a particular case of the equation (E). Hence the assumption $\sigma > \alpha$ in the entire work [4] is not required.

THEOREM 2.5. *Suppose that* $p(t)$ *is as in* (A_1) *and* (H_2), (H_4) *and* (H_6) *hold. If every solution of the equation* (E) *oscillates or tends to zero as* $t \to \infty$, *then* (H_1) *holds.*

Proof. Assume

$$\int_{t_2}^{\infty} Q(t)\,\mathrm{d}t < \infty. \tag{9}$$

By (9), (H_2) and (H_4), we find $t_1 > 0$ such that $t \geq t_1$ implies

$$K\int_t^{\infty} Q(t)\,\mathrm{d}t < \frac{1-p}{20}, \qquad K\int_t^{\infty} R(t)\,\mathrm{d}t < \frac{1-p}{20}, \qquad |F(t)| < \frac{1-p}{20},$$

where $K = \max\{k_1, G(1)\}$, k_1 is the Lipschitz constant of G in $\left[\frac{1-p}{10}, 1\right]$. Let $X = \left\{u \in BC([t_1,\infty),\mathbb{R}) : \frac{1-p}{10} \leq u(t) \leq 1\right\}$. (X,d) is a complete metric space with respect to the metric $d(u,\nu) = \sup\{|u(t)-\nu(t)| : t \geq t_1\}$. For $y \in X$, we define

$$T_y(t) = \begin{cases} T_y(t_1+\rho) & \text{for } t \in [t_1, t_1+\rho], \\ p(t)y(t-\tau) + \int_t^{\infty} Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s & \\ \quad - \int_t^{\infty} R(s)G\big(y(s-\alpha)\big)\,\mathrm{d}s + F(t) + \frac{1-p}{5} & \text{for } t \geq t_1+\rho. \end{cases}$$

Clearly T maps X into X and $d(Tu,T\nu) \leq \left(\frac{1+9p}{10}\right)d(u,\nu)$. Hence T is a contraction and by Banach Contraction Principle (see [3; p. 30]), T admits a fixed point, which is required positive solution of the equation (E) giving us a contradiction. Thus the theorem is proved. □

COROLLARY 2.6. *Suppose that* $p(t)$ *is as in* (A_1) *and* $\sigma < \alpha$. *If* (H_2)–(H_4) *and* (H_6) *hold, then* (H_1) *is both a necessary and sufficient condition for every solution of the equation* (E) *to be oscillatory or tending to zero as* $t \to \infty$.

This follows from Theorem 2.4 and 2.5.

Remark 9. The above Corollary generalizes/extends/improves the entire work in [2], where a sublinear condition is imposed on G.

Remark 10. When $p(t)$ is in the ranges (A_2) or (A_3), a theorem similar to Theorem 2.5 can be developed and we can find positive solutions of the equation (E). A result similar to Corollary 2.6 for the ranges (A_2) and (A_3) may also be stated.

EXAMPLE.

$$\big(y(t) - py(t-\tau)\big)' + \big(p\,\mathrm{e}^{-\tau} - 1\big)\,\mathrm{e}^{\sigma}\,y(t-\sigma) + \mathrm{e}^{-2t}\,y(t-\alpha) = \mathrm{e}^{-t-\alpha} \qquad \text{for} \quad t > \max\{\alpha, \sigma\}, \tag{10}$$

where $p > 2\,\mathrm{e}^{\tau}$. Here in (10) all the conditions of Theorem 2.4 are satisfied except the fact that $p(t)$ lies in the range (A_4) and we see that (10) has a unbounded non-oscillatory solution $y(t) = \mathrm{e}^{t}$ tending to ∞ as $t \to \infty$.

The above example is a source of motivation for our next result.

THEOREM 2.7. *Suppose that $p(t)$ satisfies (A_4). Let (H_1)–(H_4) hold. Then every bounded solution of the equation* (E) *oscillates or tends to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 2.2 and noting that $0 = \lim\limits_{t\to\infty} z(t) \le (1-p_1)\limsup\limits_{t\to\infty} y(t)$, we get the desired result. □

THEOREM 2.8. *Suppose that $p(t)$ is as in (A_4) and (H_2), (H_4) and (H_6) hold. If every bounded solution of* (E) *oscillates or tends to zero as $t \to \infty$, then (H_1) holds.*

Proof. The proof here is similar to that of Theorem 2.5 with the following changes:

$$K\int_t^{\infty} Q(t)\,\mathrm{d}t < \frac{p_1}{N+p_1}, \qquad K\int_t^{\infty} R(t)\,\mathrm{d}t < \frac{p_1}{N+p_1}, \qquad |F(t)| < \frac{p_1}{N+p_1},$$

where $N > \max\left\{\frac{p_1(3-p_1)}{(p_1-1)}, \frac{3p_1+p_1p_2+2p_2-p_1^2}{p_1-1}\right\}$.

Suppose $L = \frac{N+3p_1+p_1p_2+2p_2}{p_1(N+p_1)}$. It is clear that $L < 1$. Let

$$X = \left\{u \in BC\big([t_1,\infty),\mathbb{R}\big) : \tfrac{p_1}{N+p_1} \le u(t) \le L\right\}.$$

Let $\lambda = \frac{(p_1+2)p_2}{(N+p_1)}$. Define for $y \in X$,

$$T_y(t) = \begin{cases} T_y(t_1+\rho) & \text{for } t \in [t_1, t_1+\rho], \\[4pt] \frac{1}{p(t+\tau)}y(t+\tau) - \frac{1}{p(t+\tau)}\int\limits_{t+\tau}^{\infty} Q(s)G\big(y(s-\sigma)\big)\,\mathrm{d}s & \\ \quad + \frac{1}{p(t+\tau)}\int\limits_{t+\tau}^{\infty} R(s)G\big(y(s-\alpha)\big)\,\mathrm{d}s & \\ \quad - \frac{F(t+\tau)}{p(t+\tau)} + \frac{\lambda}{p(t+\tau)}, & \text{for } t \ge t_1+\rho \end{cases}$$

$\|Tu - T\nu\| \le \mu\,\|u-\nu\|$ where $\mu = \frac{N+3p_1}{p_1(N+p_1)} < 1$. Hence the theorem is proved. □

COROLLARY 2.9. *Suppose that $p(t)$ satisfies (A_4). Let (H_2), (H_3), (H_4) and (H_6) hold. Then every bounded solution of the equation* (E) *oscillates or tends to zero as $t \to \infty$ if and only if (H_1) holds.*

The proof follows from Theorems 2.7 and 2.8.

THEOREM 2.10. *Suppose that $\sigma > \alpha$, $p(t)$ is as in (A_5). Assume (H_1)-(H_3), then every bounded solution of*

$$\big(y(t) - p(t)y(t-\tau)\big)' + Q(t)G\big(y(t-\sigma)\big) - R(t)G\big(y(t-\alpha)\big) = 0 \tag{11}$$

oscillates.

Proof. Let $y(t)$ be any bounded positive solution of (11) for $t \geq t_0$. Then proceeding as in the proof of Theorem 2.2 with $F(t) = 0$, we obtain

$$w'(t) = -G\big(y(t-\sigma)\big)\big\{Q(t) - R(t-\sigma+\alpha)\big\} \leq 0\,.$$

Hence $w(t) > 0$ or $w(t) < 0$ for $t \geq t_1 \geq t_0$. $\lim\limits_{t\to\infty} w(t) = \lim\limits_{t\to\infty} z(t) = \ell$, where $-\infty < \ell < \infty$. Then clearly $\lim\limits_{t\to\infty} y(t) = 0$ consequently, $\ell = 0$ by Lemma 2.1. As $z(t) > w(t) > 0$, therefore $\liminf\limits_{t\to\infty} y(t) > 0$, which is a contradiction. Hence the theorem is proved. □

THEOREM 2.11. *Suppose that $p(t)$ is as in (A_6) and $\alpha > \sigma$. Assume that (H_2) and (H_4) hold. Let $R(t)$ be monotonic decreasing. Suppose that*

$$\int\limits_{\rho}^{\infty} Q^*(t)\ \mathrm{d}t = \infty\,, \qquad \textit{where} \quad Q^*(t) = \min\big\{Q(t), Q(t-\tau)\big\}\,, \tag{H_7}$$

$$\begin{gathered} G(u)G(\nu) \geq G(u\nu)\,, \quad G(u) + G(\nu) \geq \delta G(u+\nu)\,, \quad G(-u) = -G(u) \\ \textit{for } u > 0\,, \ \nu > 0\,, \ \textit{where } \delta > 0 \textit{ is a constant,} \end{gathered} \tag{H_8}$$

$$Q^*(t) \geq R(t-\sigma+\alpha-\tau)\,. \tag{H_9}$$

Then every solution of the equation (E) *oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be an eventually positive solution of (E) for $t > t_y$. Then we proceed as in the proof of Theorem 2.4 and arrive at (3), (4) and (5). Consequently $\lim\limits_{t\to\infty} w(t) = \ell$, where $-\infty \leq \ell < \infty$. But $z(t) > 0$ and

$$z(t) - F(t) = w(t) - \int\limits_{t}^{t+\alpha-\sigma} R(s)G\big(y(s-\alpha)\big)\ \mathrm{d}s \leq w(t)$$

for large t. We claim $y(t)$ is bounded, otherwise $y(t)$ is unbounded implies $z(t)$ is unbounded. Hence there exists an increasing sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$, $z(t_n) \to \infty$ as $n \to \infty$ and $z(t_n) = \max\{z(t):\ t_1 \leq t \leq t_n\}$.

Thus

$$z(t_n) = w(t_n) + F(t_n) - \int_{t_n}^{t_n-\sigma+\alpha} R(s)G\big(y(s-\alpha)\big)\,\mathrm{d}s \\ \le w(t_n) + F(t_n)\,.$$

Thus $y(t)$ is bounded, which implies $\lim\limits_{t\to\infty} z(t) = \ell$ and $\ell < 0$ is not possible since $z(t) > 0$ for large t when $0 \le \ell < \infty$. If $\ell = 0$, then $\lim\limits_{t\to\infty} y(t) = 0$, and if $\ell > 0$, then for $t \ge t_2 > t_1$, $z(t) > \lambda > 0$. Using (5), monotonic nature of R, definition of $Q^*(t)$ and (H_7), one may obtain

$$\begin{aligned} 0 &= w'(t) + G\big(y(t-\sigma)\big)\big\{Q(t) - R(t-\sigma+\alpha)\big\} \\ &\quad + G\big(-p(t-\sigma)\big)w'(t-\tau) + G\big(-p(t-\sigma)\big)G\big(y(t-\sigma-\tau)\big)\times \\ &\qquad\qquad \big\{Q(t-\tau) - R(t-\sigma+\alpha-\tau)\big\} \\ &\ge w'(t) + G(p_2)w'(t-\tau) + \big\{Q^*(t) - R(t-\sigma-\tau+\alpha)\big\}G\big(y(t-\sigma)\big) \\ &\quad + G\big(-p(t-\sigma)y(t-\sigma-\tau)\big)\big\{Q^*(t) - R(t-\tau-\sigma+\alpha)\big\} \\ &\ge w'(t) + G(p_2)w'(t-\tau) + \big\{Q^*(t) - R(t-\tau-\sigma+\alpha)\big\}\big\{\delta G\big(z(t-\sigma)\big)\big\} \\ &\ge w'(t) + G(p_2)w'(t-\tau) + \delta G(\lambda)\big[Q^*(t) - R(t-\tau+\alpha-\sigma)\big]\,. \end{aligned}$$

Integrating the above inequality from t_2 to ∞ and using (H_2), (H_7), (H_9) we arrive at the contradiction $w(t) + G(p_2)w(t-\tau) < 0$ for large t. The proof for the case $y(t) < 0$ is similar. Thus the theorem is proved. □

Remark 11. The prototype of G satisfying (H_8) is

$$G(u) = \big(\beta + |u|^\lambda\big)|u|^\mu \operatorname{sgn} u\,, \qquad \text{where} \quad \lambda > 0\,, \ \ \mu > 0\,, \ \ \beta \ge 1\,.$$

Remark 12. In Theorem 2.10, if we assume $Q(t)$ is monotonously increasing in place of (H_8), then also the theorem holds because of Remark 4. Hence the above theorem substantially improves [4; Theorems 4, 6]. Also it may be noted that (H_7) implies (H_1).

3. Final comments

In this section before we close we give some comments which may be helpful for further research. In the equation (E), if we take $\alpha > \sigma$, then the nonlinear function G could be linear, sublinear or superlinear (see Theorem 2.4). However, if we take $\alpha < \sigma$, then G could be linear or sublinear (see Theorem 2.2). That means in the case $\alpha > \sigma$ the superlinear case of G needs further study, particularly when the solution is unbounded. The case when $\alpha = \sigma$ reduces the equation (E) to the equation (1) and that does not come under the purview of present discussion, but one may refer [5], [6] as regards to the behaviour of solutions of the equation (1). It is clear by this paper that when $G(u) = u$ that is for the linear case, the assumption $\alpha > \sigma$ or $\alpha < \sigma$ is not required though many authors have assumed $\alpha > \sigma$ in their work (see [1], [4], [7], [8]). It would be interesting if one removes the restriction (H_5) on G for $\alpha < \sigma$ case (see Theorem 2.2.). Moreover for the ranges (A_4), (A_5) and (A_7) of $p(t)$ one may study the oscillatory behaviour of unbounded solution of the equation (E) for further research.

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