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WEAKLY ATOMIC LATTICES WITH STONEAN CONGRUENCE LATTICE

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ABSTRACT. We characterize congruence Boolean and congruence Stonean weakly atomic lattices replacing G. Grätzer's and T. Katriňák's weak projectivity conditions by their local forms.

1. Introduction

Lattices having a Boolean congruence lattice were characterized by G. Grätzer and E. T. Schmidt [4], and T. Katriňák [11] characterized lattices whose congruence lattice is Stonean. In both results the so called "projectivity conditions" (Grätzer's weak modularity, Katriňák's almost weak modularity) play an important role.

Starting from the above mentioned results we give a new characterization for weakly atomic lattices with Boolean respectively Stonean congruence lattices. We generalize earlier results due to G. Grätzer, T. Katriňák, Iqbalunnisa and E. T. Schmidt.

2. Preliminaries

A lattice L with 0 is called *pseudocomplemented* if for each $x \in L$ there exists an $x^* \in L$ such that for any $y \in L$, $y \wedge x = 0 \iff y \leqq x^*$.

If $x^* \vee x^{**} = 1$ for all $x \in L$, and L is distributive, then L is called a *Stone lattice.*

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It is well known that the congruence lattice $\mathrm{Con}\, L$ of a lattice L is distributive and pseudocomplemented. We note that the identities $x^* = x^{***}$, $(x \wedge y)^{**} = x^{**} \wedge y^{**}$ $(x, y \in L)$ hold in any pseudocomplemented lattice, and the identity $(x \vee y)^{**} = x^{**} \vee y^{**}$ is satisfied by any Stone lattice ([3]).

An ordered pair $a, b \in L$ with $b \leqq a$ is denoted by a/b and it is called a *quotient of* L. If $b \leqq d \leqq c \leqq a$, then c/d is called a *subquotient of* a/b and we write $c/d \subseteq a/b$. If $b \prec a$, then a/b is called a *prime quotient.*

We shall use the notation $a/b \to c/d$ for the weak projectivity of quotients (see [5] or [3; Chapt. 3]). The principal congruences of a lattice can be built up by using weakly projective quotients. More precisely, we have the following theorem.

THEOREM 2.1. (R. P. Dilworth [2]) *Let L be an arbitrary lattice and $a, b, c, d \in L$, $b < a$, $d \leqq c$. Then we have $(c, d) \in \theta(a, b)$ if and only if there exists a sequence of elements $c = e_0 \geqq e_1 \geqq \cdots \geqq e_n = d$ such that for all $i = 0, \ldots, n-1$ the relation $a/b \to e_i/e_{i+1}$ holds.*

Clearly, if c/d is a prime quotient, then $(c, d) \in \theta(a, b) \iff a/b \to c/d$.

DEFINITION 2.2.

(i) The lattice L is called *weakly modular* (or *weakly projective*) if for any quotients a/b, c/d of L, $a/b \to c/d$ and $c \neq d$ imply the existence of a subquotient $a_1/b_1 \subseteq a/b$ with $c/d \to a_1/b_1$.

(ii) L is said to be *almost weakly modular* (see [8]) whenever $a/b \to u/v$ and $u \neq v$ imply the existence of a subquotient $a_1/b_1 \subseteq a/b$ with $a_1 \neq b_1$ such that for any quotient r/s with $r \neq s$ and $a_1/b_1 \to r/s$ there exists a quotient z/t with $r/s \to z/t$, $u/v \to z/t$ and $z \neq t$. (See the diagram below.)

The quotient a_1/b_1 from (ii) will be called *the dominant subquotient of the quotient a/b (relative to u/v).*

$$\begin{tikzcd} a_1/b_1 \subseteq a/b \arrow[r] \arrow[d] & u/v \arrow[d] \\ r/s \arrow[r] & z/t \end{tikzcd}$$

DEFINITION 2.3.

(i) The congruence $\theta \in \mathrm{Con}\, L$ is said to be *separable* if for any $a, b \in L$, $a < b$, there exists a chain $a = z_0 \leqq z_1 \leqq \cdots \leqq z_n = b$ such that for each $i = 1, \ldots, n$, either $(z_{i-1}, z_i) \in \theta$ holds or there is no subquotient $r/s \subseteq z_i/z_{i-1}$, satisfying $r \neq s$ and $(s, r) \in \theta$.

(ii) $\theta \in \operatorname{Con} L$ is said to be *weakly separable* (see [11] or [8]) if for any $a, b \in L$, $a < b$, there exists a chain $a = z_0 \leqq z_1 \leqq \cdots \leqq z_n = b$ such that for each $i = 1, \dots, n$, either

(a) $z_i/z_{i-1} \to u/v$ and $(u, v) \in \theta$ imply $u = v$,

or

(b) for every subquotient $r/s \subseteq z_i/z_{i-1}$, $r \neq s$, there exists a quotient u/v, satisfying $r/s \to u/v$, $u \neq v$ and $(u, v) \in \theta$.

Remark 2.4. It is readily seen that condition (a) of Definition 2.3(ii) is equivalent to $(z_{i-1}, z_i) \in \theta^*$, and condition (b) is fulfilled if and only if there is no subquotient $r/s \subseteq z_i/z_{i-1}$, $r \neq s$, satisfying $(s, r) \in \theta^*$. Hence θ is weakly separable if and only if θ^* is separable.

By the Grätzer-Schmidt theorem, the congruence lattice of a lattice L is Boolean if and only if L is weakly modular and every $\theta \in \operatorname{Con} L$ is separable. Analogously T. Katriňák [11] (see also [8]) proved the following theorem:

THEOREM 2.5. ([11; Theorem 4]) *Let L be an arbitrary lattice. Then* $\operatorname{Con} L$ *is a Stone lattice if and only if L is almost weakly modular and every $\theta \in \operatorname{Con} L$ is weakly separable.*

The projectivity notions below will be important in our development.

DEFINITION 2.6.

(i) A lattice L is said to be *locally weakly modular* if for any prime quotients a/b, c/d of L, $a/b \to c/d$ implies $c/d \to a/b$.

(ii) L is called *locally almost weakly modular* (see [7] and [8]) if for any prime quotients p/q, r/s and t/u of L with $p/q \to r/s$ and $p/q \to t/u$, there exists a prime quotient v/w such that $r/s \to v/w$ and $t/u \to v/w$.

Remark 2.7. The following implications are obvious:

$$\begin{aligned} L \text{ is weakly modular} &\Longrightarrow L \text{ is locally weakly modular} \\ &\Longrightarrow L \text{ is locally almost weakly modular.} \\ L \text{ is weakly modular} &\Longrightarrow L \text{ is almost weakly modular} \\ &\Longrightarrow L \text{ is locally almost weakly modular.} \end{aligned}$$

3. Weakly atomic lattices

A lattice L is called *weakly atomic* if for any $a, b \in L$, $a < b$, there exist $c, d \in L$ such that $a \leqq c \prec d \leqq b$. L is called *atomic* if for any $x \in L$, $x \neq 0$, the interval $[0, x]$ contains at least one atom of L. The identical and the universal relation on L are denoted by $\triangle$ and $\bigtriangledown$ respectively.

First, we show that in a weakly atomic lattice the weak projectivity conditions and the weak separability of congruences are closely related.

PROPOSITION 3.1. *A weakly atomic lattice L is weakly modular if and only if it is locally weakly modular and almost weakly modular in the same time.*

Proof. The "only if part" is clear, according to Remark 2.7.

In order to prove the "if part", let us consider a weakly atomic lattice L which is locally weakly modular and almost weakly modular in the same time, and take $a, b, c, d \in L$, $b < a$, $d < c$ such that $a/b \to c/d$. Let a_1/b_1 be the dominant subquotient of a/b. Since L is weakly atomic, a_1/b_1 contains a prime subquotient r/s. Clearly, we have $a_1/b_1 \to r/s$. As L is almost weakly modular, there exists a quotient z/t with $r/s \to z/t$, $c/d \to z/t$ and $z \neq t$. Let z_1/t be a prime subquotient of z/t. Then $z/t \to z_1/t_1$ implies $r/s \to z_1/t_1$ and $c/d \to z_1/t_1$. Because L is locally weakly modular, we get $z_1/t_1 \to r/s$. Now $c/d \to z_1/t_1$ and $z_1/t_1 \to r/s$ imply $c/d \to r/s$. As $r/s \subseteq a/b$, the former relation proves that L is weakly modular. □

THEOREM 3.2. *Let L be a weakly atomic lattice with the property that for all elements $p \prec q$, $p, q \in L$, the principal congruences $\theta(p, q)$ are weakly separable. Then,*

(i) *L is almost weakly modular if and only if L is locally almost weakly modular.*
(ii) *L is weakly modular if and only if L is locally weakly modular.*

Proof. In view of Remark 2.7 it is enough to prove the "if" part of the above sentences.

(i) Take $a, b \in L$, $b < a$, and assume that $a/b \to u/v$ for some $u, v \in L$, $u \neq v$. Since L is weakly atomic, there exists a prime quotient $p/q \subseteq u/v$. Then $u/v \to p/q$. Set $\varphi = \theta(p, q)$, then φ is weakly separable by our assumption. Since now, according to Remark 2.4, φ^* is separable, there is a chain $b = z_0 \leqq z_1 \leqq \dots \leqq z_n = a$ such that for each $i = 1, \dots, n$, either $(z_{i-1}, z_i) \in \varphi^*$ or there is no subquotient $r/s \subseteq z_i/z_{i-1}$, $r \neq s$, satisfying $(s, r) \in \varphi^*$. We claim that there is at least one $1 \leqq i_0 \leqq n$ such that z_{i_0}/z_{i_0-1} satisfies the latter case.

Indeed, if $(z_{i-1}, z_i) \in \varphi^*$ held for all $1 \leqq i \leqq n$, then we would get $(a, b) \in \varphi^*$ implying $(p, q) \in \varphi^* \wedge \theta(p, q) = \varphi^* \wedge \varphi = \triangle$, a contradiction.

Since L is weakly atomic, there exists a prime quotient $a_1/b_1 \subseteq z_{i_0}/z_{i_0-1}$. Clearly $(a_1, b_1) \notin \varphi^*$, whence we get $\theta(a_1, b_1) \wedge \varphi \neq \triangle$ i.e. $\theta(a_1, b_1) \wedge \theta(p, q) \neq \triangle$. Take now a quotient r/s with $a_1/b_1 \to r/s$ and $r \neq s$. Then r/s contains a prime subquotient r_1/s_1 and we have $a_1/b_1 \to r_1/s_1$. As $\theta(a_1, b_1) \wedge \theta(p, q) \neq \triangle$, there is a prime quotient e/f such that $(f,e) \in \theta(a_1,b_1) \wedge \theta(p,q)$. Hence $a_1/b_1 \to e/f$ and $p/q \to e/f$. Since L is locally almost weakly modular, $a_1/b_1 \to r_1/s_1$ and $a_1/b_1 \to e/f$ imply the existence of a prime quotient z/t with $r_1/s_1 \to z/t$ and $e/f \to z/t$. Now $r/s \to r_1/s_1$ and $r_1/s_1 \to z/t$ gives $r/s \to z/t$, and the relations $u/v \to p/q$, $p/q \to e/f$, $e/f \to z/t$ imply $u/v \to z/t$. Because we have showed the existence of a quotient z/t with $z \neq t$, $r/s \to z/t$ and $u/v \to z/t$, the lattice L is almost weakly modular.

(ii) Let L be a locally weakly modular lattice. Since now, in view of Remark 2.7, L is locally almost weakly modular as well, the above (i) implies that L is almost weakly modular. Therefore, by applying Proposition 3.1 we obtain that L is weakly modular. □

COROLLARY 3.3. *Let L be a weakly atomic lattice. Then*

(i) $\operatorname{Con} L$ *is Boolean if and only if L is locally weakly modular and any $\theta \in \operatorname{Con} L$ is separable.*

(ii) $\operatorname{Con} L$ *is a Stone lattice if and only if L is locally almost weakly modular and any $\theta \in \operatorname{Con} L$ is weakly separable.*

P r o o f. The "only if part" is clear for both (i) and (ii). It is also not hard to prove the "if part" of them:

(i) If every $\theta \in \operatorname{Con} L$ is separable, then, according to Remark 2.4, each $\theta \in \operatorname{Con} L$ is weakly separable, too. As L is weakly atomic and locally weakly modular, in view of Theorem 3.2(ii), L is also weakly modular, therefore (i) follows by applying the Grätzer-Schmidt theorem.

(ii) We simply apply Theorem 3.2(i) and Theorem 2.5. □

LEMMA 3.4. *Let L be a weakly atomic and locally almost weakly modular lattice and let $a, b \in L$ such that $a \prec b$. Then for any congruence $\theta \in \operatorname{Con} L$ we have either $(a,b) \in \theta^*$ or $(a,b) \in \theta^{**}$.*

P r o o f. We claim that $\theta(a,b) \wedge \theta^* \neq \triangle$ and $\theta(a,b) \wedge \theta^{**} \neq \triangle$ could not hold in the same time.

In contrary, if both of them are satisfied, then there exist $c, d, e, f \in L$ with $c \prec d$ and $e \prec f$ such that $(c,d) \in \theta(a,b) \wedge \theta^*$ and $(e,f) \in \theta(a,b) \wedge \theta^{**}$, whence we get $b/a \to d/c$ and $b/a \to f/e$. As L is locally almost weakly modular, there exists a prime quotient h/g of L such that $d/c \to h/g$ and $f/e \to h/g$. Hence we obtain $(h,g) \in \theta(c,d) \wedge \theta(e,f) \leqq \theta^* \wedge \theta^{**} = \triangle$ — a contradiction.

Thus we have either $\theta(a,b) \wedge \theta^* = \triangle$ or $\theta(a,b) \wedge \theta^{**} = \triangle$. In the first case we obtain $(a,b) \in \theta^{**}$ and in the second $(a,b) \in \theta^*$. □

For a pseudocomplemented lattice L its *Boolean part* is defined as $\mathrm{B}(L) = \{x \in L : x = x^{**}\}$. $(\mathrm{B}(L), \wedge, \veebar)$ is a Boolean algebra, where $a \veebar b$ is defined to be $(a^* \wedge b^*)^*$. (For more details, see [6], [11], [3] or [13].)

The following proposition shows that in the case of a weakly atomic lattice the weak modularity conditions can be replaced by conditions concerning the principal congruences corresponding to prime quotients.

PROPOSITION 3.5. *Let L be a weakly atomic lattice. Then*

(i) *L is locally weakly modular if and only if for each $a \prec b$ the congruence $\theta(a,b)$ is an atom of* $\operatorname{Con} L$.

(ii) *L is locally almost weakly modular if and only if for each $a \prec b$ the congruence $\theta^{**}(a,b)$ is an atom of* $\mathrm{B}(\operatorname{Con} L)$.

(iii) *If $\operatorname{Con} L$ is atomic, then L is locally almost weakly modular if and only if all $\theta(a,b)$ with $a \prec b$ contain exactly one atom of* $\operatorname{Con} L$.

Proof.

(i) Assume that L is locally weakly modular, and let $a, b \in L$ with $a \prec b$ and $\theta \leqq \theta(a,b)$, $\theta \neq \Delta$. Since L is weakly atomic, there are $c, d \in L$ such that $(c,d) \in \theta$ and $c \prec d$. As now $(c,d) \in \theta(a,b)$, we get $b/a \rightarrow d/c$ and this implies $d/c \rightarrow b/a$, because L is locally weakly modular. Thus we get $\theta(a,b) \leqq \theta(c,d) \leqq \theta$, i.e. $\theta = \theta(a,b)$, proving that θ is an atom of $\operatorname{Con} L$. The converse is obvious.

(ii) Let L be locally almost weakly modular, $a, b \in L$, $a \prec b$, and take a $\theta \in \mathrm{B}(\operatorname{Con} L)$, $\theta \neq \Delta$ with $\theta \leqq \theta^{**}(a,b)$. Now, in view of Lemma 3.4, we have either $\theta(a,b) \leqq \theta^{*}$ or $\theta(a,b) \leqq \theta^{**}$. The first case can be excluded, since it implies $\theta \leqq \theta^{**}(a,b) \leqq \theta^{***} = \theta^{*}$, i.e. $\theta = \theta \wedge \theta^{*} = \Delta$, a contradiction. As $\theta^{**} = \theta$, the second case gives $\theta^{**}(a,b) \leqq (\theta^{**})^{**} = \theta$. Hence $\theta = \theta^{**}(a,b)$, therefore $\theta^{**}(a,b)$ is an atom of $\mathrm{B}(\operatorname{Con} L)$.

Conversely, assume that for any $a, b \in L$, $a \prec b$, the congruence $\theta^{**}(a,b)$ is an atom of $\mathrm{B}(\operatorname{Con} L)$ and let $b/a \rightarrow p/q$ and $b/a \rightarrow r/s$ for some prime quotients p/q and r/s. Then $\theta(p,q) \leqq \theta(a,b)$, $\theta(r,s) \leqq \theta(a,b)$ and hence $\theta^{**}(p,q) \leqq \theta^{**}(a,b)$, $\theta^{**}(r,s) \leqq \theta^{**}(a,b)$. As $\theta^{**}(a,b)$ is an atom in $\mathrm{B}(\operatorname{Con} L)$ and since $\theta^{**}(p,q), \theta^{**}(r,s) \in \mathrm{B}(\operatorname{Con} L)$, we get $\theta^{**}(p,q) = \theta^{**}(r,s) = \theta^{**}(a,b)$. We must have $\theta(p,q) \wedge \theta(r,s) \neq \Delta$, otherwise $\theta(p,q) \wedge \theta(r,s) = \Delta$ would imply $\Delta = \big(\theta(p,q) \wedge \theta(r,s)\big)^{**} = \theta^{**}(p,q) \wedge \theta^{**}(r,s) = \theta^{**}(a,b)$, a contradiction. Since L is weakly atomic, there exist $u, t \in L$, $u \prec t$, such that $(u,t) \in \theta(p,q) \wedge \theta(r,s)$. Hence $p/q \rightarrow t/u$ and $r/s \rightarrow t/u$, and this proves that L is locally almost weakly modular.

(iii) Let L be locally almost weakly modular and suppose that for a pair $a \prec b$ the congruence $\theta(a,b)$ contains two different atoms α_1 and α_2 of $\operatorname{Con} L$. Then $\alpha_1^{**}, \alpha_2^{**} \in \mathrm{B}(\operatorname{Con} L)$ and $\alpha_1^{**} \leqq \theta^{**}(a,b)$, $\alpha_2^{**} \leqq \theta^{**}(a,b)$. Since, by the above (ii), $\theta^{**}(a,b)$ is an atom of $\mathrm{B}(\operatorname{Con} L)$, we get $\alpha_1^{**} = \alpha_2^{**} = \theta^{**}(a,b)$. However, $\alpha_1 \wedge \alpha_2 = \Delta$ implies $\theta(a,b) \leqq \theta^{**}(a,b) = \alpha_1^{**} \wedge \alpha_2^{**} = (\alpha_1 \wedge \alpha_2)^{**} = \Delta$ — a contradiction. As $\operatorname{Con} L$ is an atomic lattice, $\theta(a,b)$ must contain exactly one atom.

Conversely, assume that for $a \prec b$ the congruence $\theta(a,b)$ contains a single atom $\alpha \in \operatorname{Con} L$ and let $b/a \to p/q$ and $b/a \to r/s$ for some prime quotients p/q and r/s. Then $\theta(p,q) \leqq \theta(a,b)$ and $\theta(r,s) \leqq \theta(a,b)$. Since each of $\theta(p,q)$, $\theta(r,s)$ and $\theta(a,b)$ contains a single atom, we conclude that $\alpha \leqq \theta(p,q)$ and $\alpha \leqq \theta(r,s)$. As L is weakly atomic, there is a prime quotient t/u such that $\alpha = \theta(u,t)$. Hence $p/q \to t/u$ and $r/s \to t/u$, proving that L is locally almost weakly modular. □

COROLLARY 3.6. *Let L be a weakly atomic lattice.*

(i) *If L is locally weakly modular, then* $\operatorname{Con} L$ *is atomic.*

(ii) *If L is locally almost weakly modular, then* $\mathrm{B}(\operatorname{Con} L)$ *is atomic.*

Proof. Take any $\theta \in \operatorname{Con} L \setminus \{\Delta\}$. As L is weakly atomic, there exist $a, b \in L$, $a \prec b$, such that $(a,b) \in \theta$.

(i) If L is locally weakly modular, then, in view of Proposition 3.5(i), $\theta(a,b)$ is an atom of $\operatorname{Con} L$. Since $\theta(a,b) \leqq \theta$, the lattice $\operatorname{Con} L$ is atomic.

(ii) As L is locally almost weakly modular, Proposition 3.5(ii) gives that the congruence $\theta^{**}(a,b)$ is an atom in $\mathrm{B}(\operatorname{Con} L)$. If $\theta \in \mathrm{B}(\operatorname{Con} L)$, then we get $\theta^{**}(a,b) \leqq \theta^{**} = \theta$, proving that $\mathrm{B}(\operatorname{Con} L)$ is atomic. □

Combining the statements of Corollary 3.3, Proposition 3.5 and Corollary 3.6 we obtain:

COROLLARY 3.7. *Let L be a weakly atomic lattice. Then*

(i) $\operatorname{Con} L$ *is Boolean if and only if for all $a, b \in L$ with $a \prec b$ the congruence $\theta(a,b)$ is an atom of* $\operatorname{Con} L$ *and any congruence of L is separable.*

(ii) $\operatorname{Con} L$ *is a Stone lattice if and only if for all $a, b \in L$ with $a \prec b$ the congruence $\theta^{**}(a,b)$ is an atom of* $\mathrm{B}(\operatorname{Con} L)$ *and any congruence of L is weakly separable.*

(iii) $\operatorname{Con} L$ *is an atomic Stone lattice if and only if for all $a, b \in L$ with $a \prec b$ the congruence $\theta(a,b)$ contains exactly one atom of* $\operatorname{Con} L$ *and any congruence of L is weakly separable.*

Remark 3.8. We note that Corollary 3.3(ii) implies that a semidiscrete lattice has a Stonean congruence lattice if and only if L is locally almost weakly modular — an assertion established in [8]. Corollary 3.7(i) gives that a finite lattice has a Boolean congruence lattice if and only if for every $a, b \in L$ with $a \prec b$ the principal congruence $\theta(a,b)$ is an atom of $\operatorname{Con} L$ — statement which is implicitly contained in [14]. Corollary 3.7(iii) constitute a generalization of Iqbalunnisa's result [10].

It is interesting that a separable congruence θ is not weakly separable in general, however, in [12] it was showed that in a weakly modular lattice the required relationship is true. Sharpening this result, here we prove:

PROPOSITION 3.9. *Let L be an almost weakly modular and weakly atomic lattice and let $\theta \in \operatorname{Con} L$. If θ is separable, then it is weakly separable as well.*

Proof. Assume that $\theta \in \operatorname{Con} L$ is separable and take any $a, b \in L$, $a < b$. Then there is a chain $a = z_0 \leqq z_1 \leqq \cdots \leqq z_n = b$ such that for each $i = 1, \dots, n$, either $(z_{i-1}, z_i) \in \theta$ or there is no subquotient $r/s \subseteq z_i/z_{i-1}$ with $r \neq s$ and $(s, r) \in \theta$. We show that θ is weakly separable, too.

Indeed, if $(z_{i-1}, z_i) \in \theta^{**}$, then there is no subquotient $r/s \subseteq z_i/z_{i-1}$ satisfying $r \neq s$ and $(s, r) \in \theta^*$, i.e. the condition (b) of Definition 2.3(ii) is obviously satisfied. Since $\theta \leqq \theta^{**}$, the same is true whenever $(z_{i-1}, z_i) \in \theta$ holds. If $(z_{i-1}, z_i) \in \theta^*$, then $z_i/z_{i-1} \to u/v$ and $(u, v) \in \theta$ imply $u = v$, i.e. condition (a) of Definition 2.3(ii) is satisfied. Therefore, to prove that θ is weakly separable, it is enough to show that for each $i = 1, \dots, n$, either $(z_{i-1}, z_i) \in \theta^*$ or $(z_{i-1}, z_i) \in \theta^{**}$ holds.

In contrary, assume that for some $i_0 \in \{1, \dots, n\}$ none of the latter conditions is satisfied. Then clearly, $\theta(z_{i_0-1}, z_{i_0}) \wedge \theta^* \neq \Delta$ and $\theta(z_{i_0-1}, z_{i_0}) \wedge \theta^{**} \neq \Delta$. Since L is weakly atomic, there are prime quotients u/v and w/y such that $z_{i_0}/z_{i_0-1} \to u/v$, $(u, v) \in \theta^*$ and $z_{i_0}/z_{i_0-1} \to w/y$, $(w, y) \in \theta^{**}$ hold. Since L is almost weakly modular, there is a dominant subquotient a_1/b_1 of z_{i_0}/z_{i_0-1} satisfying the condition of Definition 2.2(ii) (for $a = z_{i_0-1}$, $b = z_{i_0}$). Let p/q be a prime subquotient of a_1/b_1. Then, in view of Lemma 3.4, we have either $(q, p) \in \theta^*$ or $(q, p) \in \theta^{**}$. As $a_1/b_1 \to p/q$, there are quotients z/t and z'/t' with $z \neq t$ and $z' \neq t'$ such that $p/q \to z/t$, $u/v \to z/t$ and $p/q \to z'/t'$, $w/y \to z'/t'$, according to Definition 2.2(ii). Now, in the case $(q, p) \in \theta^*$ we get $(z', t') \in \theta^* \wedge \theta^{**} = \Delta$ — a contradiction, and similarly, the case $(q, p) \in \theta^{**}$ gives $(z, t) \in \theta^{**} \wedge \theta^* = \Delta$ — a contradiction again. Hence for each $i \in \{1, \dots, n\}$ we have either $(z_{i-1}, z_i) \in \theta^*$ or $(z_{i-1}, z_i) \in \theta^{**}$, and this completes the proof. $\square$

LEMMA 3.10. *If L is a weakly atomic lattice such that all congruences of L are separable, then for each $\theta \in \operatorname{Con} L$ we have $\theta = \bigvee\{\theta(a, b) :\ a \prec b,\ (a, b) \in \theta\}$.*

Proof. Take any $\theta \in \operatorname{Con} L$ and let $\theta_I = \bigvee\{\theta(a, b) :\ a \prec b,\ (a, b) \in \theta\}$. Clearly, $\theta_I \leqq \theta$. Take any $a, b \in \theta$, $a \neq b$. Since by our assumption the congruence θ_I is separable, there exists a chain $a = z_0 \leqq z_1 \leqq \cdots \leqq z_n = b$ in L such that for each $i = 1, \dots, n$, either $(z_{i-1}, z_i) \in \theta_I$ or there is no subquotient $r/s \subseteq z_i/z_{i-1}$ with $r \neq s$ and $(s, r) \in \theta_I$. As L is weakly atomic, for all $z_{i-1} < z_i$ there are $c, d \in L$ such that $z_{i-1} \leqq c \prec d \leqq z_i$. Since $a \leqq c < d \leqq b$, we have $(c, d) \in \theta$, thus we get $(c, d) \in \theta_I$ and this excludes the above mentioned second case. Therefore we obtain $(z_{i-1}, z_i) \in \theta_I$ for all $i = 1, \dots, n$ implying $a, b \in \theta_I$. Hence $\theta \leqq \theta_I$, and this completes the proof. $\square$

If any element of a lattice L is a join of completely join-irreducible elements of L, then L is called a *CJ-generated lattice.* It is easy to see that each $\theta(a,b)$ with $a \prec b$ is a completely join-irreducible element of $\operatorname{Con} L$. Hence $\operatorname{Con} L$ is a CJ-generated lattice, whenever the assumptions of Lemma 3.10 hold. As any algebraic CJ-generated lattice is completely distributive ([1]), we obtain:

COROLLARY 3.11. *If L is a weakly atomic lattice with separable congruences, then $\operatorname{Con} L$ is completely distributive.*

A complete and distributive pseudocomplemented lattice L is called *completely Stonean* if $\bigvee_{i\in I} x_i^{**} = \Big(\bigvee_{i\in I} x_i\Big)^{**}$ holds for any system of elements $x_i \in L$, $i \in I$. A characterization of lattices with completely Stonean congruence lattice can be found in [13]. Here we add the following:

THEOREM 3.12. *Let L be a weakly atomic and locally almost weakly modular lattice. If all the congruences of L are separable, then $\operatorname{Con} L$ is a completely distributive and completely Stonean lattice.*

Proof. In view of Corollary 3.11, $\operatorname{Con} L$ is a completely distributive lattice. Since the inequality $\bigvee_{i\in I} x_i^{**} \leqq \Big(\bigvee_{i\in I} x_i\Big)^{**}$ holds in any complete lattice, we have to prove only that for any system $\theta_i \in \operatorname{Con} L$, $i \in I$, and any $u,v \in L$, $(u,v) \in \Big(\bigvee_{i\in I} \theta_i\Big)^{**}$ implies $(u,v) \in \bigvee_{i\in I} \theta_i^{**}$. As by our assumption all the congruences of L are separable, we get $\theta(u,v) = \bigvee\{\theta(a,b) :\ a \prec b,\ (a,b) \in \theta(u,v)\}$ according to Lemma 3.10. Then, in view of Lemma 3.4, for any $(a,b) \in \theta(u,v)$ with $a \prec b$ and any $i \in I$ we have either $(a,b) \in \theta_i^{*}$ or $(a,b) \in \theta_i^{**}$. We claim that there exists at least one $i_0 \in I$ such that $(a,b) \in \theta_{i_0}^{**}$.

In contrary, assume that $(a,b) \in \theta_i^{*}$ for all $i \in I$. Then $\theta(a,b) \wedge \theta_i = \triangle$, $i \in I$, implies $\bigvee_{i\in I} \theta_i \leqq \theta^*(a,b)$, whence we get $\Big(\bigvee_{i\in I} \theta_i\Big)^{**} \leqq \theta^{***}(a,b) = \theta^*(a,b)$. However, $\theta(a,b) \leqq \theta(u,v) \leqq \Big(\bigvee_{i\in I} \theta_i\Big)^{**}$ implies $\theta(a,b) \leqq \theta^*(a,b)$, i.e. $\theta(a,b) = \theta(a,b) \wedge \theta^*(a,b) = \triangle$, a contradiction.

Thus our claim is true and it implies $\theta(a,b) \leqq \bigvee_{i\in I} \theta_i^{**}$ for all $(a,b) \in \theta(u,v)$, $a \prec b$. Hence we obtain $\theta(u,v) = \bigvee\{\theta(a,b) :\ a \prec b,\ (a,b) \in \theta(u,v)\} \leqq \bigvee_{i\in I} \theta_i^{**}$, i.e. $(u,v) \in \bigvee_{i\in I} \theta_i^{**}$ and this completes the proof. □

Problems

1) Characterize those weakly atomic lattices whose congruence lattices are completely Stonean by the mean of prime quotients.
2) The same problem for weakly atomic lattices with relatively Stonean congruence lattices. (A starting point can be for instance [8].)
3) Characterize those weakly atomic lattices whose congruence lattices are (relative) (L_n)-congruence lattices. (A starting point can be [7] and [9].)

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