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OPTIMAL CONTROL FOR $n \times n$ HYPERBOLIC SYSTEMS INVOLVING OPERATORS OF INFINITE ORDER

H. A. El-Saify — G. M. Bahaa

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ABSTRACT. In this paper, the following $n \times n$ mixed Dirichlet problem of hyperbolic type

$$\frac{\partial^2}{\partial t^2} y_i(\overline{u}) + \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} y_i(\overline{u}) + \sum_{j=1}^{n} a_{ij} y_j(\overline{u}) = f_i + u_i \qquad \text{in} \quad Q\,,$$

$$\mathrm{D}^\omega y_i(\overline{u}) = 0 \qquad \text{on } \Sigma \quad \text{for } |\omega| = 0,1,2,\dots\,, \quad |\omega| \le \alpha - 1\,, \quad i = 1,2,\dots,n\,, \quad \text{(D)}$$

$$y_i(x,0;\overline{u}) = y_{i,0}(x)\,, \quad \frac{\partial}{\partial t} y_i(x,0;\overline{u}) = y_{i,1}(x) \qquad \text{in} \quad \mathbb{R}^N\,,$$

where f_i are given functions, $\overline{u} = (u_i)_{i=1}^n$ and a_{ij} are coefficients matrix such that

$$a_{ij} = \begin{cases} 1 & \text{if } i \ge j\,, \\ -1 & \text{if } i < j\,, \end{cases}$$

$$\mathrm{D}^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \dots (\partial x_N)^{\alpha_N}}\,, \qquad \alpha = (\alpha_1, \dots, \alpha_N)\,,$$

is a multi-index for differentiation, $|\alpha| = \sum_{i=1}^{N} \alpha_i$, will be discussed, which involve infinite order elliptic operator A having the form

$$A y_i(\overline{u}) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} y_i(\overline{u})\,.$$

The optimality conditions for this system are given. The problem with Neumann conditions also will be formulated.

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Introduction

By using the theory of J. L. Lions [4] and Ju. A. Dubinskii [1], I. M. Gali et. al. [3] discussed the optimal control problem governed by systems of infinite order hyperbolic operators. They found the optimality conditions for these systems, which consists of one equation.

The authors presented in [8] a set of inequalities defining an optimal control of $n \times n$ systems with infinite number of variables, of hyperbolic type.

The questions treated in this paper are related to the above result but in different direction, by taking the case of operators of infinite order with finite dimension i.e., we discuss the $n \times n$ systems of hyperbolic type involving the operator of infinite order.

First we prove the existence and uniqueness of solutions for these systems with Dirichlet conditions; then we find the optimality conditions of the optimal control problem governed by these systems. The problem with Neumann conditions is also considered here.

I. Some function spaces

The object of this section is to give the definition of some function spaces of infinite order, and the chains of the constructed space which will be used later.

We define the Sobolev space $W^{\infty}\{a_\alpha, 2\}(\mathbb{R}^N)$ (which we shall denote by $W^{\infty}\{a_\alpha, 2\}$) of infinite order of periodic functions $\phi(x)$ defined on all boundary Γ of $\mathbb{R}^N$, $N \geq 1$, as follows [1]:

$$W^{\infty}\{a_\alpha, 2\} = \Big\{\phi(x) \in C^{\infty}(\mathbb{R}^N) : \sum_{|\alpha|=0}^{\infty} a_\alpha \| \mathrm{D}^\alpha \phi\|_2^2 < \infty\Big\},$$

where $a_\alpha \geq 0$ is a numerical sequence and $\|\cdot\|_2$ is the canonical norm in the space $L^2(\mathbb{R}^N)$ (all functions are assumed to be real valued).

The space $W^{-\infty}\{a_\alpha, 2\}$ is defined as the formal conjugate space to the space $W^{\infty}\{a_\alpha, 2\}$, namely

$$W^{-\infty}\{a_\alpha, 2\} = \Big\{\psi(x) : \psi(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha \mathrm{D}^\alpha \psi_\alpha(x)\Big\},$$

where $\psi_\alpha \in L^2(\mathbb{R}^N)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha \|\psi_\alpha\|_2^2 < \infty$.

The duality pairing of the spaces $W^{\infty}\{a_\alpha, 2\}$ and $W^{-\infty}\{a_\alpha, 2\}$ is postulated by the formula

$$(\phi, \psi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{\mathbb{R}^N} \psi_\alpha(x) \mathrm{D}^\alpha \phi(x) \, \mathrm{d}x,$$

where

$$\phi \in W^{\infty}\{a_\alpha, 2\}, \qquad \psi \in W^{-\infty}\{a_\alpha, 2\}.$$

From above, $W^{\infty}\{a_\alpha, 2\}$ is everywhere dense in $L^2(\mathbb{R}^N)$ with topological inclusions and $W^{-\infty}\{a_\alpha, 2\}$ denotes the topological dual space with respect to $L^2(\mathbb{R}^N)$, so we have the following chain

$$W^{\infty}\{a_\alpha, 2\} \subseteq L^2(\mathbb{R}^N) \subseteq W^{-\infty}\{a_\alpha, 2\}.$$

Analogous to the above chain we have:

$$W_0^{\infty}\{a_\alpha, 2\} \subseteq L^2(\mathbb{R}^N) \subseteq W_0^{-\infty}\{a_\alpha, 2\},$$

where $W_0^{\infty}\{a_\alpha, 2\}$ is the set of all functions of $W^{\infty}\{a_\alpha, 2\}$ which vanish on the boundary Γ of $\mathbb{R}^N$, i.e.,

$$W_0^{\infty}\{a_\alpha, 2\} = \Big\{\phi \in C_0^{\infty}(\mathbb{R}^N) : \|\phi\|^2 = \sum_{|\alpha|=0}^{\infty} a_\alpha \|\mathrm{D}^\alpha \phi\|_2^2 < \infty, \quad \mathrm{D}^\omega \phi|_\Gamma = 0, \ |\omega| \le \alpha\Big\}.$$

We now introduce $L^2(0,T; L^2(\mathbb{R}^N))$ (briefly $L^2(Q)$) to denote the space of measurable functions $t \mapsto \phi(t)$ such that

$$\|\phi\|_{L^2(Q)} = \left(\int_0^T \|\phi(t)\|_2^2 \, \mathrm{d}t\right)^{\frac{1}{2}} < \infty.$$

Endowed with the scalar product $(f,g) = \int_0^T (f(t), g(t))_{L^2(\mathbb{R}^N)} \, \mathrm{d}t$ is a Hilbert space. In the same manner we define the space $L^2(0,T; W^{\infty}\{a_\alpha, 2\})$, $L^2(0,T; W_0^{\infty}\{a_\alpha, 2\})$ and $L^2(0,T; W^{-\infty}\{a_\alpha, 2\})$, $L^2(0,T; W_0^{-\infty}\{a_\alpha, 2\})$ as its formal conjugate respectively.

Then it is easy to construct the Cartesian product to the above Hilbert spaces, for example

$$(W^{\infty}\{a_\alpha, 2\})^n = \prod_{i=1}^{n} (W^{\infty}\{a_\alpha, 2\})^i,$$

with norm defined by

$$\|\overline{\phi}\|_{(W^{\infty}\{a_\alpha,2\})^n} = \sum_{i=1}^{n} \|\phi_i\|_{W^{\infty}\{a_\alpha,2\}},$$

where $\overline{\phi} = (\phi_1, \dots, \phi_n) = (\phi_i)_{i=1}^n$ is a vector function and $\phi_i \in W^{\infty}\{a_\alpha, 2\}$.

Finally we have the following chains

$$\big(L^2(0,T;W^{\infty}\{a_\alpha,2\})\big)^n \subseteq \big(L^2(Q)\big)^n \subseteq \big(L^2(0,T;W^{-\infty}\{a_\alpha,2\})\big)^n$$
$$\big(L^2(0,T;W_0^{\infty}\{a_\alpha,2\})\big)^n \subseteq \big(L^2(Q)\big)^n \subseteq \big(L^2(0,T;W_0^{-\infty}\{a_\alpha,2\})\big)^n,$$

where $\big(L^2(0,T;W^{-\infty}\{a_\alpha,2\})\big)^n$ and $\big(L^2(0,T;W_0^{-\infty}\{a_\alpha,2\})\big)^n$ are the dual spaces of $\big(L^2(0,T;W^{\infty}\{a_\alpha,2\})\big)^n$ and $\big(L^2(0,T;W_0^{\infty}\{a_\alpha,2\})\big)^n$ respectively.

The following notation will be used:

$$Q = \mathbb{R}^N \times\,]0,T[\,, \qquad \Sigma = \Gamma \times\,]0,T[\,,$$

where Σ is the lateral boundary of Q.

II. Optimal control problem for $n \times n$ mixed Dirichlet problem of hyperbolic type

In this section we shall find the optimality conditions of problem (D) which is known as $n\times n$ mixed Dirichlet problem of hyperbolic type involving operator of infinite order.

First we prove the existence and uniqueness of the solution for this system, then we find the optimality conditions for this system.

So, for each $t \in\,]0,T[$, we define a family of bilinear forms on $(W^{\infty}\{a_\alpha,2\})^n$ by

$$\begin{aligned}
&\pi(t;\overline{y},\overline{\phi})\colon \big(W^{\infty}\{a_\alpha,2\}\big)^n \times \big(W^{\infty}\{a_\alpha,2\}\big)^n \to \mathbb{R},\\
&\pi(t;\overline{y},\overline{\phi}) = \sum_{i=1}^{n}\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha\, \mathrm{D}^{2\alpha}y_i(x) + \sum_{j=1}^{n}a_{ij}y_j(x)\,,\ \phi_i(x)\right)_{L^2(\mathbb{R}^N)}\\
&\qquad = \sum_{i=1}^{n}\left(Ay_i(x) + \sum_{j=1}^{n}a_{ij}y_j(x),\ \phi_i(x)\right)_{L^2(\mathbb{R}^N)} \qquad (1)\\
&\qquad = \sum_{i=1}^{n}\big(My_i(x),\ \phi_i(x)\big)_{L^2(\mathbb{R}^N)},
\end{aligned}$$

where $\overline{y} = (y_i)_{i=1}^n, \overline{\phi} = (\phi_i)_{i=1}^n \in \big(W_0^{\infty}\{a_\alpha,2\}\big)^n$, $My_i \in W_0^{-\infty}\{a_\alpha,2\}$ and M is an operator defined on $\big(W_0^{\infty}\{a_\alpha,2\}\big)^n$ by

$$
\begin{aligned}
M\Big(\overline{y}(x) &= \big(y_1(x), y_2(x), \dots, y_n(x)\big)\Big) \\
&= \Bigg(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} \, y_1(x) + \sum_{j=1}^{n} a_{1j} y_j(x), \\
&\qquad \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} \, y_2(x) + \sum_{j=1}^{n} a_{2j} y_j(x) , \\
&\qquad \vdots \\
&\qquad \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} \, y_n(x) + \sum_{j=1}^{n} a_{nj} y_j(x) \Bigg) \\
&= \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} \, y_i(x) + \sum_{j=1}^{n} a_{ij} y_j(x) \\
&= A y_i(x) + \sum_{j=1}^{n} a_{ij} y_j(x) , \qquad i = 1, 2, \dots, n ,
\end{aligned}
$$

where A is an infinite order elliptic operator. It is easy to see that M is $n \times n$ matrix of the form ([7]):

$$
\begin{pmatrix}
\sum\limits_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} + 1 & -1 & \dots & -1 \\
1 & \sum\limits_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} + 1 & \dots & -1 \\
\dots & \dots & \dots & \dots \\
1 & 1 & \dots & \sum\limits_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} + 1
\end{pmatrix} .
$$

The bilinear form (1) is coercive on $\big(W_0^\infty\{a_\alpha, 2\}\big)^n$ i.e.

$$
\Big(\forall \overline{y} \in \big(W_0^\infty\{a_\alpha, 2\}\big)^n\Big) \Big(\pi(t; \overline{y}, \overline{y}) \geq \|\overline{y}\|^2_{(W_0^\infty\{a_\alpha, 2\})^n}\Big) . \tag{2}
$$

In fact, the bilinear form (1) can be written as:

$$
\pi\big(t; \overline{y}, \overline{\phi}\big) = \sum_{i=1}^{n} \int_{\mathbb{R}^N} \sum_{|\alpha|=0}^{\infty} a_\alpha \, \mathrm{D}^\alpha \, y_i(x) \, \mathrm{D}^\alpha \, \phi_i(x) \, \mathrm{d}x + \sum_{i=1}^{n} \int_{\mathbb{R}^N} \sum_{j=1}^{n} a_{i,j} y_j(x) \phi_i(x) \, \mathrm{d}x .
$$

By taking into account the form of a_{ij}, we have

$$\pi\big(t;\overline{y},\overline{\phi}\big) = \sum_{i=1}^{n} \int_{\mathbb{R}^N} \sum_{|\alpha|=0}^{\infty} a_\alpha \, \mathrm{D}^\alpha \, y_i(x) \, \mathrm{D}^\alpha \, \phi_i(x) \, \mathrm{d}x + \sum_{i=j=1}^{n} \int_{\mathbb{R}^N} y_j(x)\phi_i(x) \, \mathrm{d}x$$
$$+ \sum_{i>j}^{n} \int_{\mathbb{R}^N} y_j(x)\phi_i(x) \, \mathrm{d}x - \sum_{i<j}^{n} \int_{\mathbb{R}^N} y_j(x)\phi_i(x) \, \mathrm{d}x \, .$$

Then

$$\begin{aligned}
\pi(t;\overline{y},\overline{y}) &= \sum_{i=1}^{n} \left(\int_{\mathbb{R}^N} \sum_{|\alpha|=0}^{\infty} a_\alpha | \, \mathrm{D}^\alpha \, y_i(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} |y_i(x)|^2 \, \mathrm{d}x \right) \\
&= \sum_{i=1}^{n} \left(\sum_{|\alpha|=0}^{\infty} a_\alpha \| \, \mathrm{D}^\alpha \, y_i(x)\|_2^2 + \|y_i(x)\|_2^2 \right) \\
&= \sum_{i=1}^{n} \|y_i(x)\|^2_{W_0^\infty\{a_\alpha,2\}} + \|\overline{y}(x)\|_2^2 \\
&\geq \|\overline{y}\|^2_{(W_0^\infty\{a_\alpha,2\})^n} \, .
\end{aligned}$$

For all $\overline{y},\overline{\phi} \in \big(W_0^\infty\{a_\alpha,2\}\big)^n$ the function $t \mapsto \pi\big(t;\overline{y},\overline{\phi}\big)$ is continuously differentiable in $]0,T[$, and

$$\pi\big(t;\overline{y},\overline{\phi}\big) = \pi\big(t;\overline{\phi},\overline{y}\big) \, . \tag{3}$$

Lemma 1. *Using theorems of* [4], [5], *if* (2) *and* (3) *hold, then for given* $f_i = f_i(x,t)$, $y_{i,0}(x)$ *and* $y_{i,1}$ *in* $L^2(Q)$, $W_0^\infty\{a_\alpha,2\}$ *and* $L^2(\mathbb{R}^N)$ *respectively, there exists a unique* $\overline{y} = (y_i)_{i=1}^n \in \big(L^2(Q)\big)^n$ *satisfying problem* (D) *which defines the state variable of our control problem.*

Outline of proof. If L is a continuous linear form on $\big(W_0^\infty\{a_\alpha,2\}\big)^n$, then by virtue the chains in Section I

$$L\big(\overline{\phi}\big) = \big(\overline{f},\overline{\phi}\big)_{(L^2(\mathbb{R}^N))^n} \, , \qquad \overline{f} \in \big(W_0^{-\infty}\{a_\alpha,2\}\big)^n .$$

From the coerciveness conditions, there exists a unique element $\overline{y} = (y_i)_{i=1}^n \in \big(L^2(Q)\big)^n$ such that

$$\left\langle \frac{\partial^2 \overline{y}}{\partial t^2}, \overline{\phi} \right\rangle + \pi\big(\overline{y},\overline{\phi}\big) = L\big(\overline{\phi}\big) \, ,$$

which is equivalent to the existence of a unique $\overline{y} = (y_i)_{i=1}^n \in \big(L^2(Q)\big)^n$ satisfying problem (D) which defines the state of our control problem. □

III. Formulation of the control problem

The space $\mathcal{U} = \left(L^2(Q)\right)^n$ is the space of controls. If $f_i \in L^2(Q)$, $y_{i,0} \in W_0^\infty\{a_\alpha, 2\}$ and $y_{i,1} \in L^2(\mathbb{R}^N)$ and if (2) and (3) hold, then for a control $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}$, the state of the system $\overline{y}(\overline{u}) = \left(y_i(\overline{u})\right)_{i=1}^n$, which depends on x, t, denoted by $\overline{y}(x,t;\overline{u})$, is given by the solution of the problem (D) i.e.,

$$\begin{gathered}
\frac{\partial^2 y_i(\overline{u})}{\partial t^2} + My_i(\overline{u}) = f_i + u_i \qquad \text{in} \quad Q\,, \\
\mathrm{D}^\omega\, y_i(\overline{u}) = 0\,, \qquad \text{on} \quad \Sigma\,, \\
y_i(x,0;\overline{u}) = y_{i,0}(x)\,, \quad \frac{\partial y_i(x,0;\overline{u})}{\partial t} = y_{i,1}(x) \qquad \text{in} \quad \mathbb{R}^N\,, \\
y_i(\overline{u}) \in L^2(Q)\,, \qquad \frac{\partial y_i(\overline{u})}{\partial t} \in L^2(Q)\,,
\end{gathered} \tag{4}$$

where the operator $\frac{\partial^2}{\partial t^2} + M$ is infinite order hyperbolic operator which maps $\left(L^2\left(0,T; W_0^\infty\{a_\alpha,2\}\right)\right)^n$ onto $\left(L^2\left(0,T; W_0^{-\infty}\{a_\alpha,2\}\right)\right)^n$ ([6]).

The observation is given by:

$$\left(z_i(\overline{u})\right)_{i=1}^n = \overline{z}(\overline{u}) = \overline{y}(\overline{u}) = \left(y_i(\overline{u})\right)_{i=1}^n$$

i.e.,

$$z_i(\overline{u}) = y_i(\overline{u}) \qquad \text{for all} \quad 1 \le i \le n\,.$$

For given $\overline{z}_d = (z_{i,d})_{i=1}^n \in \left(L^2(Q)\right)^n$, the cost function $J(\overline{u})$ is given by

$$J(\overline{u}) = \sum_{i=1}^n \|y_i(\overline{u}) - z_{i,d}\|^2_{L^2(Q)} + \sum_{i=1}^n (\mathcal{N}_i u_i, u_i)_{L^2(Q)}\,,$$

which is equivalent to

$$J(\overline{u}) = \sum_{i=1}^n \int_Q \left(y_i(\overline{u}) - z_{i,d}\right)^2 \mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^n \left(\mathcal{N}_i u_i, u_i\right)_{L^2(Q)}\,, \tag{5}$$

where

$$\mathcal{N} = (\mathcal{N}_i)_{i=1}^n \in \mathcal{L}\Big(\left(L^2(Q)\right)^n, \left(L^2(Q)\right)^n\Big)$$

is a diagonal matrix of Hermitian positive definite operators: $\mathcal{N}\overline{u} = (\mathcal{N}_i u_i)_{i=1}^n$,

$$\left(\mathcal{N}\overline{u}, \overline{u}\right)_{(L^2(Q))^n} \ge \zeta \|\overline{u}\|^2_{(L^2(Q))^n}\,, \qquad \zeta > 0\,. \tag{6}$$

Our problem is to find

$$\inf_{\overline{v} \in \mathcal{U}_{\mathrm{ad}}} J(\overline{v})\,,$$

where the set of admissible controls $\mathcal{U}_{\mathrm{ad}}$ is closed convex subset of $\left(L^2(Q)\right)^n = \mathcal{U}$.

Under the given considerations, we apply the theorems in [2], [4] to obtain:

THEOREM 1. *We assume that* (2), (3) *and* (6) *hold. The cost function is given by* (5). *The optimal control* $\overline{u} = (u_i)_{i=1}^n$ *is characterized by* (4) *and the following system of partial differential equations and inequalities*

$$\begin{aligned} \frac{\partial^2 p_i(\overline{u})}{\partial t^2} + M^* p_i(\overline{u}) &= y_i(\overline{u}) - z_{i,d} && \text{in } Q\,, \\ \mathrm{D}^\omega p_i(\overline{u}) &= 0\,, && \text{on } \Sigma\,, \\ p_i(x,T;\overline{u}) = 0\,, \quad \frac{\partial p_i(x,T;\overline{u})}{\partial t} &= 0 && \text{in } \mathbb{R}^N\,, \end{aligned} \tag{7}$$

and

$$\sum_{i=1}^n \left(p_i(\overline{u}) + \mathcal{N}_i u_i,\, v_i - u_i\right)_{L^2(Q)} \geq 0 \qquad \textit{for all} \quad \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}$$

with

$$p_i(\overline{u}),\ \frac{\partial p_i(\overline{u})}{\partial t} \in L^2(Q)\,,$$

where M^* *is the adjoint operator of* M *which is defined by*

$$M^* \phi_i = A\phi_i + \sum_{j=1}^n a_{ji}\phi_j\,,$$

a_{ji} *is the transpose of* a_{ij} *and* $p_i(\overline{u})$ *is the adjoint state.*

Outline of proof. As in [4], the control $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}$ is optimal if and only if

$$\left(\forall\, \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}\right)\left(\sum_{i=1}^n J_i'(\overline{u})(v_i - u_i) \geq 0\right),$$

which is equivalent to:

$$\sum_{i=1}^n \left(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u})\right)_{L^2(Q)} + \sum_{i=1}^n \left(\mathcal{N}_i u_i,\, v_i - u_i\right)_{L^2(Q)} \geq 0\,,$$

which may be written as

$$\sum_{i=1}^n \int_0^T \left(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u})\right)_{L^2(\mathbb{R}^N)} \mathrm{d}t + \sum_{i=1}^n \left(\mathcal{N}_i u_i, v_i - u_i\right)_{L^2(Q)} \geq 0\,. \tag{8}$$

We shall now transform (8) as follows. By scalar multiplying both sides of the first equation in (7) by $\left(y_i(\overline{v}) - y_i(\overline{u})\right)$ and integrating between $0,\ T$ we obtain

$$\sum_{i=1}^{n} \int_0^T \left(\left(\frac{\partial^2}{\partial t^2} + M^* \right) p_i(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^N)} dt$$

$$= \sum_{i=1}^{n} \int_0^T \left(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^N)} dt\,.$$

We take into account conditions (4) and (7), after applying Green's formula to the left side of (7). Notice that if $\phi_i \in L^2(Q)$, $\phi_i' \in L^2(Q)$, $\phi_i'' \in L^2\big(0,T; W_0^{-\infty}\{a_\alpha, 2\}\big)$ and if ψ_i has the same properties, then

$$\int_0^T (\phi_i'', \psi_i)\, dt = \big(\phi_i'(T), \psi_i(T)\big) - \big(\phi_i'(0), \psi_i(0)\big) - \big(\phi_i(T), \psi_i'(T)\big)$$

$$+ \big(\phi_i(0), \psi_i'(0)\big) + \int_0^T (\phi_i, \psi_i'')\, dt.$$

Then we have

$$\int_0^T \left(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^N)} dt$$

$$= \int_0^T \left(\left(\frac{\partial^2}{\partial t^2} + M^* \right) p_i(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^N)} dt$$

$$= \int_0^T \left(\left(\frac{\partial^2}{\partial t^2} + A \right) p_i(\overline{u}) + \sum_{j=1}^{n} a_{ji} p_j(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u}) \right)_{L^2(\mathbb{R}^N)} dt$$

$$= \int_0^T \left(p_i(\overline{u}),\, \left(\frac{\partial^2}{\partial t^2} + A + \sum_{j=1}^{n} a_{ij} \right) \big(y_i(\overline{v}) - y_i(\overline{u})\big) \right)_{L^2(\mathbb{R}^N)} dt$$

$$= \int_0^T \left(p_i(\overline{u}), v_i - u_i \right)_{L^2(\mathbb{R}^N)} dt\,.$$

Hence (8) becomes

$$\sum_{i=1}^{n} \int_0^T \left(p_i(\overline{u}), v_i - u_i \right)_{L^2(\mathbb{R}^N)} dt + \sum_{i=1}^{n} (\mathcal{N}_i u_i, v_i - u_i)_{L^2(Q)} \geq 0$$

$$\text{for all} \quad \overline{v} = (v_i)_{i=1}^{n} \in \mathcal{U}_{\text{ad}}\,,$$

i.e.,

$$\sum_{i=1}^{n}\big(p_i(\overline{u}) + \mathcal{N}_i u_i,\, v_i - u_i\big)_{L^2(Q)} \ge 0 \qquad \text{for all} \quad \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\mathrm{ad}}\,.$$

The theorem is proved. □

Remark 1. If we take $n = 2$, then the optimality system is given by [4]:

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2} y_1(\overline{u}) + \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \,\mathrm{D}^{2\alpha}\, y_1(\overline{u}) + y_1(\overline{u}) - y_2(\overline{u}) = f_1 + u_1 && \text{in } Q\,,\\
&\frac{\partial^2}{\partial t^2} y_2(\overline{u}) + \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \,\mathrm{D}^{2\alpha}\, y_2(\overline{u}) + y_1(\overline{u}) + y_2(\overline{u}) = f_2 + u_2 && \text{in } Q\,,\\
&\mathrm{D}^{\omega}\, y_1(\overline{u}) = 0\,, \qquad \mathrm{D}^{\omega}\, y_2(\overline{u}) = 0 && \text{on } \Sigma\,,\\
&y_1(x,0;\overline{u}) = y_{1,0}(x;\overline{u})\,, \qquad y_2(x,0;\overline{u}) = y_{2,0}(x;\overline{u}) && \text{in } \mathbb{R}^N\,,\\
&\frac{\partial y_1(x,0;\overline{u})}{\partial t} = y_{1,1}(x;\overline{u})\,, \qquad \frac{\partial y_2(x,0;\overline{u})}{\partial t} = y_{2,1}(x;\overline{u}) && \text{in } \mathbb{R}^N\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2} p_1(\overline{u}) + \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \,\mathrm{D}^{2\alpha}\, p_1(\overline{u}) + p_1(\overline{u}) + p_2(\overline{u}) = y_1(\overline{u}) - z_{1,d} && \text{in } Q\,,\\
&\frac{\partial^2}{\partial t^2} p_2(\overline{u}) + \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \,\mathrm{D}^{2\alpha}\, p_2(\overline{u}) - p_1(\overline{u}) + p_2(\overline{u}) = y_2(\overline{u}) - z_{2,d} && \text{in } Q\,,\\
&p_1(\overline{u}) = 0\,, \qquad p_2(\overline{u}) = 0 && \text{on } \Sigma\,,\\
&p_1(x,T;\overline{u}) = 0\,, \qquad p_2(x,T;\overline{u}) = 0 && \text{in } \mathbb{R}^N\,,\\
&\frac{\partial p_1(x,T;\overline{u})}{\partial t} = 0\,, \qquad \frac{\partial p_2(x,T;\overline{u})}{\partial t} = 0 && \text{in } \mathbb{R}^N\,,
\end{aligned}$$

$$\int_Q \big(p_1(\overline{u}) + \mathcal{N}_1 u_1\big)(v_1 - u_1)\big) + \big(p_2(\overline{u}) + \mathcal{N}_2 u_2\big)(v_2 - u_2)\, \mathrm{d}x\, \mathrm{d}t \ge 0\,, \quad \text{for all} \quad \overline{v} = (v_1, v_2) \in \mathcal{U}_{\mathrm{ad}}\,,$$

where $\overline{u} = (u_1, u_2) \in \mathcal{U}_{\mathrm{ad}}$ and $\overline{p}(\overline{u}) = \big(p_1(\overline{u}), p_2(\overline{u})\big)$ is the adjoint state.

IV. Optimal control problem for $n \times n$ mixed Neumann problem of hyperbolic type

First we discuss the $n \times n$ mixed Neumann problem for hyperbolic type involving operator of infinite order.

Since $\left(W_0^\infty\{a_\alpha, 2\}\right)^n$ is everywhere dense in $\left(W^\infty\{a_\alpha, 2\}\right)^n$ with the topological inclusions, $\left(W_0^\infty\{a_\alpha, 2\}\right)^n \subseteq \left(W^\infty\{a_\alpha, 2\}\right)^n$ and since the bilinear form (1) is coercive on $\left(W_0^\infty\{a_\alpha, 2\}\right)^n$, it is also coercive in $\left(W^\infty\{a_\alpha, 2\}\right)^n$ i.e.

$$\pi(t; \overline{y}, \overline{y}) \geq \lambda \|\overline{u}\|_{(W^\infty\{a_\alpha, 2\})^n}, \qquad \lambda > 0. \tag{9}$$

Also, for $\overline{y}, \overline{\phi} \in \left(W^\infty\{a_\alpha, 2\}\right)^n$ the function $t \mapsto \pi\left(t; \overline{y}, \overline{\phi}\right)$ is continuously differentiable on $]0, T[$ and

$$\pi\left(t; \overline{y}, \overline{\phi}\right) = \pi\left(t; \overline{\phi}, \overline{y}\right). \tag{10}$$

Under the above considerations, we formulate the following lemma which defines the $n \times n$ mixed Neumann problem and enables us to state our control problem.

Lemma 2. *Assuming that* (9) *and* (10) *hold, then for given* $\overline{f} = \overline{f}(x,t) \in \left(L^2(Q)\right)^n$, $y_{i,1} \in L^2(\mathbb{R}^N)$, $y_{i,0} \in W^\infty\{a_\alpha, 2\}$, *there exists a unique element* y_i *satisfying:*

$$\begin{gathered} y_i, \frac{\partial y_i}{\partial x_k}, \frac{\partial y_i}{\partial t} \in L^2(Q), \\ \frac{\partial^2}{\partial t^2} y_i + M y_i = f_i \qquad \text{in} \quad Q, \\ \frac{\partial^\omega}{\partial \nu_A^\omega} y_i = 0 \qquad \text{on} \quad \Sigma, \\ y_i(x,0) = y_{i,0}(x), \qquad \frac{\partial}{\partial t} y_i(x,0) = y_{i,1}(x) \qquad \text{in} \quad \mathbb{R}^N, \end{gathered} \tag{11}$$

where $\frac{\partial^\omega}{\partial \nu_A^\omega}$ *is the co-normal derivatives with respect to* A.

Proof. From (9), we have

$$\frac{\partial^2}{\partial t^2} y_i(t) + M y_i = f_i \qquad \text{in } Q \quad \text{for all } 1 \leq i \leq n.$$

This equation is equivalent to

$$\left(\frac{\partial^2}{\partial t^2} y_i(t), \phi_i\right)_{L^2(\mathbb{R}^N)} + (M y_i, \phi_i)_{L^2(\mathbb{R}^N)} = \left(f_i(t), \phi_i\right)_{L^2(\mathbb{R}^N)}. \tag{12}$$

Let us define

$$\left(f_i(t), \phi_i\right)_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} f_i(x,t) \phi_i(x) \, \mathrm{d}x, \qquad \phi_i \in W^\infty\{a_\alpha, 2\}.$$

In this way we obtain an element of $L^2\left(0, T; W^\infty\{a_\alpha, 2\}\right)$.

So equation (12) can be written as:

$$\int_0^T \left(\frac{\partial^2}{\partial t^2} y_i(t),\, \phi_i(x)\right)_{L^2(\mathbb{R}^N)} \mathrm{d}t + \int_0^T \left(My_i(t), \phi_i(x)\right)_{L^2(\mathbb{R}^N)} \mathrm{d}t$$

$$= \int_0^T \left(f_i(t), \phi_i(x)\right)_{L^2(\mathbb{R}^N)} \mathrm{d}t$$

$$= \int_0^T \left(\frac{\partial^2}{\partial t^2} y_i(t),\, \phi_i(x)\right)_{L^2(\mathbb{R}^N)} \mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha} \, y_i(t)\phi_i(x) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^T \int_{\mathbb{R}^N} \sum_{j=1}^{n} a_{ij} y_j(t) \phi_i(x) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \left(f_i(t), \phi_i(x)\right)_{L^2(\mathbb{R}^N)} \mathrm{d}t .$$

Applying Green's formula to the left side, we get

$$\int_Q \frac{\partial^2}{\partial t^2} y_i(t)\phi_i(x) \, \mathrm{d}x \, \mathrm{d}t + \int_Q \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{\alpha} \, y_i(t) \, \mathrm{D}^{\alpha} \, \phi_i(x) \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_\Sigma \frac{\partial^\omega y_i(t)}{\partial \nu_A^\omega} \phi_i(x) \, \mathrm{d}\Sigma + \int_Q \sum_{j=1}^{n} a_{ij} y_j(t)\phi_i(x) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_Q f_i(x,t)\phi_i(x) \, \mathrm{d}x \, \mathrm{d}t .$$

Then we have

$$\int_\Sigma \frac{\partial^\omega y_i(t)}{\partial \nu_A^\omega} \phi_i(x) \, \mathrm{d}\Sigma = 0 ,$$

i.e.,

$$\frac{\partial^\omega y_i}{\partial \nu_A^\omega} = 0 \qquad \text{on } \Sigma \quad \text{for all } i = 1, 2, \ldots, n .$$

In this case, the operator $\frac{\partial^2}{\partial t^2} + M$ is an infinite order hyperbolic operator, it maps $\left(L^2(0,T; W^{\infty}\{a_\alpha, 2\}\right)^n$ onto $\left(L^2(0,T; W^{-\infty}\{a_\alpha, 2\}\right)^n$.

Formulation of our control problem.

Let $(L^2(Q))^n$ be the space of controls. For a control $\overline{u} = (u_i)_{i=1}^n \in \left(L^2(Q)\right)^n$ the state $\overline{y}(\overline{u}) = \left(y_i(\overline{u})\right)_{i=1}^n$ is given by the solution of system:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} y_i(\overline{u}) + M y_i(\overline{u}) &= f_i + u_i \qquad \text{in} \quad Q\,, \\ \frac{\partial^\omega}{\partial \nu_A^\omega} y_i(\overline{u}) &= 0 \qquad \text{on} \quad \Sigma\,, \\ y_i(x,0;\overline{u}) = y_{i,0}(x)\,, \qquad \frac{\partial y_i(x,0;\overline{u})}{\partial t} &= y_{i,1}(x) \qquad \text{in} \quad \mathbb{R}^N\,, \\ y_i(\overline{u}),\ \frac{\partial y_i(\overline{u})}{\partial t} \in L^2(Q) &\qquad \text{for all} \quad 1 \le i \le n\,. \end{aligned} \tag{13}$$

The cost function is given by (5). Then using the general theory of J. L. Lions [4], there exists a unique optimal control $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}_{\text{ad}}$ such that

$$\left(\forall \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\text{ad}}\right)\left(J(\overline{u}) = \inf J(\overline{v})\right). \tag{14}$$

□

Moreover, we have the following theorem that gives a characterization of the optimal control.

THEOREM 2. *Problem* (14) *admits a unique solution* $\overline{u} = (u_i)_{i=1}^n \in \left(L^2(Q)\right)^n$. *Moreover it is characterized by* (13) *and the following system:*

$$\begin{aligned} \frac{\partial^2}{\partial t^2} p_i(\overline{u}) + M^* p_i(\overline{u}) &= y_i(\overline{u}) - z_{i,d} \qquad \textit{in} \quad Q\,, \\ \frac{\partial^\omega}{\partial \nu_A^\omega} p_i(\overline{u}) &= 0 \qquad \textit{on} \quad \Sigma\,, \\ p_i(x,T;\overline{u}) = 0\,, \qquad \frac{\partial p_i(x,T;\overline{u})}{\partial t} &= 0 \qquad \textit{in} \quad \mathbb{R}^N\,, \\ p_i(\overline{u}),\ \frac{\partial p_i(x,T;\overline{u})}{\partial t} &\in L^2(Q)\,, \end{aligned} \tag{15}$$

$$\sum_{i=1}^n \int_Q \left(p_i(\overline{u}) + \mathcal{N}_i u_i\right)(v_i - u_i)\,\mathrm{d}x\,\mathrm{d}t \ge 0 \qquad \textit{for all} \quad \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\text{ad}}\,,$$

where $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}_{\text{ad}}$, $p_i(\overline{u})$ *is the adjoint state variable for* $1 \le i \le n$.

Outline of the proof. Since $J(\overline{u})$ is differentiable and $\mathcal{U}_{\text{ad}}$ is bounded, the optimal control $\overline{u} = (u_i)_{i=1}^n \in \mathcal{U}_{\text{ad}}$ is characterized by

$$\left(\forall \overline{v} = (v_i)_{i=1}^n \in \mathcal{U}_{\text{ad}}\right)\left(\sum_{i=1}^n J_i'(\overline{u})(v_i - u_i) \ge 0\right),$$

which is equivalent to

$$\sum_{i=1}^{n}\big(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(Q)} + \sum_{i=1}^{n}\big(\mathcal{N}_i u_i, v_i - u_i\big)_{L^2(Q)} \ge 0\,. \tag{16}$$

Now, forming the scalar product of the first equation in (15) with $y_i(\overline{v}) - y_i(\overline{u})$ and integrating between 0, T, we have

$$\begin{aligned}
&\int_0^T \big(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(\mathbb{R}^N)}\,\mathrm{d}t \\
&= \int_0^T \Big(\frac{\partial^2}{\partial t^2} p_i(\overline{u}) + M^* p_i(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u})\Big)_{L^2(\mathbb{R}^N)}\,\mathrm{d}t \\
&= \int_0^T \Big(\frac{\partial^2}{\partial t^2} p_i(\overline{u}) + A p_i(\overline{u}) + \sum_{j=1}^{n} a_{ji} p_j(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u})\Big)_{L^2(\mathbb{R}^N)}\,\mathrm{d}t \\
&= \int_0^T \Big(\frac{\partial^2}{\partial t^2} p_i(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u})\Big)_{L^2(\mathbb{R}^N)}\,\mathrm{d}t + \int_0^T \big(A p_i(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(\mathbb{R}^N)}\,\mathrm{d}t \\
&\qquad + \int_0^T \Big(\sum_{j=1}^{n} a_{ji} p_j(\overline{u}),\, y_i(\overline{v}) - y_i(\overline{u})\Big)_{L^2(\mathbb{R}^N)}\,\mathrm{d}t\,.
\end{aligned}$$

Applying Green's formula to the right hand side, we get

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N} \Big(p_i(\overline{u}) \frac{\partial^2}{\partial t^2}\big(y_i(\overline{v}) - y_i(\overline{u})\big)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^N} \Big(p_i(\overline{u}) A\big(y_i(\overline{v}) - y_i(\overline{u})\big)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T\int_{\mathbb{R}^N} \Big(p_i(\overline{u})\Big(\sum_{j=1}^{n} a_{ij}\big(y_j(\overline{v}) - y_j(\overline{u})\big)\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_{\Sigma} \Big(\frac{\partial^\omega}{\partial \nu_A^\omega} p_i(\overline{u})\big(y_i(\overline{v}) - y_i(\overline{u})\big)\Big)\,\mathrm{d}\Sigma + \int_{\Sigma} \Big(p_i(\overline{u}) \frac{\partial^\omega}{\partial \nu_A^\omega}\big(y_i(\overline{v}) - y_i(\overline{u})\big)\Big)\,\mathrm{d}\Sigma\,.
\end{aligned}$$

Using the conditions in (13) and (15), we have

$$\int_0^T \big(y_i(\overline{u}) - z_{i,d},\, y_i(\overline{v}) - y_i(\overline{u})\big)_{L^2(\mathbb{R}^N)}\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^N} p_i(\overline{u})(v_i - u_i)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Therefore (16) becomes:

$$\sum_{i=1}^{n} \int_Q \big(p_i(\overline{u}) + \mathcal{N}_i u_i\big)(v_i - u_i)\, \mathrm{d}x\, \mathrm{d}t \geq 0 \qquad \text{for all} \quad \overline{v} = (v_i)_{i=1}^{n} \in \mathcal{U}_{\mathrm{ad}}\,,$$

which completes the proof. □

Remark 2. Let $n = 2$, then the optimality system is given by

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2} y_1(\overline{u}) - \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha}\, y_1(\overline{u}) + y_1(\overline{u}) - y_2(\overline{u}) = f_1 + u_1 && \text{in} \quad Q\,,\\
&\frac{\partial^2}{\partial t^2} y_2(\overline{u}) - \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha}\, y_2(\overline{u}) + y_1(\overline{u}) + y_2(\overline{u}) = f_2 + u_2 && \text{in} \quad Q\,,\\
&\frac{\partial^\omega}{\partial \nu_A^\omega} y_1(\overline{u}) = 0\,, \qquad \frac{\partial^\omega}{\partial \nu_A^\omega} y_2(\overline{u}) = 0 && \text{on} \quad \Sigma\,,\\
&y_1(x,0;\overline{u}) = y_{1,0}(x)\,, \qquad y_2(x,0;\overline{u}) = y_{2,0}(x) && \text{in} \quad \mathbb{R}^N\,,\\
&\frac{\partial y_1(x,0;\overline{u})}{\partial t} = y_{1,1}(x)\,, \qquad \frac{\partial y_2(x,0;\overline{u})}{\partial t} = y_{2,1}(x) && \text{in} \quad \mathbb{R}^N\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2} p_1(\overline{u}) - \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha}\, p_1(\overline{u}) + p_1(\overline{u}) + p_2(\overline{u}) = y_1(u) - z_{1,d} && \text{in} \quad Q\,,\\
&\frac{\partial^2}{\partial t^2} p_2(\overline{u}) - \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha \, \mathrm{D}^{2\alpha}\, p_2(\overline{u}) - p_1(\overline{u}) + p_2(\overline{u}) = y_2(\overline{u}) - z_{2,d} && \text{in} \quad Q\,,\\
&\frac{\partial^\omega p_1(\overline{u})}{\partial \nu_A^\omega} = 0\,, \qquad \frac{\partial^\omega p_2(\overline{u})}{\partial \nu_A^\omega} = 0 && \text{on} \quad \Sigma\,,\\
&p_1(x,T;\overline{u}) = 0\,, \qquad p_2(x,T;\overline{u}) = 0 && \text{in} \quad \mathbb{R}^N\,,\\
&\frac{\partial p_1(x,T;\overline{u})}{\partial t} = 0\,, \qquad \frac{\partial p_2(x,T;\overline{u})}{\partial t} = 0 && \text{in} \quad \mathbb{R}^N\,,
\end{aligned}$$

$$\int_Q \big(p_1(\overline{u}) + \mathcal{N}_1 u_1\big)(v_1 - u_1) + \big(p_2(u) + \mathcal{N}_2 u_2\big)(v_2 - u_2)\, \mathrm{d}x\, \mathrm{d}t \geq 0$$

$$\text{for all} \quad \overline{v} = (v_1, v_2) \in \mathcal{U}_{\mathrm{ad}}\,,$$

where $\overline{u} = (u_1, u_2) \in \mathcal{U}_{\mathrm{ad}}$, $\overline{p}(\overline{u}) = \big(p_1(\overline{u}), p_2(\overline{u})\big)$ is the adjoint state.

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