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ON THE NUMBER OF LATTICE POINTS IN CERTAIN PLANAR SEGMENTS

GERALD KUBA

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ABSTRACT. Let $\mathcal{D}_0 \subset \mathbb{R}^2$ be a compact domain whose boundary is a simple closed curve composed of finitely many pieces such that on each piece the radius of curvature exists everywhere, is bounded and non-zero, and is continuously differentiable with respect to the tangent angle. Further, let $\mathcal{D}$ be a plane domain obtained by applying a rigid motion to $\mathcal{D}_0$ and let $\mathcal{D}(a,b) := \{(x,y) \in \mathcal{D} :\ y \geq ax+b\}$, where $a, b \in \mathbb{R}$. Generalizing Huxley's famous theorem we show that when a is taken from a large class $\mathcal{R}$ of irrational numbers and b is arbitrary, for a real parameter λ

$$\#(\lambda\mathcal{D} \cap \mathbb{Z}^2) = \lambda^2 \operatorname{area} \mathcal{D} + \mathbf{O}(\lambda^{0.63}) \qquad (\lambda \to \infty).$$

Thereby the $\mathbf{O}$-constant depends only on the basic domain $\mathcal{D}_0$ and the class $\mathcal{R}$.

Additionally, we are able to extend the applicability of the standard method of estimating rounding error sums of the shape

$$\Psi(f;u,v;\lambda) := \sum_{u\lambda \leq n \leq v\lambda} \psi\left(\lambda f\left(\frac{n}{l}\right)\right) \qquad (\lambda \to \infty),$$

where $\psi(z) = z - [z] - 1/2$ and f is a real-valued function defined on an interval $[u,v] \subset \mathbb{R}$ with continuous derivatives up to order 3 and the property that f'' does not vanish on $[u,v]$. By Huxley's method, $\Psi(f;u,v;\lambda) \ll \lambda^{0.63}$ *under the additional condition that* f''' *does not vanish on* $[u,v]$.

We show that this condition, which has always been interpreted as technical, is superfluous.

1. Introduction and statement of the main result

Let $\mathcal{D}_0 \subset \mathbb{R}^2$ be a compact domain whose boundary is a simple closed curve composed of finitely many pieces such that on each piece the radius of curvature exists everywhere, is bounded and non-zero, and is continuously differentiable

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with respect to the tangent angle. Let $\mathcal{D}$ be a plane domain obtained by applying a rigid motion to $\mathcal{D}_0$, i.e. $\mathcal{D} = \mathcal{D}_0 \cdot \mathbf{A} + \boldsymbol{v}$, where $\mathbf{A}$ is a real orthogonal 2×2-matrix with determinant 1 and $\boldsymbol{v} \in \mathbb{R}^2$ is a translation vector.

The following deep result of planar lattice point theory has been proved by Huxley (cf. [2]).

There exists an effective constant C such that for every expansion factor $\lambda \geq 2$

$$|\#(\lambda\mathcal{D} \cap \mathbb{Z}^2) - \lambda^2 \operatorname{area}\mathcal{D}| \leq C\lambda^{\frac{46}{73}}(\log\lambda)^{\frac{315}{146}} .$$

C depends on $\mathcal{D}_0$, but not on the rotation matrix $\mathbf{A}$ or the translation vector $\boldsymbol{v}$.

With reference to this great theorem and for the sake of simplicity we will call any domain like $\mathcal{D}_0$ a *Huxley domain.*

The most important Huxley domain of course is a circle and in this case Huxley's theorem is the sharpest-known result concerning the famous *circle problem.*

The aim of the present paper is to achieve an analogous result if the domain $\mathcal{D}$ is replaced by segments $\{(x,y) \in \mathcal{D} : \ y \geq ax + b\}$ $(a, b \in \mathbb{R})$.

There will be no problem concerning b which may be arbitrary without influencing the constant C. On the other hand, the slope a of the boundary line $y = ax + b$ has to be chosen carefully. Clearly, with respect to the symmetry of the lattice, we may assume without loss of generality $0 \leq a \leq 1$. Of course, the desired generalization of Huxley's theorem is impossible if a is rational. Thus we assume that a is irrational. Consequently, there lies at most one lattice point on any line $y = ax + \lambda b$ and hence one may alternately consider the subdomains of $\mathcal{D}$ where $y > ax + b$, $y \leq ax + b$, or $y < ax + b$. Of course, the assumption only that a is irrational would be insufficient. What we really have to assume is that a is rather badly aproximable by rationals. Then the numbers a which must not occur are only few from a measure-theoretic standpoint.

Let $D_N(a) := D_N\big((na)_{n=1,\dots,N}\big)$ denote the discrepancy of the irrational a (cf. [4]).

For a constant $H \geq 1$ let $\Re_H$ be the set of all irrationals $a \in [0,1]$ such that the inequality $D_N(a) \leq HN^{-\frac{3}{8}}$ holds for every $N \in \mathbb{N}$.

The famous theorem of Thue-Siegel-Roth implies that for every algebraic irrational a and arbitrarily small $\varepsilon > 0$ there is a $H_{a,\varepsilon}$ with $D_N(a) \leq H_{a,\varepsilon}N^{-1+\varepsilon}$ for all $N \in \mathbb{N}$. Hence for every algebraic $a \in [0,1]\setminus\mathbb{Q}$ there is a H with $a \in \Re_H$. (For instance, $\sqrt{2}-1, \sqrt{3}-1 \in \Re_4$ by [4; Theorem 3.4].) But the sets $\Re_H$ are far away from being small. Since (for every $N \in \mathbb{N}$) D_N is a continuous function on $[0,1]\setminus\mathbb{Q}$ and $\Re_H = \bigcap_{N\in\mathbb{N}} D_N^{-1}\big([0, HN^{-\frac{3}{8}}]\big)$, there is a closed set $A_H \subset [0,1]$

such that $\mathfrak{R}_H = A_H \setminus \mathbb{Q}$, whence the set $\mathfrak{R}_H$ is always measurable. Further, $[0,1] \setminus \bigcup_{H\in\mathbb{N}} \mathfrak{R}_H$ is a Lebesgue null set because, by a well-known result due to Khintchine (cf. [4]), $D_N(a) \ll N^{-1+\varepsilon}$ $(N \to \infty)$ for almost all $a \in \mathbb{R}$. Consequently, since $\mathfrak{R}_H \subset \mathfrak{R}_{H'}$ if $H \leq H'$, the Lebesgue measure of the set $[0,1]\setminus\mathfrak{R}_H$ is arbitrarily small when H is sufficiently large.[1]

Now the main result of the present paper is the following theorem.

THEOREM 1. *Let $\mathcal{A}$ be the set of all real orthogonal 2×2-matrices with determinant 1 and, for $H \geq 1$, $\mathfrak{R}_H := \{a \in [0,1]\setminus\mathbb{Q} : (\forall N \in \mathbb{N})(D_N(a) \leq HN^{-\frac{3}{8}})\}$. Further let $\mathcal{D}_0 \subset \mathbb{R}^2$ be a Huxley domain. Then there exists an effective constant C depending only on $\mathcal{D}_0$ and H such that for every expansion factor $\lambda \geq 2$, for every $a \in \mathfrak{R}_H$, for every $b \in \mathbb{R}$, for every $\mathbf{A} \in \mathcal{A}$, and for every $\mathbf{v} \in \mathbb{R}^2$*

$$\left|\#\left(\lambda\mathcal{D}(a,b;\mathbf{A},\mathbf{v}) \cap \mathbb{Z}^2\right) - \lambda^2 \operatorname{area}\mathcal{D}(a,b;\mathbf{A},\mathbf{v})\right| \leq C\lambda^{\frac{46}{73}}(\log\lambda)^{\frac{315}{146}},$$

where

$$\mathcal{D}(a,b;\mathbf{A},\mathbf{v}) := \left\{(x,y) \in \mathcal{D}_0\cdot\mathbf{A}+\mathbf{v} :\ y \geq ax+b\right\}.$$

2. Preparation of the proof

Let the *rounding error function* ψ be defined by

$$\psi(z) = z - [z] - 1/2 \qquad (z \in \mathbb{R}),$$

where [ ] are the Gauss brackets. The following two lemmata provide good estimates of rounding error sums that we need in order to prove Theorem 1.

LEMMA 1. *Let $H \geq 1$ and $a \in \mathfrak{R}_H$. Then for $\lambda \geq 2$ and arbitrary $u, v, b \in \mathbb{R}$ we have*

$$\left|\sum_{u\lambda\leq n\leq v\lambda} \psi(an+b)\right| \leq 2H\left(1+|u|+|v|\right)\lambda^{\frac{5}{8}}.$$

Proof. By Koksma's inequality (cf. [3; Theorem 5.1]) we have for every $b \in \mathbb{R}$, every $N \in \mathbb{N}$, and every sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers

$$\left|\sum_{n=1}^{N} \psi(x_n+b)\right| \leq 2ND_N\left((x_n)_{n=1,\dots,N}\right).$$

[1] Nevertheless, every set $\mathfrak{R}_H$ is *nowhere dense* in $[0,1]\setminus\mathbb{Q}$ and thus in $\mathbb{R}$, too. This is true because if $\mathcal{L}$ is the set of all *Liouville numbers*, which is dense in $\mathbb{R}\setminus\mathbb{Q}$, then $\mathfrak{R}_H \cap \mathcal{L} = \emptyset$ since $D_N(\alpha) = \Omega(N^{-\varepsilon})$ $(N \to \infty)$ for every $\varepsilon > 0$ and all $\alpha \in \mathcal{L}$.

Consequently,

$$\left|\sum_{n=1}^{N}\psi(\pm an+b)\right|\le 2HN^{\frac{5}{8}},$$

which immediately implies the assertion. □

The next lemma follows by combining Huxley [2; Theorems 18.2.1, 18.2.2].

Lemma 2. *Let $C_1, C_2 \ge 1$ be constants and let M, M', T be positive real parameters satisfying $M \le M' < 2M$ and $T^{\frac{4}{9}} \le M \le C_1 T^{\frac{1}{2}}$. Further, let $F(t)$ be a three times continuously differentiable function on $1 \le t \le 2$ satisfying $1/C_2 \le |F^{(r)}(t)| \le C_2$ for $1 \le t \le 2$ and $r = 1,2,3$. Then there exists a constant C_3 depending only on C_1 and C_2 such that if $T \ge 2$, then*

$$\left|\sum_{M\le m\le M'}\psi\Big(\frac{T}{M}F\Big(\frac{m}{M}\Big)\Big)\right|\le C_3 T^{\frac{23}{73}}(\log T)^{\frac{315}{146}}.$$

The following lemma is a generalization of Huxley's main theorem cited in Section 1.

Lemma 3. *Fix $k, l \in \mathbb{N}$ and let $\mathcal{D}_0$ and $\mathcal{H}_0$ be two Huxley domains. Then there exists an effective constant C_0 such that for every rotation matrix $\mathbf{A} \in \mathcal{A}$ and all translation vectors $\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^2$ the following is true. If $\mathcal{H}$ is a Huxley domain with*

$$\partial\mathcal{H} \subset \partial(\mathcal{D}_0\cdot\mathbf{A}+\mathbf{v})\cup\partial(\mathcal{H}_0+\mathbf{v}_1)\cup\dots\cup\partial(\mathcal{H}_0+\mathbf{v}_k)$$

such that $\partial\mathcal{H}$ is the union of at most l smooth pieces,[2] *then the inequality*

$$|\#(\lambda\mathcal{H}\cap\mathbb{Z}^2)-\lambda^2\,\mathrm{area}\,\mathcal{H}|\le C_0\lambda^{\frac{46}{73}}(\log\lambda)^{\frac{315}{146}}$$

holds for every expansion factor $\lambda \ge 2$.

Proof. Since the number of the smooth pieces C_i of $\mathcal{H}$ is bounded by l, we can take over Huxley's original proof ([2; pp. 389–393]) word for word.

The final lemma guarantees that the sets $\Re_H$ are always bounded away from 0 and 1. □

[2]At the first sight this additional assumption seems superfluous. But consider the following counterexample. Define convex domains $\mathcal{D}_0$ and $\mathcal{H}_0$ such that $\partial\mathcal{D}_0$ is parametrized by $r(\varphi) = 1$ $(0 < \varphi \le 2\pi)$ and $\partial\mathcal{H}_0$ is parametrized by $r(\varphi) = 1$ $(1/\pi < \varphi \le 2\pi)$ and $r(\varphi) = 1 + \varphi^8 \sin(1/\varphi)$ $(0 < \varphi \le 1/\pi)$. Then both domains are Huxley domains since $\mathcal{D}_0$ is a circle and $\mathcal{H}_0$ has a sufficiently smooth boundary where the radius of curvature ϱ smoothly pendulates within the range $8/9 \le \varrho \le 8/7$. But neither $\mathcal{D}_0 \cap \mathcal{H}_0$ nor $\mathcal{D}_0 \cup \mathcal{H}_0$ is a Huxley domain because $\partial\mathcal{D}_0$ meets $\partial\mathcal{H}_0$ non-tangentially at $\varphi = 1/(n\pi)$ $(n \in \mathbb{N})$. Now, for arbitrary $N \in \mathbb{N}$, consider the domain $\mathcal{H}$ which is bounded by the curve $r(\varphi)$ $(0 < \varphi \le 2\pi)$ with $r(\varphi) = 1 + \varphi^8 \sin(1/\varphi)$ when $2\pi n \le 1/\varphi \le (2n+1)\pi$ $(n = 1, \dots, N)$ and $r(\varphi) = 1$ otherwise. Then $\mathcal{H}$ is a Huxley domain with $\partial\mathcal{H} \subset \partial\mathcal{D}_0 \cup \partial\mathcal{H}_0$, but the minimal number of smooth pieces of $\partial\mathcal{H}$ equals $2N$.

Lemma 4. *For $H \geq 1$ let $N \in \mathbb{N}$ such that $N \geq (2H)^{\frac{8}{3}}$. Then $\Re_H \subset \left[\frac{1}{2N}, 1 - \frac{1}{2N}\right]$.*

Proof. Note that, by assumption, $N \geq 6$ and let $a \in \Re_H$. Since there is nothing to show if $\frac{1}{N} \leq a \leq 1 - \frac{1}{N}$ suppose firstly that $a < \frac{1}{N}$. Then we have $na \in [0, Na] \subset [0,1]$ for every $n = 1, 2, \dots, N$ and hence, by the definition of the discrepancy and with $\mathbb{I}_{\mathcal{M}}$ denoting the indicator function of the set $\mathcal{M}$,

$$1 - Na = \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}_{[0,Na]}(na) - Na \right| \leq D_N(a) \leq HN^{-\frac{3}{8}} \leq \frac{1}{2},$$

whence $a \geq \frac{1}{2N}$. If on the other hand $a > 1 - \frac{1}{N}$, then the same argument applied to $1 - a$ instead of a yields $a \leq 1 - \frac{1}{2N}$ since $D_N(1-a) = D_N(a)$. □

3. Lattice points in segments of a circle

For fixed $r > 0$ and arbitrary $a \in \Re_H$ $(H \geq 1)$, define *circular segments*

$$\sigma(a,d;r) := \left\{(x,y) \in \mathbb{R}^2 : \ (x^2 + y^2 \leq r^2) \ \wedge \ (y \leq ax + d)\right\},$$

where $-r\sqrt{1+a^2} < d < -r$, so that $\sigma(a,d;r)^\circ \neq \emptyset$ and the slope of any tangent to the circular piece of the boundary of σ is always positive (and finite).

Then we can write

$$\sigma(a,d;r) := \left\{(x,y) \in \mathbb{R}^2 : \ (x_1 \leq x \leq x_2) \ \wedge \ \left(f(x) \leq y \leq g(x)\right)\right\},$$

where $g(x) := ax + d$, $f(x) := -\sqrt{r^2 - x^2}$ and $0 < x_1 < x_2 < r$ such that $f(x_1) = g(x_1)$ and $f(x_2) = g(x_2)$. Then the slope of the tangents mentioned above is given by the first derivative of the function f.

We are going to apply Lemma 2 in order to derive a formula for the number of lattice points in the domains $\lambda\sigma(a,d;r)$. Thereby it is inevitable to make an assumption like the following.

(*) *There are constants c_1, c_2, $0 < c_1 < c_2 < \infty$, such that $c_1 \leq f'(x) \leq c_2$ $(x_1 \leq x \leq x_2)$.*

Note that the bounds for the first derivative of f yield new bounds for the higher derivatives. Actually, (*) implies $rc_3 \leq x_1 < x_2 \leq rc_4$ with

$$c_3 := \frac{c_1}{\sqrt{1 + c_1^2}} \qquad \text{and} \qquad c_4 := \frac{c_2}{\sqrt{1 + c_2^2}}.$$

Then via $f''(x) = \left(f'(x)\right)^3 r^2/x^3$ and $f'''(x) = 3\left(f'(x)\right)^5 r^2/x^4$ we obtain the coarse but immediate estimations

$$0 < \frac{c_1^3}{c_4^3 r} \leq f'' \leq \frac{c_2^3}{c_3^3 r} < \infty \qquad \text{and} \qquad 0 < \frac{3c_1^5}{c_4^4 r^2} \leq f''' \leq \frac{3c_2^5}{c_3^4 r^2} < \infty. \qquad (**)$$

PROPOSITION 1. *Under the above premises, and assuming* $(*)$, *we have for* $\alpha, \beta \in [0,1]$ *and as* $\lambda \to \infty$,

$$\#\big(\lambda\sigma(a,d;r) \cap (\alpha + \mathbb{Z}) \times (\beta + \mathbb{Z})\big) = \lambda^2 \operatorname{area} \sigma(a,d;r) + O\Big(\lambda^{\frac{46}{73}} (\log \lambda)^{\frac{315}{146}}\Big),$$

where the O-constant depends on r, c_1, c_2, *and* H, *but not on* α, β, d, *or* $a \in \Re_H$.

Proof. Let $\lambda \geq 2 + r + 32/(rc_3)^5$ so that then $\lambda^2 r \geq 2$ and $\lambda^{\frac{1}{5}} \geq 2/x_1$ and $\lambda x_1/2 \leq \lambda x_1 - \alpha$. We have

$$\lambda\sigma(a,d;r) = \big\{(x,y) \in \mathbb{R}^2 : \ (\lambda x_1 \leq x \leq \lambda x_2) \ \wedge \ \big(\lambda f(x/\lambda) \leq y \leq \lambda g(x/\lambda)\big)\big\}.$$

Consequently,

$$\begin{aligned}
&\#\big(\lambda\sigma(a,d;r) \cap (\alpha + \mathbb{Z}) \times (\beta + \mathbb{Z})\big) \\
&= \sum_{\lambda x_1 - \alpha \leq n \leq \lambda x_2 - \alpha} \#\Big\{m \in \mathbb{Z} : \ -\beta + \lambda f\big(\tfrac{n+\alpha}{\lambda}\big) \leq m \leq -\beta + \lambda g\big(\tfrac{n+\alpha}{\lambda}\big)\Big\} \\
&= \sum_{\lambda x_1 - \alpha \leq n \leq \lambda x_2 - \alpha} \Big(\big[-\beta + \lambda g\big(\tfrac{n+\alpha}{\lambda}\big)\big] + \big[\beta - \lambda f\big(\tfrac{n+\alpha}{\lambda}\big)\big] + 1\Big) \\
&= S(\lambda, x_1, x_2, \alpha) - \Psi_1(\lambda, x_1, x_2, \alpha) - \Psi_2(\lambda, x_1, x_2, \alpha),
\end{aligned}$$

where

$$\begin{aligned}
S(\lambda, x_1, x_2, \alpha) &:= \sum_{\lambda x_1 - \alpha \leq n \leq \lambda x_2 - \alpha} \lambda\Big(g\Big(\frac{n+\alpha}{\lambda}\Big) - f\Big(\frac{n+\alpha}{\lambda}\Big)\Big), \\
\Psi_1(\lambda, x_1, x_2, \alpha) &:= \sum_{\lambda x_1 - \alpha \leq n \leq \lambda x_2 - \alpha} \psi\Big(\beta - \lambda f\Big(\frac{n+\alpha}{\lambda}\Big)\Big), \\
\Psi_2(\lambda, x_1, x_2, \alpha) &:= \sum_{\lambda x_1 - \alpha \leq n \leq \lambda x_2 - \alpha} \psi\Big(-\beta + \lambda g\Big(\frac{n+\alpha}{\lambda}\Big)\Big).
\end{aligned}$$

We apply Lemma 1 to the last sum and obtain

$$\Psi_2(\lambda, x_1, x_2, \alpha) \ll \lambda^{\frac{5}{8}} \leq \lambda^{\frac{46}{73}} \qquad (\lambda \to \infty),$$

where the $\ll$-constant depends only on x_1, x_2, and H, hence only on r, c_1, c_2, and H.

The first sum can be handled by applying the Euler summation formula (cf. [3]). Then we have

$$\begin{aligned}
S(\lambda, x_1, x_2, \alpha) = \lambda &\int_{\lambda x_1 - \alpha}^{\lambda x_2 - \alpha} \Big(g\Big(\frac{u+\alpha}{\lambda}\Big) - f\Big(\frac{u+\alpha}{\lambda}\Big)\Big)\, du \\
&+ \int_{\lambda x_1 - \alpha}^{\lambda x_2 - \alpha} \psi(u)\Big(g'\Big(\frac{u+\alpha}{\lambda}\Big) - f'\Big(\frac{u+\alpha}{\lambda}\Big)\Big)\, du.
\end{aligned}$$

Obviously, the first integral equals

$$\lambda^2 \int_{x_1}^{x_2} \big(g(u) - f(u)\big)\, du = \lambda^2 \operatorname{area} \sigma(a, d; r)\,,$$

and, via (*) and $\left| \int_v^w \psi(u)\, du \right| \le \frac{1}{8}$ and the second mean value theorem, the absolute value of the second is not greater than $(a + c_2)/8 \le 1 + c_2$.

Thus it remains to estimate $\Psi_1(\lambda, x_1, x_2, \alpha)$. Let $M_0 := \lambda x_1 - \alpha$, and choose $J \in \mathbb{N}$ with $2^{J-1} M_0 \le \lambda x_2 - \alpha < 2^J M_0$. Now, define a dyadic sequence $M_j = 2^j M_0$ $(j < J)$ and put $M_J := [\lambda x_2 - \alpha] + 1$. Then

$$\Psi_1(\lambda, x_1, x_2, \alpha) = \sum_{j=0}^{J-1} \sum_{M_j \le m < M_{j+1}} \psi\left(\frac{T_j}{M_j} F_j\left(\frac{m}{M_j} \right) \right),$$

where for $j = 0, 1, \dots, J-1$,

$$F_j(u) := \beta \frac{M_j}{T_j} - \lambda \frac{M_j}{T_j} f\left(\frac{M_j u + \alpha}{\lambda} \right) \qquad (1 \le u \le 2).$$

Now set $T_j := \lambda M_j$ $(0 \le j < J)$ in order to apply Lemma 3 to each of the J inner sums. Then we have

$$F_j^{(n)}(u) = -\left(\frac{M_j}{\lambda} \right)^n f^{(n)}\left(\frac{M_j u + \alpha}{\lambda} \right) \qquad (n \in \mathbb{N}).$$

Since for $0 \le j < J$, $M_j \in [\lambda x_1 - \alpha, \lambda x_2 - \alpha] \subset [\lambda x_1/2, \lambda x_2] \subset \lambda[r c_3/2, r c_4]$, via (*) and (**) it is easy to find a constant $C_2 = C_2(r, c_1, c_2) \ge 1$ such that $1/C_2 \le |F_j^{(n)}| \le C_2$ for $n = 1, 2, 3$ and $j = 0, 1, \dots, J-1$. Further, since $\lambda^{\frac{1}{5}} \ge 2/x_1$, the inequality $T_j^{\frac{4}{9}} \le M_j \le C_1 T_j^{\frac{1}{2}}$ is true for every j if we set $C_1 := 1 + \sqrt{r}$.

Therefore, by Lemma 2 (note that $\lambda^3 \ge r\lambda^2 \ge T_j \ge 2$)

$$\begin{aligned} |\Psi_1(\lambda, x_1, x_2, \alpha)| &\le C_3 \left(\sum_{j=0}^{J-1} T_j^{\frac{23}{73}} \right) \big(\log(r\lambda^2)\big)^{\frac{315}{146}} \\ &\le C_3 \cdot 5 \cdot (\lambda 2^J M_0)^{\frac{23}{73}} \cdot 11 \cdot (\log \lambda)^{\frac{315}{146}} \le 69 r^{\frac{23}{73}} C_3 \lambda^{\frac{46}{73}} (\log \lambda)^{\frac{315}{146}}\,. \end{aligned}$$

This finishes the proof of Proposition 1. □

4. Proof of Theorem 1

Notation. For compact $\mathcal{M} \subset \mathbb{R}^2$ let $\operatorname{diam} \mathcal{M} = \sup\{|P-Q| : P, Q \in \mathcal{M}\}$ denote the *diameter* of $\mathcal{M}$. Further, for abbreviation, if $a, b \in \mathbb{R}$ let

$$\mathcal{M}(a,b) := \{(x,y) \in \mathcal{M} : y \geq ax+b\},$$
$$\mathcal{M}^+(a,b) := \{(x,y) \in \mathcal{M} : y > ax+b\}.$$

Finally, if $P, Q \in \mathbb{R}^2$ let $[P,Q]$ denote the *straight line segment with endpoints* P, Q,

$$[P,Q] = \{tQ + (1-t)P : 0 \leq t \leq 1\}.$$

Now let $a \in \Re_H$ and $b \in \mathbb{R}$. In order to prove Theorem 1 we put $\mathcal{D} = \mathcal{D}_0 \cdot \mathbf{A} + \mathbf{v}$ so that $\mathcal{D}(a,b) = \mathcal{D}(a,b;\mathbf{A},\mathbf{v})$. Since there is at most one lattice point on a straight line with slope a, we may exclude the trivial case $\mathcal{D}^+(a,b) = \emptyset$.

Since $\mathcal{D}(a,b)$ may not be connected, we consider its (finitely many) components. Some of them may be singletons, but at least one component has a non-empty interior provided that $\mathcal{D}^+(a,b) \neq \emptyset$. Clearly there is a $M \in \mathbb{N}$ depending only on $\mathcal{D}_0$ such that for the number $n = n(a,b,\mathbf{A},\mathbf{v})$ of all components of $\mathcal{D}(a,b)$ we always have $n \leq M$. Then we can write $\mathcal{D}(a,b) = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m \cup \mathcal{F}$, where $\mathcal{E}_1, \ldots, \mathcal{E}_m$ are pairwise disjoint, compact and connected sets with non-empty interior, $\mathcal{F}$ is a finite set of points on the line $y = ax+b$ with $\mathcal{F} \cap (\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m) = \emptyset$, and $m + |\mathcal{F}| = n$. Then we observe that there exists a constant $K \in \mathbb{N}$ depending only on $\mathcal{D}_0$ such that for every $i = 1, \ldots, m$ we have

$$\mathcal{E}_i \setminus \mathcal{D}^+(a,b) = \bigcup_{j=1}^{k_i} [P_j, Q_j],$$

where $[P_j, Q_j]$ $(j = 1, \ldots, k_i)$ are pairwise disjoint subsets of the line $y = ax+b$ and $0 \leq k_i \leq K$. (Note that $k_i > 0$ for every $i = 1, \ldots, m$ if $m \geq 2$. If $m = 1$ and $k_1 = 0$, then there is nothing to show because this case is equivalent to $\mathcal{D}(a,b) = \mathcal{D}$, so that then Theorem 1 equals Huxley's original Theorem.)

Thus the boundary of every set $\mathcal{E}_i$ is put together by a piece of the boundary of $\mathcal{D}$ and k_i straight line segments $[P_j, Q_j]$. Hence every set $\mathcal{E}_i$ becomes a Huxley domain $\mathcal{D}_i$, i.e. $\mathcal{D}_i(a,b) = \mathcal{E}_i$, if the segments $[P_j, Q_j]$ are all replaced by suitable circular arcs connecting P_j and Q_j. The pairwise disjoint Huxley domains $\mathcal{D}_1, \ldots, \mathcal{D}_m$ which allow the representation $\mathcal{D}(a,b) = \mathcal{D}_1(a,b) \cup \cdots \cup \mathcal{D}_m(a,b) \cup \mathcal{F}$ may be chosen in the following way. With respect to Lemma 4 we choose a small positive constant c_H depending only on H such that $\Re_H \subset [c_H, 1-c_H]$. Further we fix

$$r := \frac{2}{c_H} \operatorname{diam} \mathcal{D}_0$$

and, for $i = 1, \dots, m$ and $j = 1, \dots, k_i$, choose suitable $\mathbf{v}_{ij} \in \mathbb{R}^2$ and $d_{ij} < 0$ with $1 < d_{ij}^2/r^2 < 1 + a^2$ such that

$$\mathcal{D}_i \setminus \mathcal{D}_i^+(a,b) = \bigcup_{j=1}^{k_i} \left(\sigma(a, d_{ij}; r) + \mathbf{v}_{ij}\right) \qquad (i = 1, \dots, m),$$

where the circular segments $\left(\sigma(a, d_{ij}; r) + \mathbf{v}_{ij}\right)$ $(i = 1, \dots, m,\ j = 1, \dots, k_i)$ are pairwise disjoint.

Note that this can be done in a way that the radius r is fixed as above. The freedom we need for fitting the circular segments arises from the freedom to choose the d_{ij}'s. Actually, for every $i = 1, \dots, m$ and $j = 1, \dots, k_i$ we have

$$\operatorname{diam} \mathcal{D}_0' \geq \operatorname{diam} \sigma(a, d_{ij}; r) = 2r\sqrt{1 - \frac{d_{ij}^2}{r^2} \frac{1}{1 + a^2}},$$

so that we always can find a d_{ij} with the corresponding segment fitting because

$$2r\sqrt{1 - \frac{1}{1 + a^2}} = \frac{4}{\sqrt{1 + a^2}} \frac{a}{c_H} \operatorname{diam} \mathcal{D}_0 \geq 2 \operatorname{diam} \mathcal{D}_0 .$$

So the boundary of any domain $\mathcal{D}_i$ is always put together by first taking a piece of the boundary of the basic domain $\mathcal{D}_0$ and k_i pieces of one unique circle, and then applying rigid motions to all pieces. Hence, for every $i = 1, \dots, m$ we can apply Lemma 3 with $k = K$, $l = K + \mu$, where μ is the minimal number of smooth pieces of $\partial \mathcal{D}_0$, $\mathcal{H}_0 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$, $\mathcal{H} = \mathcal{D}_i$, and $\mathbf{v}_j$ $(j = 1, \dots, k)$ suitable to make up the circular segments $\sigma(a, d_{ij}; r) + \mathbf{v}_{ij}$ $(j = 1, \dots, k_i)$ out of the one disc $\mathcal{H}_0$.

Thus we obtain

$$|\#(\lambda \mathcal{D}_i \cap \mathbb{Z}^2) - \lambda^2 \operatorname{area} \mathcal{D}_i| \leq C_0 \lambda^{\frac{46}{73}} (\log \lambda)^{\frac{315}{146}} \qquad (\lambda \geq 2), \tag{4.1}$$

where the constant C_0 depends only on K, $\mathcal{D}_0$ and $\mathcal{H}_0$, which actually means that it depends only on $\mathcal{D}_0$ and H.

Next we show that Proposition 1 can be applied to all segments $\sigma(a, d_{ij}; r)$. Let $\sigma = \sigma(a, d_{ij}; r)$ and let φ_1 and φ_2 denote the circle tangent angles in the left and right vertex of σ, respectively. Further let φ_0 denote the angle of the straight line bounding the segment σ. (All angles are to be considered relative to the horizontal.) Then, by the definition of the universal radius r and with $\delta := \operatorname{diam} \sigma$ and $c := c_H$,

$$r \geq \frac{2\delta}{c} \geq \frac{\delta}{2 \sin\left(\frac{c}{3}\right)} = \frac{\delta}{2 \cos\left(\frac{\pi}{2} - \frac{c}{3}\right)},$$

whence

$$\varphi_1 + \frac{\pi}{2} - \varphi_0 = \arccos\Big(\frac{\delta}{2r}\Big) \geq \frac{\pi}{2} - \frac{c}{3}.$$

Then, since $\tan\varphi_0 = a$ and $c \leq a \leq 1$, we have

$$\tan\varphi_1 \geq \varphi_1 \geq \arctan a - \frac{c}{3} \geq \frac{\pi}{4}a - \frac{c}{3} \geq \frac{c}{3}.$$

On the other hand, for the second angle φ_2 we have

$$\varphi_2 = \varphi_0 + (\varphi_0 - \varphi_1) \leq 2\varphi_0 = 2\arctan a \leq 2\arctan(1-c),$$

whence

$$\tan\varphi_2 \leq \frac{2(1-c)}{1-(1-c)^2} \leq \frac{2}{c}.$$

As a consequence, if κ is the slope of any tangent to the circular piece of the boundary of the segment σ, then

$$0 < \frac{c_H}{3} \leq \kappa \leq \frac{2}{c_H} < \infty.$$

Thus, by Proposition 1, we have for every $i = 1,\dots,m$, $j = 1,\dots,k_i$ and $\lambda \geq 2$,

$$\big|\#\big(\lambda\big(\sigma(a,d_{ij};r) + \boldsymbol{v}_{ij}\big) \cap \mathbb{Z}^2\big) - \lambda^2 \operatorname{area}\sigma(a,d_{ij};r)\big| \leq C_4 \lambda^{\frac{46}{73}} (\log\lambda)^{\frac{315}{146}}, \quad (4.2)$$

where the constant C_4 depends only on r and c_H, i.e. only on $\mathcal{D}_0$ and H.

Now, always having in mind that $\#\{(x,y) \in \mathbb{Z}^2 : \ y = ax + \lambda b\} \leq 1$, we have for every $\lambda \geq 2$,

$$\#\big(\lambda\mathcal{D}(a,b) \cap \mathbb{Z}^2\big) = \sum_{i=1}^{m} \#\big(\lambda\mathcal{D}_i^+(a,b) \cap \mathbb{Z}^2\big) + \gamma \qquad (\gamma \in \{0,1\})$$

and

$$\lambda\mathcal{D}_i^+(a,b) = \lambda\mathcal{D}_i \setminus \bigcup_{j=1}^{k_i} \lambda\big(\sigma(a,d_{ij};r) + \boldsymbol{v}_{ij}\big) \qquad (i = 1,\dots,m),$$

so that by (4.1) and (4.2), Theorem 1 follows.

5. Lattice points in Huxley sectors

Let $\mathcal{D}$ be a Huxley domain, $E \in \mathbb{R}^2$ an arbitrary point, and $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$ planar vectors. Then we consider the sector $\mathcal{D}(E;\boldsymbol{v},\boldsymbol{w})$ given by

$$\mathcal{D}(E;\boldsymbol{v},\boldsymbol{w}) := \{X \in \mathcal{D} : \ (\exists\, t_1, t_2 \geq 0)(X = E + t_1\boldsymbol{v} + t_2\boldsymbol{w})\}.$$

Clearly, we have to place restrictions on the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in order to achieve a satisfying generalization of our result on segments of Huxley domains to sectors. For $H \geq 1$ define

$$\mathcal{V}_H := \Big\{(v_1,v_2) \in \big(\mathbb{R}\setminus\{0\}\big)^2 : \ \big(|v_1/v_2| \in \Re_H\big) \ \vee \ \big(|v_2/v_1| \in \Re_H\big)\Big\}.$$

Now, the main result of this section is the following theorem.

THEOREM 2. *Let $\mathcal{D}_0 \subset \mathbb{R}^2$ be a Huxley domain and $H \ge 1$. Then there exists a constant C' such that for all points $E \in \mathbb{R}^2$, for all vectors $\mathbf{v}, \mathbf{w} \in \mathcal{V}_H$, for every rotation matrix $\mathbf{A} \in \mathcal{A}$, for all $\alpha, \beta \in [0,1]$, for every expansion factor $\lambda \ge 2$, and with $\mathcal{D} = \mathcal{D}_0 \cdot \mathbf{A}$,*

$$\left|\#\big(\lambda\mathcal{D}(E;\mathbf{v},\mathbf{w}) \cap (\alpha+\mathbb{Z})\times(\beta+\mathbb{Z})\big) - \lambda^2 \operatorname{area}\mathcal{D}(E;\mathbf{v},\mathbf{w})\right| \le C'\lambda^{\frac{46}{73}}(\log\lambda)^{\frac{315}{146}}.$$

P r o o f . Let, for abbreviation, $\Gamma := (\alpha+\mathbb{Z})\times(\beta+\mathbb{Z})$. Clearly, we may assume that the vectors $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. Further, we may assume that the point E lies in the interior of the domain $\mathcal{D}$, because otherwise we obtain the result by applying once or twice Theorem 1 together with a possible help of suitable reflections. Then we have $\mathcal{D}(E;\mathbf{v},\mathbf{w})^\circ \ne \emptyset$. We may assume without loss of generality that the domain $\mathcal{D}(E;\mathbf{v},\mathbf{w})$ is connected, because otherwise we consider its components. Now, following the ideas in Section 4, it is not difficult to find a Huxley domain $\mathcal{D}^*$ such that

$$\overline{\mathcal{D}^* \setminus \mathcal{D}(E;\mathbf{v},\mathbf{w})} = \bigcup_{i=1}^{k} \sigma_i,$$

where $\sigma_1,\dots,\sigma_k$ are pairwise disjoint compact segments of circles with one universal radius r, and the straight line segments $\sigma_k \cap \mathcal{D}(E;\mathbf{v},\mathbf{w})$ always being parallel to $\mathbf{v}$ or $\mathbf{w}$. The number k is clearly bounded by a constant depending only on $\mathcal{D}_0$. Since $\mathbf{v},\mathbf{w} \in \mathcal{V}_H$, we have, by applying Theorem 1 to the basic domain $x^2+y^2 \le r^2$ and with a possible help of suitable translations and reflections, for every segment σ_i

$$\#(\lambda\sigma_i \cap \Gamma) = \lambda^2 \operatorname{area}\sigma_i + O\Big(\lambda^{\frac{46}{73}}(\log\lambda)^{\frac{315}{146}}\Big) \tag{5.1}$$

with the O-constant depending only on H and r. (Note that Proposition 1 only would not imply (5.1) because it is insufficient for arbitrary segments of circles.)

Now we apply Lemma 3 with $\mathcal{H}_0 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le r^2\}$ and $\mathcal{H} = \mathcal{D}^*$. This yields

$$\#(\lambda\mathcal{D}^* \cap \Gamma) = \lambda^2 \operatorname{area}\mathcal{D}^* + O\Big(\lambda^{\frac{46}{73}}(\log\lambda)^{\frac{315}{146}}\Big). \tag{5.2}$$

Further we have,

$$\#\big(\lambda\mathcal{D}(E;\mathbf{v},\mathbf{w}) \cap \Gamma\big) = \#(\lambda\mathcal{D}^* \cap \Gamma) - \sum_{i=1}^{k} \#(\lambda\sigma_i \cap \Gamma) + O(1). \tag{5.3}$$

Now by inserting the right hand sides of (5.1) and (5.2) into (5.3) we reach our goal since

$$\operatorname{area}\mathcal{D}^* - \sum_{i=1}^{k} \operatorname{area}\sigma_i = \operatorname{area}\mathcal{D}(E;\mathbf{v},\mathbf{w}).$$

A natural application of Theorem 2 is one to sectors of circles. Let $H \geq 1$ and define for $R \geq 2$ and $\kappa > 0$ with $\kappa \in \Re_H$ or $1/\kappa \in \Re_H$,

$$\mathcal{S}(R;\kappa) := \{(x,y) \in \mathbb{R}^2 : (x \geq 0) \wedge (0 \leq y \leq \kappa x) \wedge (x^2+y^2 \leq R^2)\}.$$

Then, by symmetry and Theorem 2 with $\mathcal{D}_0 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq R^2\}$, $E = (0,0)$, $\boldsymbol{v} = (1,\kappa)$, and $\boldsymbol{w} = (1,-\kappa)$, we derive (with the O-constant depending only on H)

$$\#\big(\mathcal{S}(R;\kappa) \cap \mathbb{Z}^2\big) = \frac{\arctan\kappa}{2}R^2 + \frac{1}{2}R + O\Big(R^{\frac{46}{73}}(\log R)^{\frac{315}{146}}\Big). \qquad (\star)$$

Note that this result goes beyond the scope of the problem Nowak [5] deals with since there are considered only sectors $x^2+y^2 \leq R^2$, $\alpha \leq y/x \leq \beta$ with $0 < \alpha < \beta$. □

Further, $(\star)$ implies the following nice corollary related to the circle problem.

COROLLARY 1. *For a natural number k, k not a square, define the arithmetic function*

$$A_k(n) := \#\{(x,y) \in \mathbb{N}^2 : (x^2+y^2 = n) \wedge (y^2 \leq kx^2)\} \qquad (n \in \mathbb{N}).$$

Then as $N \to \infty$,

$$\frac{1}{N}\sum_{n=1}^{N} A_k(n) = \frac{\arctan\sqrt{k}}{2} - \frac{1}{2\sqrt{N}} + O\Big(N^{-\frac{50}{73}}(\log N)^{\frac{315}{146}}\Big),$$

the O-constant depending on k.

An analogous result related to the divisor problem is the next, which we close this section with.

COROLLARY 2. *For algebraic irrationals α, β, $0 < \alpha < \beta$, define the arithmetic function*

$$B_{\alpha,\beta}(n) := \#\{(x,y) \in \mathbb{N}^2 : (x \cdot y = n) \wedge (\alpha < y/x < \beta)\} \qquad (n \in \mathbb{N}).$$

Then as $N \to \infty$,

$$\frac{1}{N}\sum_{n=1}^{N} B_{\alpha,\beta}(n) = \frac{1}{2}\log\Big(\frac{\beta}{\alpha}\Big) + O\Big(N^{-\frac{50}{73}}(\log N)^{\frac{315}{146}}\Big),$$

the O-constant depending on α and β.

6. Application to fractional part sums

In this final section we consider sums

$$\Psi(f;u,v;\lambda) := \sum_{u\lambda \le n \le v\lambda} \psi\Big(\lambda f\Big(\frac{n}{\lambda}\Big)\Big),$$

where λ is a large real parameter and f is a real-valued function defined on an interval $[u,v] \subset \mathbb{R}$ with continuous derivatives up to order 3 and the property that f'' does not vanish on $[u,v]$. (See Nowak [6] for recent results concerning such sums.) By Huxley's method ([2; Theorems 18.2.1, 18.2.2]),

$$\Psi(f;u,v;\lambda) \ll \lambda^{\frac{46}{73}} (\log\lambda)^{\frac{315}{146}} \qquad (\lambda \to \infty) \qquad (\diamondsuit)$$

under the additional condition that f''' does not vanish on $[u,v]$.

This condition has always been interpreted as technical (cf. Nowak [6]) and indeed it is superfluous as shown by the following theorem, which we conclude this article with.

THEOREM 3. *Fix $\alpha,\beta \in \mathbb{R}$ and $f\colon [\alpha,\beta] \to \mathbb{R}$, and assume that f is three times continuously differentiable on (an open neighborhood of) $[\alpha,\beta]$ with $f'' \neq 0$ there.*

Then the inequality $(\diamondsuit)$ holds uniformly in u, v $(\alpha \le u \le v \le \beta)$.

Proof. Fix $\kappa = \sqrt{2} + \big[|f'(\alpha)|\big] + \big[|f'(\beta)|\big]$. Then $\kappa > 1, |f'(\alpha)|, |f'(\beta)|$ and, by [4; Theorem 3.4], $1/\kappa \in \Re_H$ with $H = 4 + [\kappa]$. Further, for $u,v \in [\alpha,\beta]$, $u < v$, define linear functions g_u, g_v,

$$g_u(x) = f(u) - \kappa(x-u), \quad g_v(x) = f(v) + \kappa(x-v) \qquad (x \in \mathbb{R}),$$

so that $g_u(u) = f(u)$ and $g_v(v) = f(v)$. Then there is a unique (and easily computable) $x_0 \in\,]u,v[$ such that $g_u(x_0) = g_v(x_0) < f(x_0)$. Let $g_{u,v} := \max\{g_u, g_v\}$. Then $g_{u,v}(x) < f(x)$ for all $x \in\,]u,v[$ and $g_{u,v}(u) = f(u)$, $g_{u,v}(v) = f(v)$.

Note that $\mathcal{G}(f) := \big\{(x,f(x)) : \alpha \le x \le \beta\big\}$ can be read as a piece of the boundary of a Huxley domain because for the radius of curvature ϱ we have

$$\varrho = \frac{(1+\tan^2\tau)^{\frac{3}{2}}}{f''\big(f'^{-1}(\tan\tau)\big)} \cdot \frac{f''(\alpha)}{|f''(\alpha)|},$$

where τ is the tangent angle (relative to the horizontal).

Now consider the sectors

$$\mathcal{S}(f;\kappa,u,v) := \big\{(x,y) \in \mathbb{R}^2 : (u \le x \le v) \wedge \big(g_{u,v}(x) \le y \le f(x)\big)\big\}$$
$$(\alpha \le u < v \le \beta).$$

Obviously, $\mathcal{S}(f;\kappa,u_1,v_1) \supset \mathcal{S}(f;\kappa,u_2,v_2)$ if $\alpha \le u_1 \le u_2 < v_2 \le v_1 \le \beta$. Then, with the help of a suitable fixed Huxley domain $\mathcal{D}$ with $\partial\mathcal{D} \supset \mathcal{G}(f)$ and $\mathcal{D} \supset \mathcal{S}(f;\kappa,\alpha,\beta)$, we obtain, by applying Theorem 2 with $E = \big(x_0, g_{u,v}(x_0)\big)$, $\boldsymbol{v} = (-1,\kappa)$, $\boldsymbol{w} = (1,\kappa)$, and $H = 4 + [\kappa]$,

$$\#\big(\lambda\mathcal{S}(f;\kappa,u,v) \cap \mathbb{Z}^2\big) = \lambda^2 \operatorname{area}\mathcal{S}(f;\kappa,u,v) + O\Big(\lambda^{\frac{46}{73}}(\log\lambda)^{\frac{315}{146}}\Big) \qquad (\lambda \to \infty)\,. \tag{6.1}$$

Note that the O-constant depends on $\mathcal{G}(f)$ but not on u or v!

On the other side,

$$\begin{aligned}\#\big(\lambda\mathcal{S}(f;\kappa,u,v) \cap \mathbb{Z}^2\big) = & \sum_{\lambda u \le n \le \lambda v} \lambda\Big(f\Big(\frac{n}{\lambda}\Big) - g_{u,v}\Big(\frac{n}{\lambda}\Big)\Big) - \Psi(f;u,v;\lambda) \\ & - \sum_{\lambda u \le n \le \lambda x_0} \psi\Big(-\lambda g_u\Big(\frac{n}{\lambda}\Big)\Big) - \sum_{\lambda x_0 < n \le \lambda v} \psi\Big(-\lambda g_v\Big(\frac{n}{\lambda}\Big)\Big)\,.\end{aligned}$$

Consequently, by applying the Euler summation formula to the first sum and Lemma 1 (with $a = \sqrt{2} - 1$ and $a = 2 - \sqrt{2}$, respectively) to the last two sums, we derive

$$\#\big(\lambda\mathcal{S}(f;\kappa,u,v) \cap \mathbb{Z}^2\big) = \lambda^2 \operatorname{area}\mathcal{S}(f;\kappa,u,v) - \Psi(f;u,v;\lambda) + O\big(\lambda^{\frac{46}{73}}\big) \quad (\lambda \to \infty) \tag{6.2}$$

with the O-constant depending on α, β, and H.

Thus Theorem 3 follows by comparing (6.1) and (6.2). □

Final remark. The exponent $-3/8$ in the definition of the sets $\mathfrak{R}_H$ is a kind of house number and intentionally not chosen optimal. (Theorems 1 and 2 obviously remain unchanged when $-3/8$ is replaced by any fixed number $-\theta$ with $77/208 \le \theta \le 3/8$.) We have chosen $-3/8$ because it is a nice exponent and it leaves space for possibly further improvements of Huxley's method which would automatically improve the bounds in Theorems 1 and 2. Actually, in the meantime a further improvement has been announced. In a yet unpublished paper [1] Huxley shows that the bound $\lambda^{\frac{46}{73}}(\log\lambda)^{\frac{315}{146}}$ can be sharpened to $\lambda^{\frac{131}{208}}(\log\lambda)^{\frac{18627}{8320}}$. Consequently, Theorems 1 to 3 are still true with the sharper bound (and a fortiori with the bound $\lambda^{0.63}$). Further improvements of our results, without reducing the sets $\mathfrak{R}_H$, are of course only possible up to a bound $\lambda^{5/8}$, but anyhow the exponent $5/8$ is so small that it certainly lies far beyond the scope of Huxley's method.

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Institut für Mathematik u.a.St.
Universität für Bodenkultur
Peter Jordan-Straße 82
A–1190 Wien
AUSTRIA
E-mail: kuba@edv1.boku.ac.at