Paulina Szczuka

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MAXIMUMS OF STRONG ŚWIĄTKOWSKI FUNCTIONS

PAULINA SZCZUKA

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ABSTRACT. In this paper we characterize both the family of the maximums of strong Świątkowski functions and the lattice generated by the family of all strong Świątkowski functions.

1. Preliminaries

The letter $\mathbb{R}$ denotes the real line. The symbol $\mathrm{I}(a,b)$ denotes the open interval with endpoints a and b. For each $A \subset \mathbb{R}$ we use the symbol $\operatorname{Int} A$ to denote its interior.

Let I be an interval and $f\colon I \to \mathbb{R}$. We say that f is a *Darboux function* if it maps connected sets onto connected sets.

We say that f is a *quasi-continuous function at a point* $x \in I$ ([3]) if for all open sets $U \ni x$ and $V \ni f(x)$ we have $\operatorname{Int}(U \cap f^{-1}(V)) \neq \emptyset$. The symbols $\mathscr{C}(f)$ and $\mathscr{Q}(f)$ will stand for the set of points of continuity of f and the set of points of quasi-continuity of f, respectively. If $\mathscr{Q}(f) = I$, then we say that f is *quasi-continuous.*

We say that f is a *strong Świątkowski function* ($f \in \acute{\mathcal{S}}_s$) ([4]) if whenever $\alpha, \beta \in I$, $\alpha < \beta$, and $y \in \mathrm{I}(f(\alpha), f(\beta))$, there is an $x_0 \in (\alpha,\beta) \cap \mathscr{C}(f)$ such that $f(x_0) = y$. The symbol $\mathscr{U}(f)$ denotes $\bigcup\{(a,b) : f\restriction(a,b) \in \acute{\mathcal{S}}_s\}$.

For each nonempty set $A \subset I$ we define $\omega(f,A)$ as the *oscillation of* f *on* A, i.e., $\omega(f,A) = \sup\{|f(x) - f(t)| : x,t \in A\}$. For each $x \in I$ we define $\omega(f,x)$ as the *oscillation of* f *at* x, i.e., $\omega(f,x) = \lim_{\delta\to 0^+} \omega(f, I \cap (x-\delta, x+\delta))$.

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Let $f\colon \mathbb{R} \to \mathbb{R}$. If $A \subset \mathbb{R}$ and x is a limit point of A, then let

$$\overline{\lim}(f, A, x) = \overline{\lim_{t \to x,\, t \in A}} f(x)\,.$$

Similarly we define $\overline{\lim}(f, A, x^-)$, $\underline{\lim}(f, A, x^-)$, etc. Moreover we write $\overline{\lim}(f, x)$ instead of $\overline{\lim}(f, \mathbb{R}, x)$, etc..

2. Introduction

In 1992, T. Natkaniec proved the following result [7; Proposition 3].

THEOREM 2.1. *For every function f the following are equivalent:*

a) *there are quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$,*
b) *the set $\mathbb{R} \setminus \mathscr{Q}(f)$ is nowhere dense, and $f(x) \le \overline{\lim}(f, \mathscr{C}(f), x)$ for each $x \in \mathbb{R}$.*

(In 1996, this theorem was generalized by J. Borsík [1] for functions defined on regular second countable topological spaces.) He remarked also that if a function f can be written as the maximum of Darboux quasi-continuous functions, then

$$f(x) \le \min\Big\{\overline{\lim}\big(f, \mathscr{C}(f), x^-\big), \overline{\lim}\big(f, \mathscr{C}(f), x^+\big)\Big\} \qquad \text{for each} \quad x \in \mathbb{R}, \qquad (1)$$

and asked whether the following conjecture is true ([7; Remark 3]).

CONJECTURE 2.2. *If f is a function such that $\mathbb{R} \setminus \mathscr{Q}(f)$ is nowhere dense and condition* (1) *holds, then there are Darboux quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$.*

In 1999, A. Maliszewski showed that this conjecture is false, and proved some facts about the maximums of Darboux quasi-continuous functions ([5]). However, the problem of characterization of the maximums of Darboux quasi-continuous functions is still open.

In this paper we examine even smaller class of functions, namely the family $\acute{\mathcal{S}}_s$ of strong Świątkowski functions. We show a theorem quite analogous to Theorem 2.1. Moreover we find the smallest lattice containing $\acute{\mathcal{S}}_s$ (Theorem 4.2). The characterization we obtained is similar to that of the lattice generated by the family of quasi-continuous functions. (See [2; Theorem 1].)

3. Auxiliary lemmas

Lemma 3.1 is an immediate consequence of [6; Lemma 5.1].

LEMMA 3.1. *Let $I = [x_0, x_1]$ and $f\colon I \to \mathbb{R}$. If $\inf f[I] > -\infty$ and f is lower semicontinuous at x_0 and x_1, then there is a continuous function φ such that $\varphi \le f$ on I and $\varphi(x_i) = f(x_i)$ for $i \in \{0,1\}$.*

LEMMA 3.2. *Let $a_0 < a_1 < a_2$. If $f\restriction[a_{i-1}, a_i] \in \acute{\mathcal{S}}_s$ for $i \in \{1,2\}$ and $a_1 \in \mathscr{C}(f)$, then $f\restriction[a_0, a_2] \in \acute{\mathcal{S}}_s$.*

Proof. Let $\alpha, \beta \in [a_0, a_2]$, $\alpha < \beta$, and $y \in \mathrm{I}(f(\alpha), f(\beta))$. Clearly we may assume that $\alpha < a_1 < \beta$ and $y \ne f(a_1)$. Note that

$$\mathrm{I}(f(\alpha), f(\beta)) \subset \mathrm{I}(f(\alpha), f(a_1)) \cup \mathrm{I}(f(a_1), f(\beta)) \cup \{f(a_1)\}.$$

So, either $y \in \mathrm{I}(f(\alpha), f(a_1))$ and there is an $x_0 \in (\alpha, a_1) \cap \mathscr{C}(f) \subset (\alpha, \beta) \cap \mathscr{C}(f)$ with $f(x_0) = y$, or $y \in \mathrm{I}(f(a_1), f(\beta))$ and there is an $x_0 \in (a_1, \beta) \cap \mathscr{C}(f) \subset (\alpha, \beta) \cap \mathscr{C}(f)$ with $f(x_0) = y$. □

LEMMA 3.3. *If I is a compact interval and $I \subset \mathscr{U}(f)$, then $f\restriction I \in \acute{\mathcal{S}}_s$.*

Proof. Let $\alpha, \beta \in I$, $\alpha < \beta$, and $y \in \mathrm{I}(f(\alpha), f(\beta))$. We have

$$[\alpha, \beta] \subset I \subset \mathscr{U}(f) = \bigcup\{(a,b) :\ f\restriction(a,b) \in \acute{\mathcal{S}}_s\}.$$

So, by the Borel-Lebesgue theorem, there are open intervals $(a_1, b_1), \dots, (a_n, b_n)$ such that $[\alpha, \beta] \subset \bigcup_{i=1}^{n} (a_i, b_i)$ and $f\restriction(a_i, b_i) \in \acute{\mathcal{S}}_s$ for each i. There are nonoverlapping intervals $[c_1, d_1], \dots, [c_k, d_k]$ such that $[\alpha, \beta] = \bigcup_{j=1}^{k} [c_k, d_k]$, each interval $[c_j, d_j]$ is contained in some $[a_i, b_i]$, and $d_j \in \mathscr{C}(f)$ for $j \in \{1, \dots, k-1\}$. By Lemma 3.2, we conclude that $f\restriction[\alpha, \beta] \in \acute{\mathcal{S}}_s$. Hence $f(x) = y$ for some $x \in (\alpha, \beta) \cap \mathscr{C}(f)$. □

LEMMA 3.4. *Let $I = [x_0, x_2]$ be an interval and $f\colon I \to \mathbb{R}$. Assume $x_0, x_2 \in \mathscr{C}(f)$, $f \in \acute{\mathcal{S}}_s$, and let $J \subset (-\infty, \sup f[I])$ be a compact interval. Then we can construct functions $g_1, g_2 \in \acute{\mathcal{S}}_s$ such that $f = \max\{g_1, g_2\}$ and for $i \in \{1,2\}$, $g_i[I] \supset J$ and $g_i(x_j) = f(x_j)$ for $j \in \{0,2\}$.*

Proof. If $\inf f[I] = -\infty$, then $f[[x_0, x_2]] \supset J$, and we can set $g_1 = g_2 = f$. So, assume $\inf f[I] > -\infty$.

Since $f \in \acute{\mathcal{S}}_s$ and J is a compact interval, there is an $x_1 \in (x_0, x_2) \cap \mathscr{C}(f)$ with $f(x_1) > \max J$. Define

$$f_1(x) = \begin{cases} f(x) & \text{if } x \notin \{(x_0+x_1)/2, (x_1+x_2)/2\}, \\ \min\{\min J, f(x)\} & \text{otherwise.} \end{cases}$$

We have $\inf f_1[I] > -\infty$ and $x_0, x_1, x_2 \in \mathscr{C}(f_1)$. By Lemma 3.1, there is a continuous function φ such that $\varphi \le f_1 \le f$ on I and $\varphi(x_i) = f_1(x_i) = f(x_i)$ for $i \in \{0,1,2\}$. For $i \in \{1,2\}$ define

$$g_i(x) = \begin{cases} \varphi(x) & \text{if } x \in [x_{i-1}, x_i], \\ f(x) & \text{if } x \in [x_{2-i}, x_{3-i}]. \end{cases}$$

Clearly $f = \max\{g_1, g_2\}$. Fix $i \in \{1,2\}$. By Lemma 3.2, $g_i \in \acute{\mathcal{S}}_s$. Moreover, since

$$\varphi\big((x_{i-1}+x_i)/2\big) \le f_1\big((x_{i-1}+x_i)/2\big) \le \min J$$

and $\varphi(x_1) = f(x_1) > \max J$, we obtain

$$g_i[I] \supset \varphi\big[[x_{i-1}, x_i]\big] \supset \big[\varphi\big((x_{i-1}+x_i)/2\big), \varphi(x_1)\big] \supset J.$$

(Recall that φ is continuous.) Clearly $g_i = f$ on $\{x_0, x_2\}$. □

Theorem 3.5. *Let $g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ and $f = \max\{g_1, g_2\}$. If the sets $\mathscr{U}(g_1)$ and $\mathscr{U}(g_2)$ are dense in $\mathbb{R}$, then $\mathscr{U}(f)$ is dense in $\mathbb{R}$, too.*

Proof. Let $\bar{a} < \bar{b}$. Since the set $\mathscr{U}(g_1) \cap \mathscr{U}(g_2)$ is dense in $\mathbb{R}$, there is a nondegenerate interval $[a,b] \subset \mathscr{U}(g_1) \cap \mathscr{U}(g_2) \cap (\bar{a}, \bar{b})$. By Lemma 3.3, $g_i \restriction [a,b] \in \acute{\mathcal{S}}_s$ for $i \in \{1,2\}$. We will show that there is an open interval $(c,d) \subset \mathscr{U}(f) \cap (a,b)$.

First assume that there is an $x_0 \in (a,b) \cap \mathscr{C}(g_1) \cap \mathscr{C}(g_2)$ with $g_1(x_0) > g_2(x_0)$. Since $x_0 \in \mathscr{C}(g_1 - g_2)$ and $(g_1 - g_2)(x_0) > 0$, there is a $\delta > 0$ such that $g_1 - g_2 > 0$ on $(x_0 - \delta, x_0 + \delta)$. Then $f \restriction (x_0-\delta, x_0+\delta) = g_1 \restriction (x_0-\delta, x_0+\delta)$ and

$$(c,d) = (x_0-\delta, x_0+\delta) \cap (a,b) \subset \mathscr{U}(f) \cap (a,b).$$

Similarly we proceed if there is an $x_0 \in (a,b) \cap \mathscr{C}(g_1) \cap \mathscr{C}(g_2)$ with $g_1(x_0) < g_2(x_0)$. So, assume that

$$g_1(x) = g_2(x) \qquad \text{for each} \quad x \in (a,b) \cap \mathscr{C}(g_1) \cap \mathscr{C}(g_2). \tag{2}$$

We will show that

$$(a,b) \cap \mathscr{C}(g_1) = (a,b) \cap \mathscr{C}(g_2). \tag{3}$$

Suppose that, e.g., $x_0 \in (a,b) \cap \mathscr{C}(g_1) \setminus \mathscr{C}(g_2)$. Define

$$\gamma = \min\{1, \omega(g_2, x_0)\} > 0.$$

Since $x_0 \in \mathscr{C}(g_1)$, there is a $\tau > 0$ such that

$$|g_1(x) - g_1(x_0)| < \gamma/4 \qquad \text{for each} \quad x \in (x_0-\tau, x_0+\tau). \tag{4}$$

Choose $x_1, x_2 \in (x_0-\tau, x_0+\tau) \cap (a,b)$ such that $|g_2(x_1) - g_2(x_2)| > 7\gamma/8$. Since by Lemma 3.3, $g_2 \restriction [a,b] \in \acute{\mathcal{S}}_s$, there is an $x_3 \in \mathrm{I}(x_1, x_2) \cap \mathscr{C}(g_2)$ such that

$$g_2(x_3) \notin \big[g_1(x_0) - \gamma/4,\ g_1(x_0) + \gamma/4\big].$$

Then $|g_1(x_0) - g_2(x_3)| - \gamma/4 > 0$. There is an $x_4 \in \mathrm{I}(x_1, x_2) \cap \mathscr{C}(g_1) \cap \mathscr{C}(g_2)$ such that

$$|g_2(x_4) - g_2(x_3)| < |g_1(x_0) - g_2(x_3)| - \gamma/4 \,. \tag{5}$$

(Recall that the set $\mathscr{C}(g_1) \cap \mathscr{C}(g_2)$ is dense in $\mathbb{R}$.) By (4), (2), and (5), we obtain

$$\begin{aligned} \gamma/4 > |g_1(x_4) - g_1(x_0)| &= |g_2(x_4) - g_1(x_0)| \\ &\geq |g_1(x_0) - g_2(x_3)| - |g_2(x_4) - g_2(x_3)| > \gamma/4 \,, \end{aligned}$$

an impossibility. Thus, condition (3) holds.

Finally we will show that $(a, b) \subset \mathscr{U}(f)$. Take arbitrary $\alpha, \beta \in (a, b)$, $\alpha < \beta$, and $y \in \mathrm{I}(f(\alpha), f(\beta))$. Since $f = \max\{g_1, g_2\}$, we have

$$\mathrm{I}(f(\alpha), f(\beta)) \subset \mathrm{I}(g_1(\alpha), g_1(\beta)) \qquad \text{or} \qquad \mathrm{I}(f(\alpha), f(\beta)) \subset \mathrm{I}(g_2(\alpha), g_2(\beta)) \,.$$

Assume that, e.g., the first inclusion holds. Then there is an $x_0 \in (\alpha, \beta) \cap \mathscr{C}(g_1)$ with $g_1(x_0) = y$. By (3) and (2), we conclude that $x_0 \in \mathscr{C}(f)$ and $f(x_0) = y$. □

LEMMA 3.6. *Let $f : \mathbb{R} \to \mathbb{R}$. If the set $\mathscr{U}(f)$ is dense in $\mathbb{R}$, then there are functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ such that $f = \max\{g_1, g_2\}$ on $\mathbb{R}$,*

$$\mathscr{U}(f) \subset \mathscr{U}(g_1) \cap \mathscr{U}(g_2) \,, \tag{6}$$

and for $i \in \{1, 2\}$,

$$(\forall a \notin \mathscr{U}(f))(\forall \delta > 0) \left(g_i[(a, a + \delta) \cap \mathscr{C}(g_i)] \supset (-\infty, \overline{\lim}(f, \mathscr{C}(f), a^+))\right) , \tag{7}$$

$$(\forall a \notin \mathscr{U}(f))(\forall \delta > 0) \left(g_i[(a - \delta, a) \cap \mathscr{C}(g_i)] \supset (-\infty, \overline{\lim}(f, \mathscr{C}(f), a^-))\right) . \tag{8}$$

P r o o f . Write $\mathscr{U}(f)$ as the union of a family $\mathcal{I}$ consisting of nonoverlapping compact intervals, such that

(9) for each $x \in \mathscr{U}(f)$, there are $I_1, I_2 \in \mathcal{I}$ with $x \in \mathrm{Int}(I_1 \cup I_2)$,

(10) if $I \in \mathcal{I}$ and x is an endpoint of I, then $x \in \mathscr{C}(f)$.

For each $I \in \mathcal{I}$ define $J_I = [c_I, d_I]$, where

$$d_I = \min\{\sup f[I] - r_I, 1/r_I\}, \qquad c_I = \min\{d_I - 1, -1/r_I\} \,,$$

and $r_I = \mathrm{dist}(I, \mathbb{R} \setminus \mathscr{U}(f)) > 0$, and use Lemma 3.4 to construct strong Świątkowski functions g_{1I}, g_{2I} such that $g_{iI}[I] \supset J_I$ and $f(x) = g_{iI}(x)$ whenever x is an endpoint of I ($i \in \{1, 2\}$), and $f \restriction I = \max\{g_{1I}, g_{2I}\}$. For $i \in \{1, 2\}$ define

$$g_i(x) = \begin{cases} g_{iI}(x) & \text{if } x \in I, \ I \in \mathcal{I}, \\ f(x) & \text{otherwise.} \end{cases}$$

By (9), (10) and Lemma 3.2, we can easily see that condition (6) holds. Evidently $f = \max\{g_1, g_2\}$ on $\mathbb{R}$. Fix an $i \in \{1,2\}$. We will verify that condition (7) holds.

Let $a \notin \mathscr{U}(f)$, $\delta > 0$, and $y < \overline{\lim}(f, \mathscr{C}(f), a^+)$. Choose $y' \in (y, \overline{\lim}(f, \mathscr{C}(f), a^+))$ and let

$$\delta' = \min\{y'-y, \delta, 1/(|y|+1)\}.$$

There is an interval $I \in \mathcal{I}$ such that $I \subset (a, a+\delta')$ and $\sup f[I] > y'$. Since $r_I < \delta'$, we have

$$d_I = \min\{\sup f[I]-r_I, 1/r_I\} \geq \min\{y'-\delta', |y|+1\} \geq y$$

and

$$c_I \leq -1/r_I \leq -|y| - 1 < y.$$

Hence

$$y \in J_I = [c_I, d_I] \subset g_{iI}[I \cap \mathscr{C}(g_i)] = g_i[I \cap \mathscr{C}(g_i)] \subset g_i[(a, a+\delta) \cap \mathscr{C}(g_i)].$$

Similarly we can show that condition (8) holds. □

4. Main results

THEOREM 4.1. *For every function $f \colon \mathbb{R} \to \mathbb{R}$ the following are equivalent:*

a) *there are functions $g_1, g_2 \in \acute{\mathcal{S}}_s$ with $f = \max\{g_1, g_2\}$,*
b) *the set $\mathscr{U}(f)$ is dense in $\mathbb{R}$ and*

$$\min\{\overline{\lim}(f, \mathscr{C}(f), x^+), \overline{\lim}(f, \mathscr{C}(f), x^-)\} \geq f(x) \qquad \text{for each} \quad x \in \mathbb{R}.$$

Proof.

a) $\implies$ b). Let $f = \max\{g_1, g_2\}$, where $g_1, g_2 \in \acute{\mathcal{S}}_s$. Recall that each strong Świątkowski function is both Darboux and quasi-continuous. (See, e.g., [4].) Hence by (1), $\min\{\overline{\lim}(f, \mathscr{C}(f), x^+), \overline{\lim}(f, \mathscr{C}(f), x^-)\} \geq f(x)$ for each $x \in \mathbb{R}$. By Theorem 3.5, the set $\mathscr{U}(f)$ is dense in $\mathbb{R}$.

b) $\implies$ a). By Lemma 3.6, there are functions g_1, g_2 such that $f = \max\{g_1, g_2\}$ and conditions (6)–(8) are fulfilled. Fix an $i \in \{1,2\}$. We will show that $g_i \in \acute{\mathcal{S}}_s$.

Let $\alpha < \beta$ and $y \in \mathrm{I}(g_i(\alpha), g_i(\beta))$. Without loss of generality we may assume that $g_i(\alpha) < g_i(\beta)$. If $[\alpha, \beta] \subset \mathscr{U}(f)$, then by (6), $[\alpha, \beta] \subset \mathscr{U}(g_i)$, and by Lemma 3.3, there is an $x \in (\alpha, \beta) \cap \mathscr{C}(g_i)$ with $g_i(x) = y$. So, assume that $[\alpha, \beta] \setminus \mathscr{U}(f) \neq \emptyset$. We consider two cases.

Case 1. $\beta \notin \mathscr{U}(f)$.
By assumption,

$$y < g_i(\beta) \le f(\beta) \le \overline{\lim}(f, \mathscr{C}(f), \beta^-).$$

So by (8), there is an $x \in (\alpha, \beta) \cap \mathscr{C}(g_i)$ such that $g_i(x) = y$.

Case 2. $\beta \in \mathscr{U}(f)$.
Put $\gamma = \max([\alpha, \beta] \setminus \mathscr{U}(f))$. Then $\gamma < \beta$ and $\gamma \notin \mathscr{U}(f)$. By (7), there is an $\eta \in (\gamma, \beta) \cap \mathscr{C}(g_i)$ such that $g_i(\eta) < y$. By (6), we have $[\eta, \beta] \subset \mathscr{U}(f) \subset \mathscr{U}(g_i)$. So by Lemma 3.3, there is an $x \in (\eta, \beta) \cap \mathscr{C}(g_i) \subset (\alpha, \beta) \cap \mathscr{C}(g_i)$ such that $g_i(x) = y$. □

THEOREM 4.2. *The smallest lattice containing all strong Świątkowski functions is the family $\mathscr{L}$ consisting of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the set $\mathscr{U}(f)$ is dense in $\mathbb{R}$.*

Proof. Let $g_1, g_2 \in \mathscr{L}$. By Theorem 3.5, $\max\{g_1, g_2\} \in \mathscr{L}$. Moreover

$$\min\{g_1, g_2\} = -\max\{-g_1, -g_2\} \in \mathscr{L}.$$

So, $\mathscr{L}$ is a lattice. Since $\mathscr{L}$ contains $\acute{\mathcal{S}}_s$, it contains the smallest lattice containing $\acute{\mathcal{S}}_s$ as well.

Now we will show the opposite inclusion. Let $f \in \mathscr{L}$. By Lemma 3.6, there are functions g_1, g_2 such that $-f = \max\{g_1, g_2\}$, $\mathscr{U}(f) \subset \mathscr{U}(g_1) \cap \mathscr{U}(g_2)$, and for $i \in \{1, 2\}$ and each $x \notin \mathscr{U}(f)$

$$\underline{\lim}(g_i, \mathscr{C}(g_i), x^-) = -\infty, \qquad \underline{\lim}(g_i, \mathscr{C}(g_i), x^+) = -\infty.$$

Since the functions $-g_1$ and $-g_2$ fulfill condition b) of Theorem 4.1, there are functions $g_{11}, g_{12}, g_{21}, g_{22} \in \acute{\mathcal{S}}_s$ such that $-g_1 = \max\{g_{11}, g_{12}\}$ and $-g_2 = \max\{g_{21}, g_{22}\}$. Then

$$f = -\max\{g_1, g_2\} = \min\{-g_1, -g_2\} = \min\{\max\{g_{11}, g_{12}\}, \max\{g_{21}, g_{22}\}\}.$$

This completes the proof. □

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Bydgoszcz Academy
pl. Weyssenhoffa 11
PL-85-072 Bydgoszcz
POLAND
E-mail: Paulina@ab-byd.edu.pl