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Mathematica Slovaca, Vol. 53 (2003), No. 2, 145--172

Persistent URL: http://dml.cz/dmlcz/136881

Math. Slovaca, 53 (2003), No. 2, 145–172

Mathematica Slovaca

Dedicated to the memory of Jeanette Huynh (1988-2002), niece of the second named author.

ON THE MODULUS OF THE RIEMANN ZETA FUNCTION IN THE CRITICAL STRIP

FILIP SAIDAK — PETER ZVENGROWSKI

(*Communicated by Stanislav Jakubec*)

ABSTRACT. For the Riemann zeta function $\zeta(s)$, defined for complex $s=\sigma+it$, we write $\sigma=\frac{1}{2}+\Delta$, and we study the horizontal behaviour of $|\zeta(s)|$ in the critical strip $|\Delta|\le\frac{1}{2}$. We prove

$$\left|\zeta\left(\frac{1}{2}-\Delta+it\right)\right| \ge \left|\zeta\left(\frac{1}{2}+\Delta+it\right)\right|$$

for $0\le\Delta\le\frac{1}{2}$, $2\pi+1\le t$; and we give accurate but simple asymptotic estimates for the quotient $\alpha(\Delta,t)$ of these two quantities. Inequalities and numerical tables are presented which show just how accurate these estimates are. Several conjectures related to the Riemann Hypothesis are discussed as well.

1. Introduction

1.1. In 1838, P. G. L. Dirichlet [8] defined, for real numbers $s\ge 1$, the function $\zeta(s)$ as

$$\zeta(s) := \sum_{n\ge 1}\frac{1}{n^s} = \prod_p\left(1-\frac{1}{p^s}\right)^{-1}, \tag{1}$$

where the product representation is due to Euler [11] in 1737, and is extended over all primes. This "Euler product" relates the zeta function to primes, and during the past three centuries hundreds of papers have been written about this remarkable connection. To fully appreciate the link, one must consider $\zeta(s)$ for complex values of $s=\sigma+it$.

2000 Mathematics Subject Classification: Primary 11M06, 11M26, 11M99.
Keywords: Riemann zeta function, modulus of $\zeta(s)$, functional equation, Stirling's series, horizontal behaviour of $\zeta(s)$.

For $\sigma = \operatorname{Re}(s) > 1$, the Euler product (1) converges and defines $\zeta(s)$ as a holomorphic function of s. For $\sigma > 0$, the function $\zeta(s)$ is meromorphic with a single pole of first order at $s = 1$ (with residue 1), and has the representation

$$\zeta(s) = \frac{1}{s-1} + 1 - s \int_1^\infty \frac{x - [x]}{x^{s+1}} \, \mathrm{d}x \,, \tag{2}$$

where $[x]$ denotes the integer part of x (for more detail, see [14]). If we let $\pi(x)$ be the number of primes $\leq x$, then for all s with $\sigma > 1$, it is not difficult to show

$$\log \zeta(s) = s \int_2^\infty \frac{\pi(x)}{x(x^s - 1)} \, \mathrm{d}x \,.$$

Also, the celebrated prime number theorem[1] concerning the function $\pi(x)$, which is equivalent to the statement $\zeta(1+\mathrm{i}t) \neq 0$ for all $t \in \mathbb{R}\backslash\{0\}$ (for a simple proof of this, see Mertens [23]), can then be obtained by inverting the above relation.

1.2. Furthermore, as Euler [12] conjectured in 1748, and Riemann [29] proved in 1859, the zeta function $\zeta(s)$ satisfies the important functional equation (cf. [9], [17], [18], [19], or [34]):

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s) \Gamma(1-s) \sin \frac{\pi s}{2} \,. \tag{3}$$

Taking (2) and (3) together extends $\zeta(s)$ to a meromorphic function on the entire complex plane with a single pole at $s = 1$, and having obvious zeros (i.e. "trivial zeros") at $s = -2, -4, -6, \dots$. All other zeros, the so called "non-trivial" zeros, must lie in the "critical strip" $0 \leq \sigma \leq 1$, since clearly $\zeta(s) \neq 0$ for $\sigma > 1$, by the convergence of the Euler product in (1).

In [29], one also finds the notorious Riemann Hypothesis:

$$\bigl(0 \leq \sigma \leq 1 \quad \& \quad \zeta(\sigma + \mathrm{i}t) = 0\bigr) \implies \sigma = \frac{1}{2} \,, \tag{\textbf{RH}}$$

[1]The prime number theorem, proved independently by Hadamard [15] and de la Vallée-Poussin [35] in 1896, states

$$\pi(x) \sim \operatorname{li}(x) := \lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \right) \frac{\mathrm{d}t}{\log t} \qquad \text{as} \quad x \to \infty \,.$$

For simpler, more recent analytic proofs of this fundamental theorem, see D. J. Newman [26], or D. Zagier [38].

or, in terms of the prime counting[2] function $\pi(x)$, **(RH)** is equivalent to

$$\pi(x) = \mathrm{li}(x) + O\bigl(\sqrt{x}\log x\bigr).$$

The Riemann Hypothesis is generally considered to be the most important unsolved problem of mathematics today, mainly because of its many very deep implications. It has been verified for $\zeta(\sigma+\mathrm{i}t)$ for large ranges of t, and we know (see [4], [21], [27], [28], etc.) that the first 10^{10} zeros of $\zeta(\sigma+\mathrm{i}t)$ all lie on the line $\sigma = \frac{1}{2}$.

1.3. The functional equation plays a central rôle in the zeta function theory, however, as Grosswald writes ([14; p. 137]): "... the functional equation — probably our most powerful tool, so far — does not give us much information on what happens for $0 \le \sigma \le 1$, in the so called critical strip." The purpose of this paper is to explore the intriguing observation that $|\zeta(\sigma+\mathrm{i}t)|$ behaves very nicely with respect to σ, as contrasted to its incredibly complicated behaviour with respect to t (see Figures 1, 2 in §2.4). In our present investigations we show that, contrary to Grosswald's claim, one can apply certain sharp estimates of the Euler-MacLaurin type, together with the functional equation (3), in order to obtain proofs of some interesting new results concerning the horizontal behaviour of $|\zeta(s)|$ in the critical strip.

2. Statement of theorems

2.1. The precise statements of our main results are as follows:

THEOREM 1. *Let $s = \frac{1}{2} + \Delta + \mathrm{i}t$. For $0 \le \Delta \le \frac{1}{2}$, and $t \ge 2\pi + 1$, we have*

$$\left|\zeta\Bigl(\frac{1}{2} - \Delta + \mathrm{i}t\Bigr)\right| \ge \left|\zeta\Bigl(\frac{1}{2} + \Delta + \mathrm{i}t\Bigr)\right|. \tag{$*$}$$

Note 1. If the inequality $(*)$ in our Theorem 1 could be strengthened to show that, for $0 < \Delta \le \frac{1}{2}$, one has

$$\left|\zeta\Bigl(\frac{1}{2} - \Delta + \mathrm{i}t\Bigr)\right| > \left|\zeta\Bigl(\frac{1}{2} + \Delta + \mathrm{i}t\Bigr)\right|, \tag{$**$}$$

then the Riemann Hypothesis would follow. This is because an easy consequence of the functional equation (3) is that the non-trivial zeros (i.e. the zeros in the

[2]The best unconditional result for $\pi(x)$ is given by a theorem of Walfisz [36]:

$$\pi(x) = \mathrm{li}(x) + O\bigl\{x\exp\bigl[-A(\log x)^{3/5}(\log\log x)^{-1/5}\bigr]\bigr\}, \qquad \text{where} \quad A \in \mathbb{R}^+.$$

critical strip) either occur on $\sigma = \frac{1}{2}$, or in pairs $s = \frac{1}{2} \pm \Delta + it$, with $\frac{1}{2} \geq \Delta > 0$ (and their complex conjugates).

Note 2. The number $2\pi + 1$ defining the vertical range in all our results is simply a convenience, in fact, our proofs are all valid for $t \geq 7$. However, the theorems are false if $t \leq 2\pi$.

2.2. In the following theorem and thereafter, the symbol $\sim$ means asymptotic as $t \to \infty$. We have:

THEOREM 2 (First and Second Approximations). *Again write s in the form $s = \frac{1}{2} + \Delta + it$, and define (also see Remark in 4.1) the function $\alpha(\Delta, t)$ as*

$$\alpha(\Delta, t) := \frac{\left|\zeta\left(\frac{1}{2} - \Delta + it\right)\right|}{\left|\zeta\left(\frac{1}{2} + \Delta + it\right)\right|}.$$

For $0 \leq \Delta \leq \frac{1}{2}$, as $t \to \infty$, we have:

$$\alpha(\Delta, t) \sim \left(\frac{|s|}{2\pi}\right)^{\Delta} \sim \left(\frac{t}{2\pi}\right)^{\Delta}. \tag{†}$$

Remark 1. With this notation our Theorem 1 shows that $\alpha(\Delta, t) \geq 1$ for all $t \geq 2\pi + 1$.

Remark 2. In Section 9 below (Theorem 4 in particular) we explain why the second approximation (the one on the far right of (†)) is always better than the first approximation.

Let us now state two other results we shall prove — Theorem 3 and Theorem 4. Theorem 3 gives a still closer approximation to the function $\alpha(\Delta, t)$, while Theorem 4 proves a result that orders all of our estimations. We have:

THEOREM 3 (Third Approximation). *For $s = \sigma + it = \frac{1}{2} + \Delta + it$, where $0 \leq \Delta \leq \frac{1}{2}$, we have*

$$\alpha(\Delta, t) \sim \beta(\Delta, t) := \left(\frac{|s|}{2\pi}\right)^{\Delta} \left(1 - \frac{4\sigma^3 - \sigma}{12t^2}\right). \tag{††}$$

THEOREM 4. *In the usual notation, if $0 \leq \Delta \leq \frac{1}{2}$, and $2\pi + 1 \leq t$, then we have*

$$\beta(\Delta, t) \leq \alpha(\Delta, t) \leq \left(\frac{t}{2\pi}\right)^{\Delta} \leq \left(\frac{|s|}{2\pi}\right)^{\Delta}. \tag{\#}$$

2.3. In the remaining sections of this paper, we shall prove the estimates (†) of Theorem 2 first, and then improve it to (††). Theorem 1 will then be proved, followed by Theorem 4.

We should also note that $(t/2\pi)^{\Delta}$, i.e. the second estimate of (†) in our Theorem 2, appears in Titchmarsh's classic text [34; Sect. 4.12, p. 78, Sect. 5.1, p. 95]. He rewrites the functional equation (3) as

$$\zeta(s) = \chi(s)\zeta(1-s)\,, \tag{3'}$$

where

$$\chi(s) = 2^{s-1}\pi^{s}\sec\Big(\frac{\pi s}{2}\Big)\Gamma(s)^{-1}\,.$$

For $s = \sigma + it$, in any fixed strip $A \le \sigma \le B$, as $t \to \infty$, we have

$$\log\Gamma(\sigma+it) = \Big(\sigma+it-\frac{1}{2}\Big)\log(it) - it + \frac{1}{2}\log(2\pi) + O\Big(\frac{1}{t}\Big)\,,$$

and therefore

$$\Gamma(\sigma+it) = t^{\sigma+it-1/2}\,\mathrm{e}^{-\pi t/2-it+i\pi(\sigma-1/2)/2}\sqrt{2\pi}\Big(1+O\Big(\frac{1}{t}\Big)\Big)\,.$$

Hence

$$\chi(s) = \Big(\frac{2\pi}{t}\Big)^{\sigma+it-1/2}\mathrm{e}^{i(t+\pi/4)}\Big(1+O\Big(\frac{1}{t}\Big)\Big)\,.$$

From this it is clear that, as $t \to \infty$, for $\sigma < \frac{1}{2}$ we have

$$|\chi(s)| \sim \Big(\frac{t}{2\pi}\Big)^{\frac{1}{2}-\sigma}\,. \tag{T}$$

This is all Titchmarsh proves[3] in [34], and he notes that it implies

$$|\zeta(s)| = O\big(t^{1/2-\sigma}\big) \qquad \text{for} \quad \sigma \le -\delta < 0\,,$$

and

$$|\zeta(s)| = O\big(t^{3/2+\delta}\big) \qquad \text{for} \quad \sigma \ge -\delta\,.$$

In the book he uses these bounds as a motivation for the study of the Lindelöf Hypothesis (equivalent to the statement $|\zeta(s)| = O\big(t^{\varepsilon}\big)$ for all $\varepsilon > 0$), and as a starting point for the methods of Vinogradov and Weyl-Hardy-Littlewood (see [34; Chapt. 5]).

It is important to note that the above technique suffices to get estimates such as those in our Theorem 2, but is not nearly delicate enough to prove the inequalities in our Theorem 1 and Theorem 4.

[3] Karatsuba & Voronin [18; p. 84], in their chapter on the approximate functional equation, are also content with a similarly weak estimate.

Remark. Some idea of the accuracy of our estimates can be obtained from the fact that for $0 \le \Delta \le \frac{1}{2}$ and $t \ge 2\pi + 1$, our $\left(|s|/2\pi\right)^{\Delta}$ is accurate to within the factor 1.0051; $(t/2\pi)^{\Delta}$ to within 1.00016, and (††) to within 1.000064. For $t \ge 50$ these factors are, respectively, 1.00010, 1.000032, and 1.00000028 (cf. the Appendix). For example, if $\Delta = 0.2$ and $t = 70$, then the numbers in Theorem 4, to ten decimal places, are

$$1.6195076868 \le \alpha(0.2, 70) \le 1.6195100013 \le 1.6195261956,$$

where

$$\alpha(0.2, 70) = 1.6195076876\ldots.$$

2.4. Motivation behind our research comes from various observations concerning the simplicity of the horizontal behaviour of $|\zeta(s)|$ in the critical strip.

In order to give the reader a better idea of how much simpler (than the behaviour of the modulus on the vertical lines) the horizontal behaviour of $|\zeta(\sigma + \mathrm{i}t)|$ really is, we fix $t = t_0$, and in the Figure 1 below we illustrate three prototypical cases (i.e. for t_0 "far" from a zero, "near" a zero, and "at" a zero of $\zeta(s)$, respectively) of the behaviour of $|\zeta(\sigma + \mathrm{i}t_0)|$ for $0 \le \sigma \le 1$, while the more familiar Figure 2 shows the much more complicated behaviour of $\left|\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right|$ for $0 \le t \le 100$.

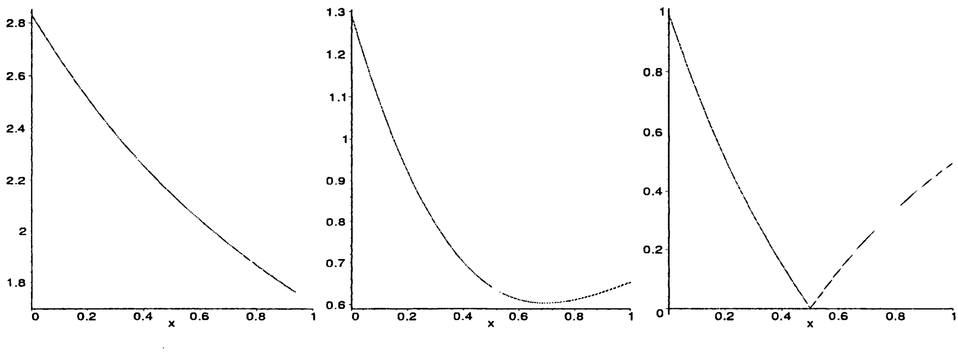

FIGURE 1. $|\zeta(\sigma + \mathrm{i}t_0)|$ for $t_0 = 17$, 24.5 and 25.011, with $0 \le \sigma \le 1$.

Note. The vertical scale varies for the graphs in Figure 1, and we also remark that the graphs in Figure 1, if plotted for a larger range of σ, are essentially no more complicated than in the range $0 \le \sigma \le 1$ depicted (of course, we always have $\left|\zeta\left(\sigma + \mathrm{i}t_0\right)\right| \to 1$, as $\sigma \to \infty$).

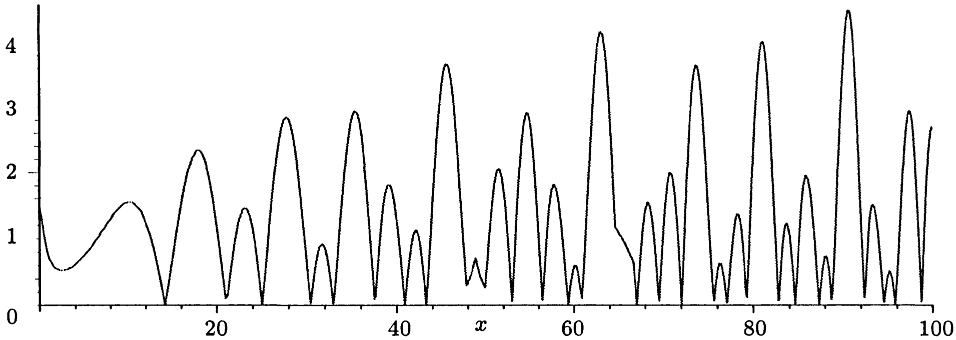

FIGURE 2. $|\zeta(\sigma + it)|$ for $\sigma = \frac{1}{2}$, and $0 \le t \le 100$.

2.5. In the light of this "evidence" it is now plausible to conjecture:

CONJECTURE 1. For any fixed $t \ge 2\pi + 1$, and $0 \le \sigma < \frac{1}{2}$, we have

$$\frac{\partial |\zeta(s)|}{\partial \sigma} < 0 . \tag{4}$$

Equivalently, $|\zeta(s)|$ is strictly monotone decreasing in this range.

Just like $(**)$, Conjecture 1 implies the Riemann Hypothesis.

3. Sharp Stirling series

3.1. The proofs of our main results are based on considerations involving the functional equation (3) for $\zeta(s)$, as well as certain series related to the Γ-function[4]. Let us start by recalling some definitions and a couple of standard lemmas we shall require in our proofs.

For complex s with $\sigma > 0$, we have the following useful formulae (they are due to Euler [13] for real s): $\Gamma(s+1) = s\Gamma(s)$,

$$\Gamma(s) = \int_0^\infty \frac{t^s}{\mathrm{e}^t} \frac{\mathrm{d}t}{t}, \qquad \Gamma(s)\zeta(s) = \int_0^\infty \frac{t^s}{\mathrm{e}^t - 1} \frac{\mathrm{d}t}{t},$$

and

$$\Gamma(s)\Gamma(-s) = \frac{-\pi}{s \sin(\pi s)},$$

[4]Some of our techniques extend those of Lindelöf [20], who was able (among other things) to give a very short proof of the following theorem of Mellin [22]: $|\zeta(1+it)| < \log t$, as $t \to \infty$.

i.e. the functional equation of $\Gamma(s)$, implied by the general formula

$$\Gamma(s)\Gamma(z) = 2\Gamma(s+z) \int_0^{\pi/2} (\cos\theta)^{2s-1}(\sin\theta)^{2z-1}\,\mathrm{d}\theta\,.$$

We also note that $\Gamma(\overline{s}) = \overline{\Gamma(s)}$ and $\zeta(\overline{s}) = \overline{\zeta(s)}$.

3.2. Now, define Bernoulli polynomials $b_k(x)$ as the unique solutions of $b_0(x) = 1$, and the recursive condition $b'_k(x) = kb_{k-1}(x)$ with $\int_0^1 b_k(t)\,\mathrm{d}t = 0$. If we let $B_k(x)$, the Bernoulli functions, be the periodic functions (of period 1) that coincide with $b_k(x)$ on $[0,1)$, then the Euler-MacLaurin Summation Formula states (see [5], [9], [25]):

Lemma 1. *For any real $a < b$, and a $(m+1)$-times differentiable function $f(t)$ on the interval $[a,b]$, in the above notation, we have*

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\,\mathrm{d}t + \sum_{k=0}^{m} \frac{(-1)^{k+1}B_{k+1}}{(k+1)!}\left(f^{(k)}(b) - f^{(k)}(a)\right) + E_m\,,$$

where

$$E_m = \frac{(-1)^m}{(m+1)!}\int_a^b B_{m+1}(t)f^{(k+1)}(t)\,\mathrm{d}t\,.$$

3.3. The error term in Lemma 1 is due to T. J. Stieltjes [31], who in 1889 also proved that, if $s = r\,\mathrm{e}^{\mathrm{i}\theta}$ and $|s| \to \infty$ with $|\theta| = |\arg(s)| \le \pi - \varepsilon$ (for $\varepsilon > 0$), then the well-known 1730 series of J. Stirling [32] has the following sharp error:

$$\log\Gamma(s) = \left(s - \tfrac{1}{2}\right)\log s - s + \tfrac{1}{2}\log(2\pi) + \phi(s) \tag{5}$$

with

$$\phi(s) := \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)s^{2k-1}} + R_{2n}\,, \tag{5$'$}$$

and

$$|R_{2n}| \le \left(\frac{1}{\cos(\theta/2)}\right)^{2n+2}\left|\frac{B_{2n+2}}{(2n+2)(2n+1)s^{2n+1}}\right|\,. \tag{5$''$}$$

Here, and in Lemma 1, B_k denotes the kth Bernoulli number (first time appearing in J a k o b B e r n o u l l i's *Ars Conjectandi* [3] of 1713), defined[5] for all $k \geq 0$ via the Taylor series expansion for $|x| < 2\pi$,

$$f(x) = \frac{x}{\mathrm{e}^x - 1} \stackrel{\text{def}}{=} 1 - \frac{1}{2}x + \frac{B_2}{2!}x^2 - \frac{B_4}{4!}x^4 + \frac{B_6}{6!}x^6 - \cdots .$$

Exponentiating the above series (5) for $\log \Gamma(s)$ immediately gives us (see[6] [6], for example) the following result, where the function $\psi(s)$ below is defined as $\psi(s) = \exp(\phi(s))$.

Lemma 2. *For all complex s with $|\arg(s)| \leq \pi - \varepsilon$, we have*

$$\Gamma(s) = s^s \mathrm{e}^{-s} \sqrt{\frac{2\pi}{s}}\psi(s), \tag{6}$$

with

$$\psi(s) = 1 + \frac{1}{12s} + \frac{1}{288s^2} - \frac{139}{51840s^3} - \frac{571}{2488320s^4} + O(s^{-5}).$$

This lemma is one of the key formulae we shall need to apply in our proofs of Theorem 2 and Theorem 3. Clearly from (5) and (6), both $\phi(s)$ and $\psi(s)$ are holomorphic in our domain $0 \leq \sigma \leq 1$, $t \geq 2\pi + 1$.

4. The functional equation

4.1. Let us now return to the functional equation (3). We rewrite it as follows

$$\zeta(s) = (2\pi)^s \Gamma(1-s)\zeta(1-s)\frac{\sin(\frac{\pi s}{2})}{\pi},$$

thus

$$\zeta\Big(\frac{1}{2} - \Delta + \mathrm{i}t\Big) = \\ = (2\pi)^{1/2-\Delta}(2\pi)^{\mathrm{i}t}\overline{\Gamma\Big(\frac{1}{2} + \Delta + \mathrm{i}t\Big)}\frac{\sin(\frac{\pi}{2}(\frac{1}{2} - \Delta + \mathrm{i}t))}{\pi}\overline{\zeta\Big(\frac{1}{2} + \Delta + \mathrm{i}t\Big)},$$

[5] We have $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_n = 0$ for all odd $n > 1$. Recursively one can compute

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730} \cdots .$$

In terms of Bernoulli functions $B_k(x)$, we have $B_k = B_k(0)$ for all k.

[6] The formula (6) is taken from [6], but one should note that, in [6; p. 8], the series for $\log \Gamma(s)$ is given incorrectly. It uses an old (W h i t t a k e r & W a t s o n [37]) definition of Bernoulli numbers (cf. [5]).

and therefore

$$\begin{aligned}&\left|\zeta\left(\frac{1}{2}-\Delta+\mathrm{i}t\right)\right| = \\ &= \frac{(2\pi)^{1/2-\Delta}}{\pi}\left|\Gamma\left(\frac{1}{2}+\Delta+\mathrm{i}t\right)\right|\left|\sin\left(\frac{\pi}{2}\left(\frac{1}{2}-\Delta+\mathrm{i}t\right)\right)\right|\left|\zeta\left(\frac{1}{2}+\Delta+\mathrm{i}t\right)\right|,\end{aligned}$$

whence

$$\alpha(\Delta,t) = \frac{(2\pi)^{1/2-\Delta}}{\pi}\left|\Gamma\left(\frac{1}{2}+\Delta+\mathrm{i}t\right)\right|\left|\sin\left(\frac{\pi}{2}\left(\frac{1}{2}-\Delta+\mathrm{i}t\right)\right)\right|. \tag{7}$$

This formula will be the starting point of all our proofs.

Remark. Strictly speaking, the original definition of the function $\alpha(\Delta,t)$ in our Theorem 2 (see Section 2.1) makes no sense when $\zeta\left(\frac{1}{2}+\Delta+\mathrm{i}t\right)=0$. However, the above equation (7) shows that such points are removable singularities, and that the function $\alpha(\Delta,t)$ is in fact the modulus of the holomorphic function $A(\Delta,t)$, defined as $A(\Delta,t) = \frac{(2\pi)^{1/2-\Delta}}{\pi}\Gamma\left(\frac{1}{2}+\Delta+\mathrm{i}t\right)\sin\left(\frac{\pi}{2}\left(\frac{1}{2}-\Delta+\mathrm{i}t\right)\right)$. Since this function has no zeros for $t>0$, our $\alpha(\Delta,t)$ is a real analytic function in Δ, t.

5. Three technical lemmas

5.1. In order to prove (†) of Theorem 2, we shall need the following lemmas (everywhere below, $\exp(s)=\mathrm{e}^s$ will be used interchangeably):

Lemma 3. *For all complex s, write $s=\sigma+\mathrm{i}t=\left(\frac{1}{2}+\Delta\right)+\mathrm{i}t=r\,\mathrm{e}^{\mathrm{i}\theta}$. Then we have*

$$|\Gamma(s)| \sim \sqrt{\frac{2\pi}{\mathrm{e}}}\,\frac{|s|^{\Delta}}{\exp(t\theta+\Delta)}. \tag{8}$$

Proof. By the well-known Stirling's formula (see (6), or [9]) we know:

$$\Gamma(s) \sim s^{s-\frac{1}{2}}\,\mathrm{e}^{-s}\sqrt{2\pi} = \sqrt{2\pi}\exp\left(\left(s-\frac{1}{2}\right)\log s - s\right),$$

where

$$\begin{aligned}\left(s-\frac{1}{2}\right)\log s - s &= (\Delta+\mathrm{i}t)(\log r+\mathrm{i}\theta) - \frac{1}{2} - \Delta - \mathrm{i}t \\ &= \left(\Delta\log r - t\theta - \frac{1}{2} - \Delta\right) + \mathrm{i}(t\log r + \Delta\theta - t).\end{aligned}$$

Thus

$$\begin{aligned}|\Gamma(s)| &\sim \sqrt{2\pi}\exp\left(\Delta\log r - t\theta - \frac{1}{2} - \Delta\right) \\ &= \sqrt{2\pi}\,\frac{r^{\Delta}}{\exp\left(t\theta+\frac{1}{2}+\Delta\right)} = \sqrt{\frac{2\pi}{\mathrm{e}}}\,\frac{|s|^{\Delta}}{\exp(t\theta+\Delta)}.\end{aligned}$$

□

5.2. In addition to this lemma, we have:

Lemma 4. *For* $0 \le \Delta \le \frac{1}{2}$,

$$\left|\sin\left(\frac{\pi}{4} - \frac{\pi}{2}\Delta + \mathrm{i}\,\frac{\pi t}{2}\right)\right| \sim \frac{1}{2}\exp\left(\frac{\pi t}{2}\right). \tag{9}$$

Proof. For any complex number $s = \sigma + \mathrm{i}t$, one has

$$\sin(\sigma + \mathrm{i}t) = \sin(\sigma)\cosh(t) + \mathrm{i}\cos(\sigma)\sinh(t),$$

whence

$$|\sin(s)|^2 = \sin^2(\sigma)\cosh^2(t) + \cos^2(\sigma)\sinh^2(t) = \sin^2(\sigma) + \sinh^2(t).$$

This immediately implies

$$\left|\sin\left(\frac{\pi}{4} - \frac{\pi}{2}\Delta + \mathrm{i}\,\frac{\pi t}{2}\right)\right|^2 = \sin^2\left(\frac{\pi}{4} - \frac{\pi\Delta}{2}\right) + \sinh^2\left(\frac{\pi t}{2}\right),$$

where the second term can be estimated as

$$\sinh^2\left(\frac{\pi t}{2}\right) = \left(\frac{\mathrm{e}^{\pi t/2} - \mathrm{e}^{-\pi t/2}}{2}\right)^2 \sim \frac{1}{4}\exp(\pi t), \qquad \text{as} \quad t \to \infty,$$

while for the first term one has

$$\sin^2\left(\frac{\pi}{4} - \frac{\pi\Delta}{2}\right) \le \frac{1}{2}.$$

Taking square roots proves the result. □

5.3. For $s = \sigma + \mathrm{i}t$, in the same range, the third result we shall need is the following estimate, with θ as in Lemma 3:

Lemma 5. *For* $0 \le \Delta \le \frac{1}{2}$, *and all* $t \ge 2\pi + 1$, *we have*

$$\theta = \frac{\pi}{2} - \frac{\frac{1}{2} + \Delta}{t} + O\left(\frac{1}{t^3}\right). \tag{10}$$

Proof. Notice that if $s = r\,\mathrm{e}^{\mathrm{i}\theta}$, then $\tan(\theta) = \frac{t}{\sigma} = \frac{t}{\frac{1}{2}+\Delta}$ is large positive for $t \ge 2\pi + 1$ and $0 \le \Delta \le \frac{1}{2}$, and so $\delta = \cot(\theta) = \frac{\frac{1}{2}+\Delta}{t}$ is small in that range. Hence

$$\theta = \operatorname{arccot}(\delta) = \frac{\pi}{2} - \arctan(\delta) = \frac{\pi}{2} - \delta + \frac{\delta^3}{3} - \frac{\delta^5}{5} + \cdots.$$

□

6. Proof of Theorem 2

6.1. It is now easy to complete the proof of our first main result.

THEOREM 2. *For* $0 \le \Delta \le \frac{1}{2}$ *we have*

$$\alpha(\Delta,t) = \frac{\left|\zeta\left(\frac{1}{2}-\Delta+\mathrm{i}t\right)\right|}{\left|\zeta\left(\frac{1}{2}+\Delta+\mathrm{i}t\right)\right|} \sim \left(\frac{|s|}{2\pi}\right)^{\Delta} \sim \left(\frac{t}{2\pi}\right)^{\Delta}. \tag{$\dagger$}$$

Proof. Recall (7), i.e.

$$\alpha(\Delta,t) = \frac{(2\pi)^{1/2-\Delta}}{\pi}\left|\Gamma\left(\frac{1}{2}+\Delta+\mathrm{i}t\right)\right|\left|\sin\left(\frac{\pi}{2}\left(\frac{1}{2}-\Delta+\mathrm{i}t\right)\right)\right|,$$

and apply Lemma 3, Lemma 4 and Lemma 5, respectively, to get

$$\begin{aligned}
\alpha(\Delta,t) &\sim \frac{\sqrt{2\pi}}{\pi}\cdot\frac{1}{(2\pi)^{\Delta}}\cdot\frac{\sqrt{2\pi}}{\sqrt{e}}\cdot\frac{|s|^{\Delta}}{\exp(t\theta+\Delta)}\cdot\frac{1}{2}\exp\left(\frac{\pi t}{2}\right)\\
&= \left(\frac{|s|}{2\pi}\right)^{\Delta}\exp\left(\frac{\pi t}{2}-t\theta-\Delta-\frac{1}{2}\right)\\
&\sim \left(\frac{|s|}{2\pi}\right)^{\Delta}\mathrm{e}^{0}.
\end{aligned}$$

This proves the first estimate. Since $|s| \sim t$, the second one follows. □

7. Proof of Theorem 3

7.1. We are now ready to prove a more refined estimate of $\alpha(\Delta,t)$:

THEOREM 3. *For* $0 \le \Delta \le \frac{1}{2}$ *we have*

$$\alpha(\Delta,t) \sim \beta(\Delta,t) := \left(\frac{|s|}{2\pi}\right)^{\Delta}\left(1-\frac{4\sigma^3-\sigma}{12t^2}\right). \tag{$\dagger\dagger$}$$

Proof. The argument we present below closely follows the one given in the proof of Theorem 2, except here we take account of all terms of order $O(t^{-2})$ in the ratio $\alpha(\Delta,t)/\left(|s|/2\pi\right)^{\Delta}$. Three asymptotic estimates must be analyzed.

7.2. The first approximation is in Lemma 4:

$$\left|\sin\left(\frac{\pi}{4}-\frac{\pi}{2}\Delta+\mathrm{i}\frac{\pi t}{2}\right)\right| \sim \frac{1}{2}\exp\left(\frac{\pi t}{2}\right). \tag{11}$$

We have already seen (proof of Lemma 4)

$$\begin{aligned}\left|\sin\left(\frac{\pi}{4}-\frac{\pi}{2}\Delta+\mathrm{i}\frac{\pi t}{2}\right)\right| &= \sqrt{\sin^2\left(\frac{\pi}{4}-\frac{\pi}{2}\Delta\right)+\sinh^2\left(\frac{\pi t}{2}\right)}\\ &= \sinh\left(\frac{\pi t}{2}\right)\sqrt{1+\tau^2}\,,\end{aligned}$$

where

$$\tau=\frac{\sin\left(\frac{\pi}{4}-\frac{\pi}{2}\Delta\right)}{\sinh\left(\frac{\pi t}{2}\right)}\,.$$

Since τ is small positive, the binomial theorem applies, and it allows us to write

$$\begin{aligned}&\frac{1}{2}\left(\mathrm{e}^{\pi t/2}-\mathrm{e}^{-\pi t/2}\right)\left(1+\frac{1}{2}\tau^2-\frac{1}{8}\tau^4+\frac{1}{16}\tau^6-\cdots\right)\\ &=\frac{1}{2}\,\mathrm{e}^{\pi t/2}\left(1-\mathrm{e}^{-\pi t}\right)\left(1+\frac{1-2\sin^2\left(\frac{\pi}{4}-\frac{\pi}{2}\Delta\right)}{\mathrm{e}^{\pi t}}+O\left(\frac{1}{\mathrm{e}^{2\pi t}}\right)\right)\\ &=\frac{1}{2}\,\mathrm{e}^{\pi t/2}\left(1-\frac{\sin\left(\frac{\pi}{2}-\pi\Delta\right)}{\mathrm{e}^{\pi t}}+O\left(\frac{1}{\mathrm{e}^{2\pi t}}\right)\right).\end{aligned}$$

This approximation is correct up to $O(\mathrm{e}^{-\pi t})$ and is negligible in comparison with $O(t^{-n})$ for any $n\geq 1$.

7.3. For the second approximation we consider the exact Stirling formula (6). In order to approximate $\psi(s)$ to $O(t^{-2})$, since $|s|\sim t$, it suffices to consider

$$1+\frac{1}{12s}+\frac{1}{288s^2}=\frac{\left(\frac{\sigma}{t}\right)^2-1+\frac{1}{12}\frac{\sigma}{t^2}+\frac{1}{288t^2}+\mathrm{i}\left(\frac{2\sigma}{t}+\frac{1}{12t}\right)}{\left(\frac{s}{t}\right)^2}\,,$$

i.e.

$$\left|1+\frac{1}{12s}+\frac{1}{288s^2}\right|=\frac{\sqrt{\left(\left(\frac{\sigma}{t}\right)^2-1+\frac{1}{12}\frac{\sigma}{t^2}+\frac{1}{288t^2}\right)^2+\left(\frac{2\sigma}{t}+\frac{1}{12t}\right)^2}}{1+\frac{\sigma^2}{t^2}}\,,$$

since $\left|\left(\frac{s}{t}\right)^2\right|=\frac{|s|^2}{t^2}=1+\frac{\sigma^2}{t^2}$. Expanding the numerator, and then using the binomial series again, and also using the fact that $\left(1+\frac{\sigma^2}{t^2}\right)^{-1}=1-\frac{\sigma^2}{t^2}+\frac{\sigma^4}{t^4}-\cdots$, it is routine to determine that

$$\left|1+\frac{1}{12s}+\frac{1}{288s^2}\right|=1+\frac{\sigma}{12t^2}+O\left(t^{-3}\right).$$

We omit the details.

7.4. Finally, to make the correction to our third approximation is the easiest; one simply takes $\theta \sim \frac{\pi}{2} - \frac{\sigma}{t} + \frac{\sigma^3}{3t^3}$ (instead of just taking $\frac{\pi}{2} - \frac{\sigma}{t}$ as was done in Theorem 2). Then $\frac{\pi t}{2} - t\theta - \Delta - \frac{1}{2} = \frac{\pi t}{2} - t\theta - \sigma \sim -\frac{\sigma^3}{3t^2}$, accurate to $O(t^{-4})$. Proceeding exactly as in Theorem 2, but with these refinements, gives

$$\begin{aligned}
\alpha(\Delta, t) &\sim \left(\frac{|s|}{2\pi}\right)^{\Delta} \exp\left(\frac{\pi t}{2} - t\theta - \Delta - \frac{1}{2}\right) |\psi(t)| \\
&\sim \left(\frac{|s|}{2\pi}\right)^{\Delta} \exp\left(-\frac{\sigma^3}{3t^2}\right)\left(1 + \frac{\sigma}{12t^2}\right) \\
&\sim \left(\frac{|s|}{2\pi}\right)^{\Delta} \left(1 - \frac{\sigma^3}{3t^2}\right)\left(1 + \frac{\sigma}{12t^2}\right) \\
&\sim \left(\frac{|s|}{2\pi}\right)^{\Delta} \left(1 - \frac{4\sigma^3 - \sigma}{12t^2}\right).
\end{aligned}$$

□

Remark 1. All approximations in the above proof are clearly up to $O(t^{-4})$ or better, except possibly the approximation for $|\psi(s)|$. However, since $\psi(s) = 1 + \frac{1}{12s} + \frac{1}{288s^2} - \frac{139}{51840s^3} + \cdots$ and since $\theta = \arg(s) \sim \frac{\pi}{2}$, the first few terms of this series decrease rapidly in modulus while having arguments approximately $0, -\frac{\pi}{2}, \pi, \frac{\pi}{2}, 0, \dots$. This means that the term $a_3 s^{-3}$ only affects $|\psi(s)|$ up to $O(|s|^{-4})$ (a consequence of the Pythagoras theorem), since it is nearly parallel to the i-axis. Thus, our estimate $|\psi(s)| \sim \left|1 + \frac{1}{12s} + \frac{1}{288s^2}\right|$ is actually valid up to $O(t^{-4})$.

Remark 2. The heuristic argument in Remark 1 is easy to carry out analytically.

Remark 3. Clearly, further refinements of this type, up to $O(t^{-6})$, $O(t^{-8})$, etc. can be derived in the same way.

8. Proof of Theorem 1

8.1. We begin by proving an important lemma:

LEMMA 6. *Let $s = \frac{1}{2} + \Delta + it$, where $0 \le \Delta \le \frac{1}{2}$, and $t \ge 2\pi + 1$. Then $\Re(\phi(s))$ is an increasing function of Δ (see (5) and (5′) for definition of $\phi(s)$).*

Proof. It suffices to show $\Re(\phi'(s)) > 0$, since $\Re(\phi'(s)) = \frac{\partial}{\partial\Delta}\Re(\phi(s))$. From (5′) we have

$$\phi'(s) = -\sum_{k=1}^{n} \frac{B_{2k}}{2ks^{2k}} + R'_{2n} = -\frac{B_2}{2s^2} - \frac{B_4}{4s^4} - \cdots - \frac{B_{2n}}{2ns^{2n}} + R'_{2n}.$$

Here we again have an upper bound for the error term due to Stieltjes (given for $n = 0$ in Edwards' [9; p. 114]):

$$R'_{2n} \le \frac{1}{\cos^{2n+3}(\theta/2)} \cdot \left| \frac{B_{2n+2}}{(2n+2)s^{2n+2}} \right| .$$

Thus $\phi'(s) = -\frac{1}{12s^2} + R'_2$, where

$$|R'_2| \le \frac{1}{(\sqrt{2}/2)^5} \cdot \frac{1}{30} \cdot \frac{1}{4|s|^4} < \frac{1}{20|s|^4},$$

since

$$\frac{1}{(\sqrt{2}/2)^5} = \left(\sqrt{2}\,\right)^5 = \sqrt{32} < 6, \qquad \text{and} \qquad 0 \le \frac{\theta}{2} < \frac{\pi}{4}.$$

Hence, we also have $|\Re(R'_2)| \le |R'_2| < \frac{1}{20(\sigma^2+t^2)^2}$. Now, evidently

$$-\frac{1}{12s^2} = -\frac{\sigma^2 - 2\,\mathrm{i}\sigma t - t^2}{12(\sigma^2+t^2)^2} \implies \Re\left(-\frac{1}{12s^2}\right) = \frac{t^2-\sigma^2}{12(\sigma^2+t^2)^2},$$

and clearly, since we have the simple inequality $\frac{t^2-\sigma^2}{12(\sigma^2+t^2)^2} > \frac{1}{20(\sigma^2+t^2)^2}$ for all σ and t in our range, $\Re(\phi'(s)) > 0$ as desired. □

Note. Let us mention that (although not needed in the above proof), following [9; p. 113], in the notation $\Pi(s) = \Gamma(s+1)$ we have

$$\phi'(s) = \frac{\Pi'(s)}{\Pi(s)} - \log s - \frac{1}{2s}, \tag{12}$$

i.e. $\phi'(s)$ is closely related to the logarithmic derivative of $\Gamma(s)$.

8.2. Now we are ready to prove (an equivalent form of) our main result:

THEOREM 1. *For $0 \le \Delta \le \frac{1}{2}$ and $t \ge 2\pi + 1$, we have*

$$\alpha(\Delta, t) \ge 1. \tag{$*$}$$

Proof. Since $\alpha(0,t) = 1$, $(*)$ is evidently implied by the stronger statement that $\alpha(\Delta, t)$ is strictly monotone increasing, equivalently that $\log \alpha(\Delta, t)$ is strictly monotone increasing, and again equivalently $\frac{\partial}{\partial \Delta} \log \alpha(\Delta, t) > 0$. Starting, as usual, from (7), we apply Lemma 2 (the exact Stirling formula) and Lemma 3, and have

$$\alpha(\Delta, t) =$$
$$= 2\left(\frac{|s|}{2\pi}\right)^{\Delta} \exp\left(-t\theta - \Delta - \frac{1}{2}\right) |\psi(s)| \sqrt{\sin^2\Big(\frac{\pi}{2}\Big(\frac{1}{2} - \Delta\Big)\Big) + \sinh^2\Big(\frac{\pi t}{2}\Big)}\,.$$

Thus, taking logarithms yields

$$\log\alpha(\Delta,t)=\log 2+\Delta\log\left(\frac{|s|}{2\pi}\right)-t\theta-\frac{1}{2}-\Delta+\log|\psi(s)|+\frac{1}{2}\log w\,,$$

where (in this proof and hereafter)

$$w=w(\Delta,t)=\sin^2\left(\frac{\pi}{2}\left(\frac{1}{2}-\Delta\right)\right)+\sinh^2\left(\frac{\pi t}{2}\right).$$

Since $\log|z|=\Re(\log z)$ for any complex number z, we have

$$\log\alpha(\Delta,t)=\log 2+\Delta\log\left(\frac{|s|}{2\pi}\right)-t\theta-\frac{1}{2}-\Delta+\Re(\phi(s))+\frac{1}{2}\log w\,. \tag{13}$$

Using Lemma 6, and also the evident fact $\Delta\cdot\frac{\partial}{\partial\Delta}\log\left(\frac{|s|}{2\pi}\right)\geq 0$, we see

$$\frac{\partial}{\partial\Delta}\log\alpha(\Delta,t)\geq\log\left(\frac{|s|}{2\pi}\right)-\frac{\partial}{\partial\Delta}\left(t\theta+\frac{1}{2}+\Delta\right)+\frac{1}{2}\frac{\partial}{\partial\Delta}\log w\,.$$

Therefore, it remains to show that for all $t\geq 2\pi+1$ one has

$$\log\left(\frac{|s|}{2\pi}\right)>-\frac{1}{2}\frac{\partial}{\partial\Delta}\log w+\frac{\partial}{\partial\Delta}(t\theta+\sigma)\,. \tag{14}$$

On one hand, we have

$$\begin{aligned}\frac{\partial}{\partial\Delta}(t\theta+\sigma)&=\frac{\partial}{\partial\sigma}\left(t\left(\frac{\pi}{2}-\arctan\left(\frac{\sigma}{t}\right)\right)+\sigma\right)=\frac{-1}{1+\frac{\sigma^2}{t^2}}+1\\&=\frac{\sigma^2}{t^2}-\frac{\sigma^4}{t^4}+\frac{\sigma^6}{t^6}-\cdots=\frac{\sigma^2}{t^2}+R\,,\end{aligned}$$

where the error R satisfies $|R|\leq(\sigma/t)^4$.

On the other hand, from the definition of $w(\Delta,t)$ it follows that

$$-\frac{1}{2}\frac{\partial}{\partial\Delta}\log w=-\frac{1}{2}\frac{\frac{\partial w}{\partial\Delta}}{w}=\frac{-\sin\left(\frac{\pi}{2}-\pi\Delta\right)}{\sqrt{\sin^2\left(\frac{\pi}{2}\left(\frac{1}{2}-\Delta\right)\right)+\sinh^2\left(\frac{\pi t}{2}\right)}}\left(-\frac{\pi}{4}\right),$$

and so

$$\left|-\frac{1}{2}\frac{\partial}{\partial\Delta}\log w\right|\leq\frac{\pi}{4}\left(2\,\mathrm{e}^{-\pi t/2}\right)<\left(\frac{\sigma}{t}\right)^4.$$

However, as one can easily verify, for $|s|>2\pi\,\mathrm{e}^{0.0212411}=6.418\ldots$, we have $\log\left(\frac{|s|}{2\pi}\right)>\frac{\sigma^2}{t^2}+\frac{2\sigma^4}{t^4}$, and hence, in this range, (14) holds. □

Remark. As mentioned in the above proof, we have actually proved the stronger statement that the function $\alpha(\Delta,t)$ is strictly montone increasing. The weaker version of Theorem 1, as stated above, was also obtained by R. Spira [Duke Math. J., **32** (1965), 247–250]. This came to the authors' attention after the present paper went to press.

9. Proof of Theorem 4

9.1. Finally, let us prove one other, deeper result concerning the quality of approximations of the important function $\alpha(\Delta, t)$, a result that is obtained by pushing our techniques to their limits. The claim of the above Theorem 2 is that, for $0 \le \Delta \le \frac{1}{2}$, and $2\pi + 1 \le t$, we have $\alpha(\Delta, t) \sim \left(|s|/2\pi\right)^{\Delta}$. This is clearly exactly true for $\Delta = 0$, and in general the approximation ($\dagger$) is indeed very good. However, the following, asymptotically equivalent formula (cf. Theorem 2)

$$\alpha(\Delta, t) \sim \left(\frac{t}{2\pi}\right)^{\Delta} \tag{15}$$

works even better (see Table 1 in Appendix). In spite of its simplicity, $(t/2\pi)^{\Delta}$ is nearly as good as $\beta(\Delta, t)$ from Theorem 3. Furthermore, it also always happens to lie between $\left(|s|/2\pi\right)^{\Delta}$ and $\alpha(\Delta, t)$. We have:

THEOREM 4. *For $0 \le \Delta \le \frac{1}{2}$ and $t \ge 2\pi + 1$, we have the inequality*

$$\beta(\Delta, t) \le \alpha(\Delta, t) \le \left(\frac{t}{2\pi}\right)^{\Delta} \le \left(\frac{|s|}{2\pi}\right)^{\Delta}. \tag{\#}$$

9.2. A couple of preliminary remarks are in order. For $\Delta = \frac{1}{2}$, let us now show how good the approximation $\left(\frac{t}{2\pi}\right)^{\Delta}$ from (15) really is. First, note that from the formula in §3.1 for $\Gamma(s)\Gamma(-s)$, we have

$$\begin{aligned} |\Gamma(\mathrm{i}t)|^2 &= \Gamma(\mathrm{i}t)\overline{\Gamma(\mathrm{i}t)} = \Gamma(\mathrm{i}t)\Gamma(\,\overline{\mathrm{i}t}\,) = \Gamma(\mathrm{i}t)\Gamma(-\mathrm{i}t) \\ &= \frac{-\pi}{(\mathrm{i}t)\sin(\mathrm{i}\pi t)} = \frac{\pi}{t\sinh(\pi t)}, \end{aligned}$$

and therefore, directly from the definition of $\alpha(\Delta, t)$, we get

$$\begin{aligned} \alpha\left(\frac{1}{2}, t\right) &= \frac{1}{\pi}\,|\Gamma(1+\mathrm{i}t)|\,\left|\sin\left(\mathrm{i}\frac{\pi t}{2}\right)\right|, \\ \alpha^2\left(\frac{1}{2}, t\right) &= \frac{1}{\pi^2}\,|\Gamma(1+\mathrm{i}t)|^2\sinh^2\left(\frac{\pi t}{2}\right) = \frac{1}{\pi^2}\,|\mathrm{i}t\Gamma(\mathrm{i}t)|^2\sinh^2\left(\frac{\pi t}{2}\right) \\ &= \frac{t^2}{\pi^2}\frac{\pi}{t\sinh(\pi t)}\frac{1}{2}\left(\cosh(\pi t) - 1\right) = \frac{t}{2\pi}\frac{\cosh(\pi t) - 1}{\sinh(\pi t)}, \\ \alpha\left(\frac{1}{2}, t\right) &= \left(\frac{t}{2\pi}\right)^{1/2}\sqrt{\frac{\cosh(\pi t) - 1}{\sinh(\pi t)}} \sim \left(\frac{t}{2\pi}\right)^{1/2}, \qquad \text{as} \quad t \to \infty. \end{aligned}$$

Even for small values of t in our range, this relatively simple approximation is extremely good. For example, if $t = 10$, then

$$\sqrt{\frac{\cosh(\pi t) - 1}{\sinh(\pi t)}} = 1 - 2.271 \times 10^{-14}.$$

Remark. Since $\cosh(\pi t) - 1 < \sinh(\pi t)$, also $\alpha\big(\frac{1}{2}, t\big) < (t/2\pi)^{1/2}$.

9.3. In this section we shall prove $\beta(\Delta, t) \le \alpha(\Delta, t)$, and in §9.4 we prove $\alpha(\Delta, t) \le (t/2\pi)^{\Delta}$. Since $(t/2\pi)^{\Delta} \le \big(|s|/2\pi\big)^{\Delta}$ trivially, this will complete the proof of Theorem 4. We start with a useful technical lemma of little intrinsic interest.

Lemma 7. *For $\frac{1}{2} \le \sigma \le 1$ and $t > \frac{1}{2}$,*

$$f(\sigma, t) := \frac{12\sigma^2 - 1}{12t^2 - 4\sigma^3 + \sigma} - \frac{\sigma^2}{|s|^2} - \frac{\sigma^2 - t^2}{12|s|^4} \ge 0. \tag{16}$$

Proof. With some algebraic work one finds

$$f(\sigma, t) = \sigma \cdot \frac{t^2\big[(12\sigma^2 - 1)(12\sigma + 4\sigma^2 - 1) - 24\sigma\big]}{12(12t^2 - 4\sigma^3 + \sigma)|s|^4} + \frac{\sigma^3\big[(12\sigma^2 - 1)(12\sigma + 4\sigma^2 - 1) + 8\sigma^2 - 2\big]}{12(12t^2 - 4\sigma^3 + \sigma)|s|^4}.$$

In the given range we have $12\sigma^2 - 1 \ge 2$, $4\sigma^2 - 1 \ge 0$, and $12t^2 - 4\sigma^3 + \sigma > 0$. It follows that $f(\sigma, t) \ge \sigma \cdot \frac{t^2 \cdot 0 + \sigma^2 \cdot 12}{12(12t^2 - 4\sigma^3 + \sigma)|s|^4} > 0$. □

To prove the inequalities in (#) it will be more convenient to compare the logarithms of the functions involved, which will be done by using the derivative of their difference.

Proof of $\beta(\Delta, t) \le \alpha(\Delta, t)$. Consider $h(\Delta, t) := \log\alpha(\Delta, t) - \log\beta(\Delta, t)$. The inequality is equivalent to $h(\Delta, t) \ge 0$. Since $h(0, t) = 0$, it will suffice to show that $\partial h/\partial\Delta > 0$ (in the usual range $0 \le \Delta \le \frac{1}{2}$, $2\pi + 1 \le t$). From (13), and the definition of $\beta(\Delta, t)$, we have

$$h(\Delta, t) = t\left(\frac{\pi}{2} - \theta\right) - \sigma + \Re(\phi(s)) + \frac{1}{2}\log w - \log\left(\frac{12t^2 - 4\sigma^2 + \sigma}{12t^2}\right)$$

where here we will take $\phi(s) = \frac{1}{12s} - \frac{1}{360s^3} + R_4$. Since $\pi/2 - \theta = \arctan(\sigma/t)$, and since $\partial h/\partial\Delta = \partial h/\partial\sigma$, clearly

$$\frac{\partial h}{\partial\Delta} = t \cdot \frac{1}{1 + \sigma^2/t^2} \cdot \frac{1}{t} - 1 - \Re\left(\frac{1}{12s^2}\right) + \Re\left(\frac{1}{120s^4}\right) + \Re(R_4') + \frac{12\sigma^2 - 1}{12t^2 - 4\sigma^3 + \sigma} + E_3,$$

where

$$E_3 = \frac{\partial}{\partial\Delta}\left(\frac{1}{2}\log w\right) = \frac{-\cos(\pi\Delta)\pi\,\mathrm{e}^{-\pi t}}{1 - \sin(\pi\Delta)\,\mathrm{e}^{-\pi t} + \mathrm{e}^{-2\pi t}}. \tag{17}$$

Thus,

$$\begin{aligned}\frac{\partial h}{\partial \Delta} &= \frac{12\sigma^2-1}{12t^2-4\sigma^3+\sigma} - \frac{\sigma^2}{|s|^2} - \frac{\sigma^2-t^2}{12|s|^4} + \Re\left(\frac{1}{120s^4}\right) + \Re(R_4') + E_3\\ &\geq \Re\left(\frac{1}{120s^4}\right) + \Re(R_4') + E_3\,,\end{aligned}$$

using Lemma 7. It is easily seen that

$$\Re\left(\frac{1}{120s^4}\right) = \Re\left(\frac{\bar{s}^4}{|s|^8}\right) = \frac{\sigma^4-6\sigma^2t^2+t^4}{120|s|^8} = \frac{1}{120|s|^4} - \frac{t^2\sigma^2}{15|s|^8}\,.$$

Then

$$\frac{\partial h}{\partial \Delta} \geq \frac{1}{120|s|^4} + E_1 + E_2 + E_3\,,$$

where

$$\begin{aligned} E_1 &= -\frac{t^2\sigma^2}{15|s|^8} &&\text{with}\quad |E_1| = \frac{\sigma^2}{15}\cdot\frac{t^2}{|s|^2}\cdot\frac{1}{|s|^6} < \frac{1}{15|s|^6}\,,\\ E_2 &= \Re(R_4') &&\text{with}\quad |E_2| < \frac{1}{(\sqrt{2}/2)^7}\cdot\frac{1}{42}\cdot\frac{1}{6|s|^6} < \frac{1}{21|s|^6}\,,\end{aligned}$$

by [9; p. 114], and E_3 is defined as in (17), whence $|E_3| < \frac{1}{1000|s|^6}$.

Hence

$$|E_1+E_2+E_3| \leq |E_1|+|E_2|+|E_3| < \frac{1}{8|s|^6}\,.$$

But in our range $\frac{1}{120|s|^4} > \frac{1}{8|s|^6}$ (indeed $|s|^2 > 15$ suffices), and so $\partial h/\partial\Delta > 0$ is proved. □

Remark. We have actually proved a somewhat stronger result, that $\alpha(\Delta,t) - \beta(\Delta,t) \geq 0$ and is strictly monotone increasing for $0 \leq \Delta \leq \frac{1}{2}$. The same is true for $\big(|s|/2\pi\big)^{\Delta} - \alpha(\Delta,t)$ and easier to prove, however, we omit a proof of this result since the next (stronger) inequality suffices to complete the proof of Theorem 4.

9.4. Proof of $\alpha(\Delta,t) \leq (t/2\pi)^{\Delta}$. Since this proof has many similarities to the previous one, we simply sketch it and explain the features that differ from §9.3. Let $h(\Delta,t) = \Delta\log\big(\frac{t}{2\pi}\big) - \log\alpha(\Delta,t)$, where we know $h(0,t) = 0$ and $h\big(\frac{1}{2},t\big) > 0$ (cf. §9.2. Remark).

We must prove $h(\Delta,t) \geq 0$ for $0 \leq \Delta \leq \frac{1}{2}$. As in §9.3, we obtain

$$\begin{aligned} h(\Delta,t) = &-\frac{\Delta}{2}\log\left(\frac{|s|^2}{t^2}\right) - t\left(\frac{\pi}{2}-\theta\right) + \sigma - \Re\left(\frac{1}{12s}\right)\\ &- \Re(R_2) - \frac{1}{2}\log w\,.\end{aligned}$$

Using the well-known expansion

$$\log\left(\frac{|s|^2}{t^2}\right) = \log\left(1+\frac{\sigma^2}{t^2}\right) = \frac{\sigma^2}{t^2} - \frac{\sigma^4}{2t^4} + \frac{\sigma^6}{3t^6} - \cdots,$$

then differentiating, and simplifying, gives us

$$\frac{\partial h}{\partial \Delta} = \frac{-6\sigma^2 + 6\sigma - 1}{12|s|^2} + E,$$

where $E = E_1 + E_2 + E_3 + E_4$ and the error terms are $O(|s|^{-4})$. More precisely, noting that in our range we always have

$$\frac{|s|^2}{t^2} \le \frac{50}{49} \qquad \text{and} \qquad \frac{|s|^4}{t^4} \le \frac{25}{24}, \tag{18}$$

it is possible to give accurate estimates for the above error terms. We have

$$E_1 = \frac{1}{2}\left(\frac{\sigma^4}{2t^4} - \frac{\sigma^6}{3t^6} + \cdots\right), \qquad \text{and so} \quad |E_1| < \frac{\sigma^4}{4t^4}. \tag{19}$$

Similarly,

$$E_2 = \frac{-\sigma^4}{2|s|^2t^2} + \frac{\sigma^2}{6|s|^4} = \frac{1 - 3\frac{|s|^2}{t^2}\sigma^2}{6|s|^4} \cdot \sigma^2,$$

where for the numerator here we have

$$\left|1 - 3\frac{|s|^2}{t^2}\sigma^2\right| \le \left|1 - 3 \cdot 1 \cdot \frac{50}{49}\right| = \frac{101}{49},$$

by (18). Hence

$$|E_2| \le \frac{101\sigma^2}{294|s|^4} < \frac{101\sigma^2}{294t^4}. \tag{20}$$

We already know that (see §8.1)

$$E_3 = -\Re(R_2'), \qquad \text{so} \quad |E_3| < \frac{1}{20|s|^4}. \tag{21}$$

Finally, the last error term can be bounded as follows

$$E_4 = \frac{\Delta\, \mathrm{e}^{-\pi t}\cos(\pi\Delta)}{1 - 2\sin(\pi\Delta)\,\mathrm{e}^{-\pi t} + \mathrm{e}^{-2\pi t}}, \qquad \text{so} \quad |E_4| < \frac{1}{10000|s|^4}, \tag{22}$$

since in our range we have $|E_4| < \mathrm{e}^{-\pi t}$, and $\pi t > \log 10000 + 2\log|s|^2$.

Thus, combining (19), (20), (21) and (22), we get

$$|E| \le |E_1| + |E_2| + |E_3| + |E_4| < \frac{1}{t^4}\left(\frac{\sigma^4}{4} + \frac{101\sigma^2}{294} + \frac{1}{20} + \frac{1}{10000}\right).$$

At this point the proof departs from the previous one, because one has $-6\sigma^2 + 6\sigma - 1 > 0$ for $\frac{1}{2} \le \sigma < \frac{1}{2} + \frac{\sqrt{3}}{6} = 0.78867\ldots$, and $-6\sigma^2 + 6\sigma - 1 < 0$ for $\frac{1}{2} + \frac{\sqrt{3}}{6} < \sigma \le 1$.

However, our estimations are strong enough[7] to show that

$$\frac{\partial h}{\partial \Delta} > 0 \qquad \text{for} \quad \frac{1}{2} \le \sigma \le \frac{3}{4},$$

$$h\Big(\frac{3}{4}, t\Big) > \frac{3}{500t^2} \quad \Big(\text{or even } \ \frac{0.00601}{t^2}\Big)$$

by integration,

$$\left| \int_{3/4}^{17/20} \frac{\partial h}{\partial \sigma}\, \mathrm{d}\sigma \right| < \frac{3}{1000t^2} \quad \Big(\text{or even } \ \frac{0.00221}{t^2}\Big),$$

and finally

$$\frac{\partial h}{\partial \Delta} < 0 \qquad \text{for} \quad \frac{17}{20} \le \sigma \le 1.$$

Combining all this with the already known facts about $h(0,t)$ and $h(\frac{1}{2},t)$ shows that $h(\Delta,t)$ increases from 0 to a positive value $> \frac{1}{500t^2}$ in the interval $0 \le \Delta \le \frac{1}{4}$, remains positive for $\frac{1}{4} \le \Delta \le \frac{7}{20}$, and decreases over $\frac{7}{20} \le \Delta \le \frac{1}{2}$ to the positive value $h\big(\frac{1}{2},t\big)$. Thus $h(\Delta,t) \ge 0$ for $0 \le \Delta \le \frac{1}{2}$, as required. □

10. Remarks about $(t/2\pi)^\Delta$

10.1. We make a couple of additional comments concerning the approximation $(t/2\pi)^\Delta$ here. First, two elementary lemmas are given. The proof of the first one requires nothing beyond simple algebra, and is therefore omitted. The second lemma is proved below. We have:

---

[7] Proofs of these four results are technical, but straight-forward. Just to illustrate what happens, we take the last one as an example. For $\frac{17}{20} \le \sigma \le 1$,

$$|E| \le \frac{1}{t^4}\left(\frac{1}{4} + \frac{101}{294} + \frac{1}{20} + \frac{1}{10000}\right),$$

and so

$$\frac{\partial h}{\partial \Delta} \le -\frac{1}{52|s|^2} + \frac{0.6437}{t^4} \le -\frac{49}{50 \cdot 52t^2} + \frac{0.6437}{t^4}$$
$$= \frac{1}{t^2}\left(-0.01885 + \frac{0.6437}{t^2}\right) = -\frac{0.005713}{t^2} < 0$$

for all t in our range, as claimed.

Lemma 8. *For $\frac{1}{2} \le \sigma \le 1$, one has*

$$0 \le 6\sigma^3 - 3\sigma^2 \le 4\sigma^3 - \sigma\,, \tag{23a}$$

$$\frac{7}{8} \le \frac{6\sigma^3 - 3\sigma^2}{4\sigma^3 - \sigma} \le 1\,. \tag{23b}$$

Lemma 9. *We have*

$$t^\Delta = |s|^\Delta \left(1 - \frac{2\sigma^3 - \sigma^2}{4t^2} + O\left(\frac{1}{t^4}\right)\right). \tag{24}$$

P r o o f. The proof of (24) is a routine application of power series, arranged so that the convergence is assured in all cases. First, write

$$\frac{|s|}{t} = \sqrt{1 + \frac{\sigma^2}{t^2}} = 1 + \frac{1}{2}\frac{\sigma^2}{t^2} - \frac{1}{8}\frac{\sigma^4}{t^4} + \frac{1}{16}\frac{\sigma^6}{t^6} - \cdots\,,$$

using the binomial theorem. Next, using $\log(1+r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \cdots$, we find

$$\log\left(\frac{|s|}{t}\right) = \frac{\sigma^2}{2t^2} - \frac{\sigma^4}{4t^4} + \frac{\sigma^6}{6t^6} \cdots\,,$$

and hence

$$\begin{aligned}\left(\frac{t}{|s|}\right)^\Delta = \left(\frac{|s|}{t}\right)^{-\Delta} &= \exp\left(-\Delta \log\left(\frac{|s|}{t}\right)\right)\\ &= 1 - \left(\frac{\Delta}{2}\right)\frac{\sigma^2}{t^2} + \left(\frac{2\Delta + \Delta^2}{8}\right)\frac{\sigma^4}{t^4}\\ &\quad - \left(\frac{8\Delta + 6\Delta^2 + \Delta^3}{48}\right)\frac{\sigma^6}{t^6} + \cdots\\ &= 1 - \frac{2\sigma^3 - \sigma^2}{4t^2} + O\left(\frac{1}{t^4}\right).\end{aligned}$$

□

10.2. Rewriting Lemma 9 as

$$\left(\frac{t}{2\pi}\right)^\Delta = \left(\frac{|s|}{2\pi}\right)^\Delta \left(1 - \frac{6\sigma^3 - 3\sigma^2}{12t^2} + O\left(\frac{1}{t^4}\right)\right),$$

and using Lemma 8, it is now clear that in the inequality (#) of Theorem 4, in particular in $\beta(\Delta,t) \le (t/2\pi)^\Delta \le \left(|s|/2\pi\right)^\Delta$, the approximation $(t/2\pi)^\Delta$ is much closer to $\beta(\Delta,t)$ than it is to $\left(|s|/2\pi\right)^\Delta$ (roughly speaking, at least

7/8 closer). Since $\beta(\Delta,t)$ is a higher order approximation, this clarifies why $(t/2\pi)^{\Delta}$ is a better approximation than $\left(|s|/2\pi\right)^{\Delta}$. The fact that $\beta(\Delta,t)\leq \alpha(\Delta,t)\leq (t/2\pi)^{\Delta}$, proved in Section 9, is then an additional bonus; it implies that very sharp estimations, both *below* and *above* $\alpha(\Delta,t)$, have been obtained.

Remark. If instead of the critical strip one considers the behaviour of the quotient $\alpha(\Delta,t)$ in wider intervals, for example[8] in $[A,B]$, where $A\leq 0<1\leq B$, then one could apply the methods of this paper to obtain asymptotic estimates similar to Theorem 2 and Theorem 3.

11. Open problems

11.1. In addition to (**) from §2.1 and Conjecture 1 from §2.4, both of which imply the Riemann Hypothesis, there are several intriguing, but seemingly quite difficult, open questions related to the main theorems we have proved in the previous sections of this paper.

Here we state a few of the more interesting ones, especially those describing ideas potentially open to further research.

PROBLEM 2. For any $t\geq 2\pi+1$ and $0\leq\Delta\leq\frac{1}{2}$, define

$$\Theta(\sigma+\mathrm{i}t):=\frac{\left|\zeta(\frac{1}{2}-\Delta+\mathrm{i}t)\right|-\left|\zeta(\frac{1}{2}+\Delta+\mathrm{i}t)\right|}{|\zeta(\mathrm{i}t)|-|\zeta(1+\mathrm{i}t)|}. \tag{25}$$

Is there a simple way to approximate $\Theta(s)$? In particular, can we measure the difference $2\sigma-\Theta(\sigma+\mathrm{i}t)$?

The numerical evidence (as in Figure 3) suggests the following:

CONJECTURE 3. For all $t\geq 2\pi+1$, the function $\Theta(\sigma+\mathrm{i}t)$, defined in (25), is monotone increasing in the interval $0\leq\Delta\leq\frac{1}{2}$.

[8] If we had $s=\sigma+\mathrm{i}t$ with $A\leq\sigma\leq B$, then the range for t would change to $t\geq C$, but all approximations could be established analogously. However, in a wider domain, Theorem 1 and Theorem 4 would no longer be true.

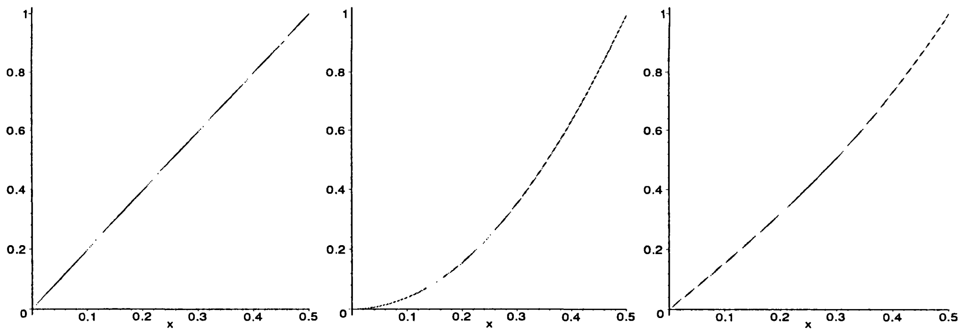

FIGURE 3. $\Theta(\sigma+\mathrm{i}t)$ for $t=10,25$ and 100, with $0\le\Delta\le\frac{1}{2}$.

11.2. Similarly, one can ask about the behaviour of the second power analogue of $\Theta(\sigma+\mathrm{i}t)$:

PROBLEM 4. For any $t\ge 2\pi+1$ and $0\le\Delta\le\frac{1}{2}$ define

$$\Theta_2(\sigma+\mathrm{i}t) := \frac{\left|\zeta\left(\frac{1}{2}-\Delta+\mathrm{i}t\right)\right|^2-\left|\zeta\left(\frac{1}{2}+\Delta+\mathrm{i}t\right)\right|^2}{|\zeta(\mathrm{i}t)|^2-|\zeta(1+\mathrm{i}t)|^2}\,. \tag{25$'$}$$

Is there a way to closely approximate the function $\Theta_2(s)$?

It is not clear to us how hard Problem 2 and Problem 4 really are. Even their special cases seem to be quite difficult. Just like in Conjecture 3, here again it seems plausible to conjecture the following interesting property of $\Theta_2(s)$.

CONJECTURE 5. For all $t\ge 2\pi+1$, the function $\Theta_2(\sigma+\mathrm{i}t)$, defined in (25$'$), is monotone increasing in the interval $0\le\Delta\le\frac{1}{2}$.

Note. Both Conjecture 3 and Conjecture 5 imply the Riemann Hypothesis.

11.3. In connection with the above Theorem 4, in particular the behaviour of the function $h(\Delta,t)$ in the interval $[0,\frac{1}{2}]$, we cannot prove, but have reasons to believe, that the following assertion is true:

CONJECTURE 6. The function $h(\Delta,t)$, defined in §9.3, has just one extreme in the interval $\left(0,\frac{1}{2}\right)$. For all $t\ge 2\pi+1$, the only value of Δ, for which $\frac{\partial h}{\partial\Delta}=0$, is $\Delta=\frac{1}{2}+\frac{\sqrt{3}}{6}$.

11.4. The purpose of this paper was to show that one can apply sharp results concerning Stirling-type series related to $\Gamma(s)$ and the functional equation of $\zeta(s)$ in order to obtain new information about the behaviour of the modulus of $\zeta(s)$ in the critical strip.

It is clear that the methods we have used throughout the paper can be applied in more general settings. This obviously includes only situations where one has a functional equation. But conversely, it appears that the existence of a functional equation is a sufficient condition for our methods to work. (In fact, in most of those general cases, the above Gamma function related Lemma 2 and Lemma 3, for example, stay unchanged, while other lemmas we have used should need only minor modifications.) This line of research seems well worth pursuing.

It is possible to prove Dirichlet L-functions analogues of our theorems. Let χ be a Dirichlet character modulo q. Then, for complex numbers s with $\Re(s) > 1$, the Dirichlet L-function $L(s,\chi)$ associated with χ is defined via the equation (see [7] or [33]):

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \tag{26}$$

where, similarly to the case of $\zeta(s)$, $L(s,\chi)$ can be analytically continued to the entire complex plane. If χ is a primitive character mod q with $\chi(-1) = 1$, then the functional equation of $L(s,\chi)$ is given simply as

$$\pi^{\frac{1-s}{2}} q^{\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) L(1-s,\overline{\chi}) = \frac{\sqrt{q}}{\tau(\chi)} \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s,\chi), \tag{27}$$

where $\tau(\chi)$ is the Gaussian sum defined as (see [7])

$$\tau(\chi) = \sum_{i=1}^{q} \chi(i) e_q(i), \qquad \text{and} \qquad |\tau(\chi)| = \sqrt{q}.$$

For $s = 1$, it is known that $L(1,\chi) \neq 0$ for all nonprincipal characters χ, and it is believed that if $L(\sigma + \mathrm{i}t, \chi) = 0$ for $0 \le \sigma \le 1$, then $\sigma = \frac{1}{2}$ (this is the well-known L-function analogue of Riemann's famous statement, given in §1.2).

We have proved the following weaker result:

THEOREM 5. *For $0 \le \Delta \le \frac{1}{2}$ and $t \ge 2\pi + 1$, we have*

$$\left|L\left(\frac{1}{2} - \Delta + \mathrm{i}t, \chi\right)\right| \ge \left|L\left(\frac{1}{2} + \Delta + \mathrm{i}t, \chi\right)\right|. \tag{28}$$

Proof of (28), together with proofs of several of its extensions to Dedekind ζ-functions and Artin L-functions will appear elsewhere.

12. Appendix

12.1. Finally, let us present some of the favourable numerical evidence underlying, and in retrospect confirming, our main results.

We shall compare the three approximations $\left(|s|/2\pi\right)^{\Delta}$, $(t/2\pi)^{\Delta}$ and $\beta(\Delta,t)$ from our paper with the actual values of $\alpha(\Delta,t)$ for $\Delta = 0.1, 0.2, 0.3, 0.4$, and 0.5 (for $\Delta = 0$ all the approximations are exact) and increasing values of t, showing how all of the estimates improve as t grows. For each fixed t, the three rows in the Table 1 listed next to t correspond to the differences $\left(|s|/2\pi\right)^{\Delta} - \alpha(\Delta,t)$, $(t/2\pi)^{\Delta} - \alpha(\Delta,t)$ and $\alpha(\Delta,t) - \beta(\Delta,t)$, respectively (blank spaces indicate that the given value is $< 10^{-50}$).

t	$\Delta = 0.1$	$\Delta = 0.2$	$\Delta = 0.3$	$\Delta = 0.4$	$\Delta = 0.5$
7	4.5×10^{-4} 8.2×10^{-5} 1.2×10^{-6}	1.1×10^{-3} 1.4×10^{-4} 5.1×10^{-6}	2.1×10^{-3} 1.6×10^{-4} 1.4×10^{-5}	3.4×10^{-3} 1.2×10^{-4} 3.2×10^{-5}	5.1×10^{-3} 2.8×10^{-10} 6.4×10^{-5}
20	5.5×10^{-5} 1.0×10^{-5} 1.7×10^{-8}	1.4×10^{-4} 1.8×10^{-5} 7.8×10^{-8}	2.6×10^{-4} 2.0×10^{-5} 2.2×10^{-7}	4.2×10^{-4} 1.5×10^{-5} 4.9×10^{-7}	6.2×10^{-4} 5.2×10^{-28} 9.7×10^{-7}
50	8.8×10^{-6} 1.6×10^{-6} 4.5×10^{-10}	2.2×10^{-5} 2.8×10^{-6} 2.0×10^{-9}	4.2×10^{-5} 3.2×10^{-6} 5.6×10^{-9}	6.7×10^{-5} 2.4×10^{-6} 1.3×10^{-8}	1.0×10^{-4} 2.5×10^{-8}
10^2	2.2×10^{-6} 4.0×10^{-7} 2.8×10^{-11}	5.6×10^{-6} 7.0×10^{-7} 1.2×10^{-10}	1.0×10^{-5} 8.0×10^{-7} 3.5×10^{-10}	1.7×10^{-5} 6.0×10^{-7} 7.9×10^{-10}	2.5×10^{-5} 1.6×10^{-9}
10^3	2.2×10^{-8} 4.0×10^{-9} 2.8×10^{-15}	5.6×10^{-8} 7.0×10^{-9} 1.2×10^{-14}	1.0×10^{-7} 8.0×10^{-9} 3.5×10^{-14}	1.7×10^{-7} 6.0×10^{-9} 7.9×10^{-13}	2.5×10^{-7} 1.6×10^{-13}

Acknowledgement

We would like to thank Garry J. Tee (University of Auckland) for very careful reading of the manuscript, and for many helpful comments.

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Received August 10, 2002
Revised October 10, 2002

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