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ZEROS OF CONTINUOUS FUNCTIONS AND THE COMPACT-OPEN TOPOLOGY

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ABSTRACT. We consider the space of all continuous real-valued functions equipped with the compact-open topology. The principal aim of this paper is the generalization of Theorem 2.1 from the paper [BALÁŽ, V.—ŠALÁT, T.: *Zeros of continuous functions and the structure of two function spaces*, Math. Slovaca **52** (2002), 397–408].

DEFINITIONS AND NOTATION. Let X be a Tychonoff (completely regular T_1) space, $C(X,\mathbb{R})$ or simply $C(X)$ be the set of all continuous functions on X to the set of all real numbers $\mathbb{R}$, let τ_{co} be the compact-open topology on $C(X)$. We define $C_0(X) = \{f \in C(X) :\ f^{-1}(\{0\}) \neq \emptyset\}$ and investigate the sets

$$H = \{f \in C(X) :\ f^{-1}(\{0\}) \text{ is perfect and nowhere dense}\},$$
$$A = \{f \in C(X) :\ f^{-1}(\{0\}) \text{ is not nowhere dense}\},$$
$$D = \{f \in C(X) :\ f^{-1}(\{0\}) \text{ is not perfect}\}.$$

If moreover X is second countable and $\mathfrak{B} = \{I_n :\ n \in \mathbb{N}\}$ is a countable base for the topology for X, then for each $n \in \mathbb{N}$ we put $A_n = \{f \in C(X) :\ I_n \subseteq f^{-1}(\{0\})\}$ and $D_n = \{f \in C(X) :\ (\exists!\, x_0 \in I_n)(f(x_0) = 0)\}$.

The definitions of all other terms are taken from Kelley [4] and Engelking [3].

Note 1.

(a) Observe that $A = \bigcup\limits_{n=1}^{\infty} A_n$ and $D = \bigcup\limits_{n=1}^{\infty} D_n$ if X is a second countable Tychonoff space.

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(b) If $f \in A_n^c$ (the complement of A_n), then there is $x \in I_n$ such that $f(x) \neq 0$, which implies that $f \in W(\{x\}, \mathbb{R}\setminus\{0\}) \subseteq A_n^c$, where $W(K,U) = \{g \in C(X) : g[K] \subseteq U\}$ for some $K \subseteq X$ compact and $U \subseteq \mathbb{R}$ open. Hence A_n^c is τ_{co}-open. Moreover the set $W(\{x\}, \mathbb{R}\setminus\{0\})$ is the preimage $P_x^{-1}[\mathbb{R}\setminus\{0\}]$ of the open set $\mathbb{R} \setminus \{0\}$ in the projection on the xth coordinate and so A_n^c is even open in the topology τ_p of pointwise convergence. Consequently each A_n is τ_p-closed and also τ_{co}-closed.

THEOREM 1. *If X is a second countable Tychonoff space, then A is of first category in $(C(X), \tau_{\text{co}})$.*

P r o o f. By Note 1(a) and 1(b) it is sufficient to show that the interior of each A_n is void. We show an equivalent statement that A_n^c is dense in $C(X)$. Let B be a τ_{co}-basic set and let $f \in B$. If $f \notin A_n$, then $f \in B \cap A_n^c \neq \emptyset$ and we are done. So suppose that $f \in A_n$. We know that on $C(X)$ the compact-open topology is the topology of uniform convergence on compacta ([4; Theorem 7.11]) and hence there is $\varepsilon > 0$ and a compact set $K \subseteq X$ such that $W(f,K,\varepsilon) \subseteq B$, where $W(f,K,\varepsilon) = \{g \in C(X) : (\forall x \in K)(|f(x) - g(x)| < \varepsilon)\}$. Define a function g on X by:

$$g(x) = f(x) + \varepsilon/2 \qquad \text{for each} \quad x \in X\,.$$

Clearly $g \in C(X,\mathbb{R})$ and $|g(x) - f(x)| = \varepsilon/2$ for each $x \in X$, which implies $g \in W(f,K,\varepsilon) \subseteq B$. On the other hand $f \in A_n$ and so for each $x \in I_n$ we have $g(x) = 0 + \varepsilon/2 = \varepsilon/2$, which means that $g \notin A_n$. Consequently $B \cap A_n^c \neq \emptyset$ and hence A_n^c is dense. □

Note 2.

(a) The construction of g used in the proof of Theorem 1 doesn't even need the space X to be Tychonoff, however, the complete regularity of X is required to obtain a "reasonable" class of continuous functions.

(b) From the construction of g we do not know whether g has a zero point at all, however, if $f \in A_n$, then we may choose two distinct points $x_1 \neq x_2$ in I_n (this can be done if X is without isolated points). Since X is a Tychonoff space, there is $h \in C(X,[0,1])$ such that $h(x_1) = 1$ and $h(x_2) = 0$. Let $g(x) = f(x) + \varepsilon/2 \cdot h(x)$ for each $x \in X$, i.e. clearly $g \in C(X)$. Moreover $0 \leq h(x) \leq 1$, that is $|g(x) - f(x)| = \varepsilon/2 \cdot h(x) \leq \varepsilon/2$ for each $x \in X$ and so $g \in W(f,K,\varepsilon) \subseteq B$. We also see that $g(x_1) = 0 + \varepsilon/2 = \varepsilon/2$, and so $g \notin A_n$. On the other hand $g(x_2) = 0 + 0 = 0$, and so $g \in C_0(X)$. Thus $g \in C_0(X) \cap B \cap A_n^c$. Such a function will be used in the later course of the paper.

Before investigating the situation for the set D we state two auxiliary facts.

PROPOSITION 1. *In a second countable locally connected space each open set has countably many components.*

Proof. Let $\mathfrak{B} = \{I_n : n \in \mathbb{N}\}$ be a countable base for the topology, U be an open set and let $\{C_\alpha : \alpha \in M\}$ be the family of all components of U. We know (see [4; Exercise 1.S]) that in a locally connected space every component of an open set is open. Therefore each C_α is open and hence for each $\alpha \in M$ we have $C_\alpha = \bigcup\{I_n : n \in N_\alpha\}$, where N_α is a nonempty subset of $\mathbb{N}$. Observe that if $n \in N_\alpha \cap N_\beta$, then $I_n \subseteq C_\alpha \cap C_\beta$, which implies that the set $C_\alpha \cup C_\beta$ is a connected set in U containing both C_α and C_β. Thus $N_\alpha \cap N_\beta = \emptyset$ whenever $\alpha \neq \beta$, i.e. the family $\{N_\alpha : \alpha \in M\}$ is a decomposition of $\mathbb{N}$ into $|M|$-many pairwise disjoint nonempty subsets. Since $\mathbb{N}$ is countable it is clear that M cannot be uncountable. Consequently U has countably many components. □

COROLLARY 2. *If X is a locally compact, locally connected separable metric space, then there is a countable base for the topology consisting of connected sets with compact closures.*

Proof. Every separable metric space is a second countable Tychonoff space and vice versa (see [4; Theorem 4.17]). Using Proposition 1 we construct a countable base consisting of connected sets. Finally, with local compactness at hand it is not hard to show that the subfamily consisting of connected sets with compact closures is still a countable base and we are done. □

THEOREM 2. *Let X be a locally compact, locally connected separable metric space without isolated points. Then D is of first category in $(C(X), \tau_{\mathrm{co}})$.*

Proof. By the preceding corollary, let $\mathfrak{B} = \{I_n : n \in \mathbb{N}\}$ be a countable base where each I_n is connected and $\overline{I_n}$ is compact. By Note 1(a) it is sufficient to show that each D_n is nowhere dense in $(C(X), \tau_{\mathrm{co}})$. Thus let $n \in \mathbb{N}$ be fixed, let B be an arbitrary τ_{co}-basic set and consider $f \in B$. If the function f has no zero point in $\overline{I_n}$, then $f[\overline{I_n}] \subseteq \mathbb{R} \setminus \{0\}$ and hence the nonempty open set $B \cap W(\overline{I_n}, \mathbb{R}\setminus\{0\})$ is contained in $B \setminus D_n$, which means that D_n is nowhere dense. So we can suppose that f has a zero point $x_0 \in \overline{I_n}$. Again, on $C(X)$ the compact-open topology is the topology of uniform convergence on compacta ([4; Theorem 7.11]), so there is $\varepsilon > 0$ and a compact K such that $W(f, K, \varepsilon) \subseteq B$. The continuity of f at x_0 implies that there is $\delta > 0$ such that $|f(x)| < \varepsilon/4$ whenever $x \in B(x_0, \delta)$ (the open ball with the center x_0 and the radius δ). From $x_0 \in \overline{I_n}$ it follows that $V = B(x_0, \delta) \cap I_n$ is a nonempty open set and so we may choose $x_1 \in V$ and $x_2 \in V$ with $x_1 \neq x_2$ (this can be done because X is without isolated points). In the Tychonoff space X there are two disjoint sets I_{n_1} and I_{n_2} in $\mathfrak{B}$ such that $x_i \in I_{n_i} \subseteq V$ for $i = 1, 2$. Finally choose x_3, x_4 such that $x_3 \in I_{n_1}$, $x_3 \neq x_1$, and $x_4 \in I_{n_2}$, $x_4 \neq x_2$. Let $A = \{x_1, x_2, x_3, x_4\}$. If we consider A with the discrete topology (which is the relative topology from X since A consists of isolated points), then the function $h' \colon A \to [-1, 1]$ defined by $h'(x_1) = h'(x_2) = 1$ and $h'(x_3) = h'(x_4) = -1$ is continuous on A. By the Tietze extension theorem there is an extension $h \in C(X,[-1,1])$ of the function h'. Now define the function g on X by:

$$g(x) = f(x) + \varepsilon/2 \cdot h(x) \qquad \text{for each} \quad x \in X\,,$$

thus $g \in C(X,\mathbb{R})$. Put $K' = K \cup \overline{I_n}$, then K' is compact. We want to show that $W(g,K',\varepsilon/4)$ is the desired open set. Let $s \in W(g,K',\varepsilon/4)$. From $K \subseteq K'$ we have $|s(x) - g(x)| < \varepsilon/4$ and $|g(x) - f(x)| = \varepsilon/2 \cdot |h(x)| \leq \varepsilon/2$ for each $x \in K$ and hence $s \in W(f,K,\varepsilon) \subseteq B$. On the other hand $A \subseteq B(x_0,\delta)$, so for each $i = 1,\dots,4$ we have $-\varepsilon/4 < f(x_i) < \varepsilon/4$. Moreover h restricted to A equals h', which implies that for $i = 1,2$

$$g(x_i) = \varepsilon/2 \cdot h(x_i) + f(x_i) > \varepsilon/2 - \varepsilon/4 = \varepsilon/4$$

and for $i = 3,4$

$$g(x_i) = f(x_i) + \varepsilon/2 \cdot h(x_i) < \varepsilon/4 - \varepsilon/2 = -\varepsilon/4\,.$$

Since $A \subseteq I_n \subseteq K'$, the value of every $s \in W(g,K',\varepsilon/4)$ is positive at x_1 and x_2 and negative at x_3 and x_4. But x_1 and x_3 are elements of a connected set I_{n_1} and s is continuous, i.e. the image $s[I_{n_1}]$ is connected, so with $s(x_1)$ and $s(x_3)$ it must contain zero. Thus s has a zero point in I_{n_1} and similarly it has a zero point in I_{n_2}, which are two disjoint subsets of I_n. We conclude that s has at least two zero points in I_n, that is $s \notin D_n$. This shows that $W(g,K',\varepsilon/4) \subseteq B \setminus D_n$ and hence D_n is nowhere dense. □

THEOREM 3. *Let X be a locally compact, locally connected second countable Tychonoff space without isolated points. Then the set H is residual in $(C(X),\tau_{\mathrm{co}})$.*

P r o o f . We know that a second countable locally compact Tychonoff space is a hemicompact k-space (see [3; Exercise 3.4.E]). Thus [5; Corollary 5.2.2] implies that $(C(X),\tau_{\mathrm{co}})$ is completely metrizable and hence it is a Baire space. Therefore it suffices to show that the complement of H is of first category. But $H^c = A \cup D$, so Theorem 1 and Theorem 2 yield the desired statement. □

Now, consider the space $C_0(X)$ with the relativized compact-open topology from $C(X)$. Define the set $H_0 = H \cap C_0(X)$. Since all the other considered sets (A, D, A_n and D_n) are subsets of $C_0(X)$, they do not change if we restrict ourselves to the space $C_0(X)$. In particular $H = C(X) \setminus (A \cup D)$ and $H_0 = C_0(X) \setminus (A \cup D)$. The first question of course is: which of the foregoing results hold if we replace $C(X)$ with $C_0(X)$?

Clearly, in the space $(C_0(X),\tau_{\mathrm{co}})$ both statements of Note 1 remain valid. Note 2(b) shows that if X is without isolated points, then an analogy of Theorem 1 is also true in $C_0(X)$. A thorough look on the proof of Theorem 2 reveals that it will work in $C_0(X)$, too, since the constructed function g belongs to $C_0(X)$. We conclude that the set $A \cup D$, which equals $C_0(X) \setminus H_0$, is of first category in $\big(C_0(X), \tau_{\mathrm{co}}\big)$. To complete our effort we will need some additional results.

PROPOSITION 3. *If T is a Baire space and S is a subset such that* $\operatorname{int} S$ *is dense in T, then S is a Baire space with respect to the relative topology from T.*

P r o o f. Consider a subset E' of S. We know that, without any assumption on the subspace S, if E' is nowhere dense in S, then it is nowhere dense in T. Therefore a set of first category in S is clearly of first category in T or in other words, if $E' \subseteq S$ is of second category in T, then it is of second category in S. So let U' be a nonempty open set in S, hence $U' = U \cap S$ for some nonempty open set U in T. Since $\operatorname{int} S$ is dense, the set $U \cap \operatorname{int} S$ is nonempty open and thus of second category in T. As U' contains $U \cap \operatorname{int} S$, it must also be of second category in T and by our initial remarks U' is of second category in S. Therefore S is a Baire space. □

PROPOSITION 4. *For a non-compact Tychonoff space X the set $C_0(X)$ is dense in $\big(C(X), \tau_{\mathrm{co}}\big)$.*

P r o o f. If $B = \bigcap_{i=1}^{m} W(K_i, U_i)$ is a τ_{co}-basic neighborhood of a function $f \in C(X)$, then the set $K = \bigcup_{i=1}^{m} K_i$ is a compact (and closed) subset of the non-compact space X and so there is a point $x_0 \in X \setminus K$ and a function $h \in C\big(X, [0,1]\big)$ with $h(x_0) = 0$ and $h[K] = \{1\}$. Hence the function g defined by $g(x) = f(x) \cdot h(x)$ coincides with f on K (thus $g \in B$) and assumes zero at x_0 (thus $g \in C_0(X)$), which shows that $C_0(X)$ is dense. □

However, in the sequel we will need a somehow stronger statement composed in the following proposition.

PROPOSITION 5. *If X is a non-compact locally connected Tychonoff space without isolated points, then* $\operatorname{int} C_0(X)$ *is dense in $\big(C(X), \tau_{\mathrm{co}}\big)$.*

P r o o f. Let $B = \bigcap_{i=1}^{m} W(K_i, U_i)$ be a τ_{co}-basic set and $f \in B$. Then $K = \bigcup_{i=1}^{m} K_i$ is a compact (and hence closed) subset of X. Since X is not compact, $X \setminus K$ is a nonempty open set, so let $x_1 \in X \setminus K$. Then there is an open connected set V with $x_1 \in V \subseteq K^c$. Choose a point $x_2 \in V$, $x_2 \neq x_1$ (this can be done because X is without isolated points). Having in mind that X is a Tychonoff space we know that there are continuous functions $h_i \in C\big(X, [0,1]\big)$ for $i = 1, \dots, 4$ with the following properties: since $\{x_1\}$ and K are disjoint closed subsets we choose h_1 such that $h_1[K] = \{1\}$ and $h_1(x_1) = 0$. Similarly for x_2 we put $h_2[K] = \{1\}$ and $h_2(x_2) = 0$. Furthermore x_1 and $K \cup \{x_2\}$ are disjoint closed subsets and so we can choose h_3 in order that $h_3\big[K \cup \{x_2\}\big] = 0$ and $h_3(x_1) = 1$. In the same manner we put $h_4\big[K \cup \{x_1\}\big] = 0$ and $h_4(x_2) = 1$. Now we define the function g on X by:

$$g(x) = f(x) \cdot h_1(x) \cdot h_2(x) + h_3(x) - h_4(x) \qquad \text{for each} \quad x \in X .$$

Clearly $g \in C(X, \mathbb{R})$. Moreover if $x \in K$, then $g(x) = f(x) \cdot 1 \cdot 1 + 0 - 0 = f(x)$ and so $g \in B$. Next, $g(x_1) = f(x_1) \cdot 0 \cdot h_2(x_1) + 1 - 0 = 1$ and $g(x_2) = f(x_2) \cdot h_1(x_2) \cdot 0 + 0 - 1 = -1$. Thus $g \in W\big(\{x_1\}, \mathbb{R}^+\big) \cap W\big(\{x_2\}, \mathbb{R}^-\big) = U$ which is a τ_{co}-basic set ($\mathbb{R}^+$ and $\mathbb{R}^-$ denote all positive and negative real numbers respectively). Finally whenever $h \in U$, then $h(x_1) > 0 > h(x_2)$ where x_1 and x_2 are elements of the connected set V, and h is continuous, which implies that $h[V]$ is connected and with $h(x_1)$ and $h(x_2)$ it must contain zero. Hence $h^{-1}\big(\{0\}\big) \neq \emptyset$ and $h \in C_0(X)$. This shows that U is a neighborhood of g contained in $C_0(X)$, i.e. $g \in \operatorname{int} C_0(X)$. Consequently $B \cap \operatorname{int} C_0(X) \neq \emptyset$ and therefore $\operatorname{int} C_0(X)$ is dense. □

COROLLARY 6. *If X is a non-compact locally compact, locally connected paracompact Tychonoff space without isolated points, then $\big(C_0(X), \tau_{\mathrm{co}}\big)$ is a Baire space.*

Proof. By [5; Theorem 5.3.1], the space $\big(C(X), \tau_{\mathrm{co}}\big)$ is a Baire space. The rest is supplied by Proposition 3 and Proposition 5. □

THEOREM 4. *Let X be a locally compact, locally connected second countable Tychonoff space without isolated points. Then the set H_0 is residual in $\big(C_0(X), \tau_{\mathrm{co}}\big)$.*

Proof. As in the proof of Theorem 3, $\big(C(X), \tau_{\mathrm{co}}\big)$ is completely metrizable. It is not difficult to show that if X is compact, then $C_0(X)$ is closed in $\big(C(X), \tau_{\mathrm{co}}\big)$, that is, $\big(C_0(X), \tau_{\mathrm{co}}\big)$ is completely metrizable and hence it is a Baire space. On the other hand, if X is not compact, then Corollary 6 implies that $\big(C_0(X), \tau_{\mathrm{co}}\big)$ is a Baire space, too. Therefore we only have to show that the set $C_0(X) \setminus H_0$ is of first category in $C_0(X)$, but that is true by the remarks preceding Proposition 3. □

To highlight the resemblances and differences, compared to our Theorem 3 and Theorem 4, we now state the original assertion of Baláž and Šalát ([1; Theorem 2.1]). Herein $C(a, b)$ denotes the complete metric space of all continuous real-valued functions on a compact interval $[a, b]$ of the reals equipped with the uniform metric. The definitions of all other occurring sets, namely $C_0(a, b)$, $H(a, b)$, $H_0(a, b)$, $A(a, b)$ and $D(a, b)$, are totally analogous to the definitions at the beginning of our paper.

THEOREM. ([1; Theorem 2.1])

(i) *The set* $H(a,b)$ *is residual in* $C(a,b)$.

(ii) *The set* $H_0(a,b)$ *is residual in* $C_0(a,b)$.

The paper [1] also contains propositions revealing other properties of A and D. We shall now investigate the general situation.

PROPOSITION 7. *If* X *is normal, then* A *is dense in* $(C_0(X),\tau_{co})$. *If moreover every singleton in* X *is a* G_δ *set, then* D *is dense, too.*

Proof. Let B be an arbitrary τ_{co}-basic set in $C_0(X)$ and let $f\in B$. Hence there is $x_0\in X$ with $f(x_0)=0$. On $C_0(X)$ the topology of uniform convergence on compacta and the compact-open topology coincide ([4; Theorem 7.11]), and so there is a compact $K\subseteq X$ and $\varepsilon>0$ such that $W(f,K,\varepsilon)\subseteq B$. Since f is continuous at x_0, there is $U\subseteq X$ open such that $|f(x)|<\varepsilon/2$ whenever $x\in U$. Furthermore there is an open set V such that $x_0\in V\subseteq\overline{V}\subseteq U$ and hence $\overline{V}$ and U^c are two disjoint closed sets, so by the Urysohn lemma there is a continuous function $h_1\in C(X,[0,1])$ such that $h_1[\overline{V}]=\{0\}$ and $h_1[U^c]=\{1\}$. Define the function g_1 on X by: $g_1(x)=h_1(x)\cdot f(x)$ for each $x\in X$. Clearly $g_1\in C_0(X)$ and we also see that $V\subseteq g_1^{-1}(\{0\})$. Thus $g_1^{-1}(\{0\})$ is not nowhere dense, so $g_1\in A$. Moreover, for each $x\in X$ we have $0\le h_1(x)\le 1$, which implies $0\le 1-h_1(x)\le 1$. Hence if $x\in U$, then

$$|g_1(x)-f(x)|=|f(x)|\cdot|1-h_1(x)|\le|f(x)|<\varepsilon/2.$$

On the other hand, if $x\in U^c$, then $g_1(x)=1\cdot f(x)=f(x)$. Consequently, for each $x\in X$ we have $|g_1(x)-f(x)|<\varepsilon/2$, which implies $g_1\in W(f,K,\varepsilon)\subseteq B$. Thus $A\cap B\neq\emptyset$ and A is therefore dense.

Concerning the set D: since the singleton $\{x_0\}$ is a closed G_δ set, [3; Corollary 1.5.11] implies the existence of a continuous function $h_2\in C(X,[0,1])$ for which $\{x_0\}=h_2^{-1}(\{0\})$. Define the function g_2 on X by: $g_2(x)=g_1(x)+\varepsilon/2\cdot h_2(x)$ for each $x\in X$. Clearly $g_2\in C_0(X)$. Next, for each $x\in X$ we have

$$|g_2(x)-f(x)|\le|g_1(x)-f(x)|+\varepsilon/2\cdot|h_2(x)|<\varepsilon/2+\varepsilon/2=\varepsilon.$$

Thus $g_2\in W(f,K,\varepsilon)\subseteq B$. Since g_1 is constantly zero on $\overline{V}$, we see that $g_2(x)=\varepsilon/2\cdot h_2(x)$ for each $x\in\overline{V}$, and therefore x_0 is the only zero point of g_2 in V. Consequently $g_2\in D$, which shows that D intersects B and we are done. □

In view of Proposition 4 and Proposition 7 the next two statements are clear, provided we realize that if X is compact, then $C(X)\setminus C_0(X)=W(X,\mathbb{R}\setminus\{0\})$ is a nonempty open set disjoint from $C_0(X)$ and hence no subset of $C_0(X)$ (neither A nor D) can be dense in $C(X)$.

COROLLARY 8(a). *Let X be a normal space. The set A is dense in $(C(X),\tau_{co})$ if and only if X is not compact.*

COROLLARY 8(b). *Let X be a normal space and let every singleton in X be a G_δ set. The set D is dense in $(C(X),\tau_{co})$ if and only if X is not compact.*

PROPOSITION 9. *Let X be a locally connected normal space without isolated points. Then A is not nowhere dense in $(C(X),\tau_{co})$. If moreover every singleton in X is a G_δ set, then D is not nowhere dense, too.*

Proof. To show a set is not nowhere dense it suffices to find a nonempty open set in which the set is dense. Let $V \subseteq X$ be a nonempty open connected set (by local connectedness such set exists). Since X is without isolated points, we can choose two distinct points $x_1 \neq x_2$ in V. Put $B = W(\{x_1\},\mathbb{R}^+) \cap W(\{x_2\},\mathbb{R}^-)$, where $\mathbb{R}^+$ and $\mathbb{R}^-$ denote all positive and negative real numbers respectively. Clearly B is a τ_{co}-basic set. Again, there is a continuous function $h \in C(X,[0,1])$ with $h(x_1) = 1$ and $h(x_2) = 0$. Define a function by $f(x) = h(x) - 1/2$ for each $x \in X$. Immediately we see that $f \in C(X)$ and $f \in B$, hence B is a nonempty open set in $(C(X),\tau_{co})$. Furthermore every function in B is positive and negative at x_1 and x_2 respectively, which are elements of a connected set V. Therefore every function in B has a zero point in X (this argument has been used several times before), that is $B \subseteq C_0(X)$. According to Proposition 7, under their respective assumptions the sets A and D are dense in B which proves the proposition. $\square$

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