Anita Dąbrowicz-Tlałka; Tadeusz Jankowski
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QUADRATIC CONVERGENCE OF MONOTONE ITERATIONS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS

Anita Dąbrowicz-Tlałka — Tadeusz Jankowski

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ABSTRACT. The general quasilinearization method is applied to differential-algebraic equations with initial conditions showing that corresponding linear monotone iterations converge to a unique solution.

1.

Consider the following differential-algebraic equations of the form

$$\begin{aligned} x'(t) &= f\big(t,x(t),y(t)\big), \qquad t \in J = [0,b], \quad x(0) = x_0, \\ 0 &= g\big(t,x(t),y(t)\big), \qquad t \in J, \end{aligned} \tag{1}$$

where $f, g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $x_0 \in \mathbb{R}$ are given. By a solution of (1) we mean a pair $(x,y) \in C^1(J,\mathbb{R}) \times C(J,\mathbb{R})$ for which equations (1) are satisfied.

To find a solution of (1) we can construct some iterations showing that under some assumptions they are convergent to this solution (see, for example [1], [4], [7]). In [6], approximate solution for (1) is constructed by corresponding numerical procedures while in [5] the existence of extremal solutions is proved for a special case of (1). Method based on lower and upper solutions combined with monotone iterations is very useful too (for details, see [8], [9]). In [3] such approach is given for problem (1) and it is proved that the corresponding monotone sequences converge quadratically to the unique solution of problem (1) if among other things we assume that f and g are Ω-convex functions. The purpose of this paper is to continue this topic, that is, we show that when we split f and g into the sum of two Ω-convex or Ω-concave functions we can obtain a result for (1) with the same conclusion as in [3]. An example is given at the end of this paper.

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2.

Let us introduce the following definition:

DEFINITION 1. A pair $(y_0, u_0) \in C^1(J,\mathbb{R}) \times C(J,\mathbb{R})$ is said to be a *lower solution* of (1) if

$$\begin{aligned} y_0'(t) &\le f\big(t, y_0(t), u_0(t)\big), \qquad t \in J, \quad y_0(0) \le x_0, \\ 0 &\le g\big(t, y_0(t), u_0(t)\big), \qquad t \in J, \end{aligned} \tag{2}$$

and a pair (z_0, w_0) is an *upper solution* of (1) if the above inequalities are reversed.

Put

$$\Omega = \big\{(t,u,v):\ y_0(t) \le u \le z_0(t),\ u_0(t) \le v \le w_0(t),\ t \in J\big\}$$

and assume that Ω is not empty.

The notation $f \in C^{0,2,2}(\Omega,\mathbb{R})$ means that $f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy} \in C(\Omega,\mathbb{R})$.

We say that a function $\alpha \in C^{0,2,2}(\Omega,\mathbb{R})$ is Ω*-convex* if $\alpha_{xx}(t,x,y) \ge 0$, $\alpha_{xy}(t,x,y) \ge 0$, $\alpha_{yy}(t,x,y) \ge 0$ for $(t,x,y) \in \Omega$, and it is Ω-concave if the above inequalities are reversed.

In this paper we shall discuss problem (1) when f and g can be split as the sum of two functions, namely $f = F + P$, $g = G + Q$ assuming that F and G are Ω-convex while P and Q are Ω-concave.

For given functions F, G, P and Q let us define some functions by the following relations for convenience,

$$\begin{aligned} V_1(t,y,z,u,w) &= F_x(t,y,u) + P_x(t,z,w), \\ V_2(t,y,z,u,w) &= F_y(t,y,u) + P_y(t,z,w), \\ W_1(t,y,z,u,w) &= G_x(t,y,u) + Q_x(t,z,w), \\ W_2(t,y,z,u,w) &= G_y(t,y,u) + Q_y(t,z,w). \end{aligned}$$

Now, we can formulate the following lemma:

LEMMA 1. *Let $f = F + P$ and $F, P \in C^{0,2,2}(\Omega,\mathbb{R})$. Assume that F is Ω-convex and P is Ω-concave. Then, for $y_0(t) \le \bar{y} \le \bar{z} \le z_0(t)$ and $u_0(t) \le \bar{u} \le \bar{w} \le w_0(t)$, we have*

$$f(t,\bar{y},\bar{u}) - f(t,\bar{z},\bar{w}) \le -V_1(t,\bar{y},\bar{z},\bar{u},\bar{u})[\bar{z}-\bar{y}] - V_2(t,\bar{z},\bar{z},\bar{u},\bar{w})[\bar{w}-\bar{u}], \qquad t \in J.$$

P r o o f . Using a mean value theorem and assumptions, we obtain

$$\begin{aligned}
&f(t,\bar{y},\bar{u}) - f(t,\bar{z},\bar{w}) \\
&= f(t,\bar{y},\bar{u}) - f(t,\bar{z},\bar{u}) + f(t,\bar{z},\bar{u}) - f(t,\bar{z},\bar{w}) \\
&= -\, f_x(t,\xi,\bar{u})(\bar{z}-\bar{y}) - f_y(t,\bar{z},\delta)(\bar{w}-\bar{u}) \\
&= -\, \big[F_x(t,\xi,\bar{u}) + P_x(t,\xi,\bar{u})\big](\bar{z}-\bar{y}) - \big[F_y(t,\bar{z},\delta) + P_y(t,\bar{z},\delta)\big](\bar{w}-\bar{u}) \\
&\le -\, \big[F_x(t,\bar{y},\bar{u}) + P_x(t,\bar{z},\bar{u})\big](\bar{z}-\bar{y}) - \big[F_y(t,\bar{z},\bar{u}) + P_y(t,\bar{z},\bar{w})\big](\bar{w}-\bar{u}),
\end{aligned}$$

where $\bar{y} < \xi < \bar{z}$ and $\bar{u} < \delta < \bar{w}$. Hence we have the assertion. It ends the proof. □

Remark 1. Similarly if $g = G + Q$, $G, Q \in C^{0,2,2}(\Omega, \mathbb{R})$ and G is Ω-convex while Q is Ω-concave, then

$$g(t,\bar{y},\bar{u}) - g(t,\bar{z},\bar{w}) \le -W_1(t,\bar{y},\bar{z},\bar{u},\bar{u})[\bar{z}-\bar{y}] - W_2(t,\bar{z},\bar{z},\bar{u},\bar{w})[\bar{w}-\bar{u}]$$

for $y_0(t) \le \bar{y} \le \bar{z} \le z_0(t)$, $u_0(t) \le \bar{u} \le \bar{w} \le w_0(t)$, $t \in J$.

The next lemma is useful to get some relations between elements of monotone sequences.

LEMMA 2. *Assume that* $K_1 \in C(J,\mathbb{R})$, $K_2, L \in C(J,\mathbb{R}_+)$ *with* $\mathbb{R}_+ = [0,+\infty)$. *If* $(p,q) \in C^1(J,\mathbb{R}) \times C(J,\mathbb{R})$ *satisfy the inequalities*

$$\begin{aligned}
p'(t) &\le K_1(t)p(t) + K_2(t)q(t), \qquad t \in J, \quad p(0) \le 0, \\
q(t) &\le L(t)p(t), \qquad t \in J,
\end{aligned}$$

then $p(t) \le 0$ *and* $q(t) \le 0$ *on* J.

P r o o f . Put $K(t) = K_1(t) + K_2(t)L(t)$, $t \in J$. Under our assumptions we see that $p'(t) \le K(t)p(t)$, $t \in J$. This yields the inequality

$$p(t) \le p(0)\, \mathrm{e}^{\int_0^t K(\tau)\,\mathrm{d}\tau}, \qquad t \in J.$$

Since $p(0) \le 0$, it proves that $p(t) \le 0$ and $q(t) \le 0$ on J. It ends the proof. □

The next theorem gives some sufficient conditions for the uniqueness of the solution of (1) but it does not guarantee the existence of the solution.

THEOREM 1. *Assume that* $f, g, f_x, f_y, g_x, g_y \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and that there exist nonnegative constants* $\tilde{K}$, $\tilde{L}$, $\tilde{k}$, $\tilde{l}$, *where* $\tilde{l} > 0$ *such that the conditions*

(i) $|f(t,x,y) - f(t,\bar{x},\bar{y})| \le \tilde{K}|x-\bar{x}| + \tilde{L}|y-\bar{y}|$,
(ii) $|g(t,x,\bar{y}) - g(t,\bar{x},\bar{y})| \le \tilde{k}|x-\bar{x}|$,
(iii) $|g(t,\bar{x},y) - g(t,\bar{x},\bar{y})| \ge \tilde{l}|y-\bar{y}|$

are satisfied for $t \in J$, $x, y, \bar{x}, \bar{y} \in \mathbb{R}$. *Then problem* (1) *has at most one solution.*

Proof. Assume that (1) has two distinct solutions (x,y) and $(\bar{x},\bar{y})$. Put $p=|x-\bar{x}|$, $q=|y-\bar{y}|$. Indeed $p(0)=0$. Then, by assumptions, we obtain

$$\begin{aligned}\tilde{l}\,|y(t)-\bar{y}(t)| &\le \big|g\big(t,x(t),\bar{y}(t)\big)-g\big(t,x(t),y(t)\big)\big|\\ &= \big|g\big(t,x(t),\bar{y}(t)\big)-g\big(t,\bar{x}(t),\bar{y}(t)\big)\big|\\ &\le \tilde{k}|x(t)-\bar{x}(t)|\,, \qquad t\in J\,,\end{aligned}$$

and hence

$$\begin{aligned}p(t) &\le \int_0^t \big|f\big(s,x(s),y(s)\big)-f\big(s,\bar{x}(s),\bar{y}(s)\big)\big|\,\mathrm{d}s\\ &\le \int_0^t \big[\tilde{K}p(s)+\tilde{L}q(s)\big]\,\mathrm{d}s\\ &\le \left(\tilde{K}+\frac{\tilde{L}\tilde{k}}{\tilde{l}}\right)\int_0^t p(s)\,\mathrm{d}s\,, \qquad t\in J\,.\end{aligned}$$

By Gronwall's inequality, $p(t)=0$, $t\in J$, and then $q(t)=0$, $t\in J$, showing that $x=\bar{x}$, $y=\bar{y}$. It ends the proof. □

Remark 2. Note that problem (1) has at most one solution in Δ if assumptions (i)–(iii) of Theorem 1 are satisfied in the set Ω instead of $J\times\mathbb{R}\times\mathbb{R}$. Here

$$\begin{aligned}\Delta=\big\{(x,y)\in C^1(J,\mathbb{R})\times C(J,\mathbb{R}) : {}& y_0(t)\le x(t)\le z_0(t)\,,\\ & u_0(t)\le y(t)\le w_0(t)\,,\ t\in J\big\}\end{aligned}$$

assuming that it is not empty.

3.

Now we can formulate the main results of this paper.

THEOREM 2. *Assume that* $F,G,P,Q\in C^{0,2,2}(\Omega,\mathbb{R})$, $f=F+P$, $g=G+Q$, *and:*

(i) $(y_0,u_0)\in C^1(J,\mathbb{R})\times C(J,\mathbb{R})$ *and* $(z_0,w_0)\in C^1(J,\mathbb{R})\times C(J,\mathbb{R})$ *are lower and upper solutions of problem* (1), *respectively such that* $y_0(t)\le z_0(t)$, $u_0(t)\le w_0(t)$, $t\in J$,

(ii) F, G *are* Ω*-convex, and* P, Q *are* Ω*-concave,*

(iii) $W_1(t,y,z,u,w)\ge 0$ *and* $V_2(t,y,z,u,w)\ge 0$ *for* $(t,y,u),(t,z,w)\in\Omega$,

(iv) $g_y(t,u,v)<0$ *for* $(t,u,v)\in\Omega$.

Then there exist monotone sequences $\{y_n, u_n\}$, $\{z_n, w_n\}$ which converge uniformly and monotonically on J to the unique solution in Δ of problem (1), and this convergence is quadratic.

Proof. Using assumption (ii), we see that

$$\begin{aligned} V_i(t,y,\bar{z},u,\bar{w}) &\le V_i(t,\bar{y},z,\bar{u},w)\,, \qquad i=1,2,\\ W_i(t,y,\bar{z},u,\bar{w}) &\le W_i(t,\bar{y},z,\bar{u},w)\,, \qquad i=1,2 \end{aligned} \tag{3}$$

if $(t,y,u),(t,\bar{y},\bar{u}),(t,z,w),(t,\bar{z},\bar{w}) \in \Omega$, $y \le \bar{y}$, $u \le \bar{u}$, $z \le \bar{z}$, $w \le \bar{w}$.

For $n = 0,1,\dots$, $t \in J$ and $y_{n+1}(0) = x_0$, $z_{n+1}(0) = x_0$, let us consider the systems

$$\begin{aligned} y'_{n+1}(t) = {} & f\big(t,y_n(t),u_n(t)\big) + V_1(t,n)\big[y_{n+1}(t)-y_n(t)\big]\\ & + V_2(t,n)\big[u_{n+1}(t)-u_n(t)\big]\,,\\ 0 = {} & g\big(t,y_n(t),u_n(t)\big) + W_1(t,n)\big[y_{n+1}(t)-y_n(t)\big]\\ & + W_2(t,n)\big[u_{n+1}(t)-u_n(t)\big] \end{aligned}$$

and

$$\begin{aligned} z'_{n+1}(t) = {} & f\big(t,z_n(t),w_n(t)\big) + V_1(t,n)\big[z_{n+1}(t)-z_n(t)\big]\\ & + V_2(t,n)\big[w_{n+1}(t)-w_n(t)\big]\,,\\ 0 = {} & g\big(t,z_n(t),w_n(t)\big) + W_1(t,n)\big[z_{n+1}(t)-z_n(t)\big]\\ & + W_2(t,n)\big[w_{n+1}(t)-w_n(t)\big]\,, \end{aligned}$$

where, for $i = 1,2$,

$$\begin{aligned} V_i(t,n) &= V_i\big(t,y_n(t),z_n(t),u_n(t),w_n(t)\big)\,,\\ W_i(t,n) &= W_i\big(t,y_n(t),z_n(t),u_n(t),w_n(t)\big)\,. \end{aligned}$$

Note that the pair (y_{n+1}, u_{n+1}) is a solution of the linear system

$$\begin{aligned} y'_{n+1}(t) &= V_1(t,n)y_{n+1}(t) + V_2(t,n)u_{n+1}(t) + \Lambda(t,n)\,, \quad y_{n+1}(0) = x_0\,,\\ 0 &= W_1(t,n)y_{n+1}(t) + W_2(t,n)u_{n+1}(t) + \Gamma(t,n)\,, \end{aligned} \tag{4}$$

$t \in J$, with

$$\begin{aligned} \Lambda(t,n) &= f\big(t,y_n(t),u_n(t)\big) - V_1(t,n)y_n(t) - V_2(t,n)u_n(t)\,,\\ \Gamma(t,n) &= g\big(t,y_n(t),u_n(t)\big) - W_1(t,n)y_n(t) - W_2(t,n)u_n(t)\,. \end{aligned}$$

In view of assumption (iv) and (3), we see that for $n = 0$

$$\begin{aligned} W_2(t,0) &= W_2\big(t,y_0(t),z_0(t),u_0(t),w_0(t)\big)\\ &\le W_2\big(t,z_0(t),z_0(t),w_0(t),w_0(t)\big)\\ &= g_y\big(t,z_0(t),w_0(t)\big) < 0\,, \qquad t \in J\,, \end{aligned}$$

so

$$u_1(t) = -\frac{W_1(t,0)}{W_2(t,0)}y_1(t) - \frac{\Gamma(t,0)}{W_2(t,0)},$$
$$y_1'(t) = V(t)y_1(t) + W(t), \qquad y_1(0) = x_0$$

with

$$V(t) = V_1(t,0) - V_2(t,0)\frac{W_1(t,0)}{W_2(t,0)},$$
$$W(t) = \Lambda(t,0) - V_2(t,0)\frac{\Gamma(t,0)}{W_2(t,0)}.$$

It proves that system (4) has a unique solution for $n = 0$. It means that y_1, u_1 are well defined. By the same way we can show that z_1, w_1 are well defined, too.

By assumption (i), we have $y_0(t) \le z_0(t)$ and $u_0(t) \le w_0(t)$ for $t \in J$. Now, we need to show that

$$\begin{aligned} y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \\ u_0(t) \le u_1(t) \le w_1(t) \le w_0(t), \end{aligned} \qquad t \in J. \tag{5}$$

To do it let $p = y_0 - y_1$, $q = u_0 - u_1$, so $p(0) \le 0$. Then, knowing that (y_0, u_0) is a lower solution of (1), we see that

$$\begin{aligned} p'(t) \le\ & f(t, y_0(t), u_0(t)) - f(t, y_0(t), u_0(t)) - V_1(t,0)[y_1(t) - y_0(t)] \\ & -V_2(t,0)[u_1(t) - u_0(t)] \\ =\ & V_1(t,0)p(t) + V_2(t,0)q(t) \end{aligned}$$

and

$$\begin{aligned} 0 \le\ & g(t, y_0(t), u_0(t)) - g(t, y_0(t), u_0(t)) - W_1(t,0)[y_1(t) - y_0(t)] \\ & -W_2(t,0)[u_1(t) - u_0(t)] \\ =\ & W_1(t,0)p(t) + W_2(t,0)q(t). \end{aligned}$$

Since $W_2(t,0) < 0$, $t \in J$, by (iii) and Lemma 2, we obtain $p(t) \le 0$, $q(t) \le 0$ on J, and as a result we get $y_0(t) \le y_1(t)$ and $u_0(t) \le u_1(t)$, $t \in J$. Similarly, we can show that $z_1(t) \le z_0(t)$ and $w_1(t) \le w_0(t)$ on J, too.

Now, we put $p = y_1 - z_1$, $q = u_1 - w_1$, so $p(0) = 0$. Then, by Lemma 1, Remark 1 and (3), we have

$$\begin{aligned} p'(t) &= f(t, y_0(t), u_0(t)) - f(t, z_0(t), w_0(t)) \\ &\quad + V_1(t,0)[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &\quad + V_2(t,0)[u_1(t) - u_0(t) - w_1(t) + w_0(t)] \\ &\le - V_1(t, y_0(t), z_0(t), u_0(t), u_0(t))[z_0(t) - y_0(t)] \\ &\quad - V_2(t, z_0(t), z_0(t), u_0(t), w_0(t))[w_0(t) - u_0(t)] \\ &\quad + V_1(t,0)[p(t) + z_0(t) - y_0(t)] + V_2(t,0)[q(t) + w_0(t) - u_0(t)] \\ &\le V_1(t,0)p(t) + V_2(t,0)q(t), \end{aligned}$$

and

$$\begin{aligned} 0 &= g(t, y_0(t), u_0(t)) - g(t, z_0(t), w_0(t)) \\ &\quad + W_1(t,0)[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &\quad + W_2(t,0)[u_1(t) - u_0(t) - w_1(t) + w_0(t)] \\ &\le - W_1(t, y_0(t), z_0(t), u_0(t), u_0(t))[z_0(t) - y_0(t)] \\ &\quad - W_2(t, z_0(t), z_0(t), u_0(t), w_0(t))[w_0(t) - u_0(t)] \\ &\quad + W_1(t,0)[p(t) + z_0(t) - y_0(t)] + W_2(t,0)[q(t) + w_0(t) - u_0(t)] \\ &\le W_1(t,0)p(t) + W_2(t,0)q(t). \end{aligned}$$

By Lemma 2, we have $y_1(t) \le z_1(t)$ and $u_1(t) \le w_1(t)$, $t \in J$. This shows that (5) is satisfied.

In the next step we have to show that (y_1, u_1) and (z_1, w_1) are lower and upper solutions of problem (1), respectively. Using (ii), (3) and Lemma 1, we obtain

$$\begin{aligned} y_1'(t) &= f(t, y_0(t), u_0(t)) - f(t, y_1(t), u_1(t)) + f(t, y_1(t), u_1(t)) \\ &\quad + V_1(t,0)[y_1(t) - y_0(t)] + V_2(t,0)[u_1(t) - u_0(t)] \\ &\le - V_1(t, y_0(t), y_1(t), u_0(t), u_0(t))[y_1(t) - y_0(t)] \\ &\quad - V_2(t, y_1(t), y_1(t), u_0(t), u_1(t))[u_1(t) - u_0(t)] \\ &\quad + V_1(t,0)[y_1(t) - y_0(t)] + V_2(t,0)[u_1(t) - u_0(t)] \\ &\quad + f(t, y_1(t), u_1(t)) \le f(t, y_1(t), u_1(t)), \end{aligned}$$

$$
\begin{aligned}
0 &= g\big(t, y_0(t), u_0(t)\big) - g\big(t, y_1(t), u_1(t)\big) + g\big(t, y_1(t), u_1(t)\big) \\
&\quad + W_1(t,0)\big[y_1(t) - y_0(t)\big] + W_2(t,0)\big[u_1(t) - u_0(t)\big] \\
&\le - W_1\big(t, y_0(t), y_1(t), u_0(t), u_0(t)\big)\big[y_1(t) - y_0(t)\big] \\
&\quad - W_2\big(t, y_1(t), y_1(t), u_0(t), u_1(t)\big)\big[u_1(t) - u_0(t)\big] \\
&\quad + W_1(t,0)\big[y_1(t) - y_0(t)\big] + W_2(t,0)\big[u_1(t) - u_0(t)\big] + g\big(t, y_1(t), u_1(t)\big) \\
&\le g\big(t, y_1(t), u_1(t)\big)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
z_1'(t) &= f\big(t, z_0(t), w_0(t)\big) - f\big(t, z_1(t), w_1(t)\big) + f\big(t, z_1(t), w_1(t)\big) \\
&\quad + V_1(t,0)\big[z_1(t) - z_0(t)\big] + V_2(t,0)\big[w_1(t) - w_0(t)\big] \\
&\ge V_1\big(t, z_1(t), z_0(t), w_1(t), w_1(t)\big)\big[z_0(t) - z_1(t)\big] \\
&\quad + V_2\big(t, z_0(t), z_0(t), w_1(t), w_0(t)\big)\big[w_0(t) - w_1(t)\big] \\
&\quad + V_1(t,0)\big[z_1(t) - z_0(t)\big] + V_2(t,0)\big[w_1(t) - w_0(t)\big] + f\big(t, z_1(t), w_1(t)\big) \\
&\ge f\big(t, z_1(t), w_1(t)\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
0 &= g\big(t, z_0(t), w_0(t)\big) - g\big(t, z_1(t), w_1(t)\big) + g\big(t, z_1(t), w_1(t)\big) \\
&\quad + W_1(t,0)\big[z_1(t) - z_0(t)\big] + W_2(t,0)\big[w_1(t) - w_0(t)\big] \\
&\ge W_1\big(t, z_1(t), z_0(t), w_1(t), w_1(t)\big)\big[z_0(t) - z_1(t)\big] \\
&\quad + W_2\big(t, z_0(t), z_0(t), w_1(t), w_0(t)\big)\big[w_0(t) - w_1(t)\big] \\
&\quad + W_1(t,0)\big[z_1(t) - z_0(t)\big] + W_2(t,0)\big[w_1(t) - w_0(t)\big] + g\big(t, z_1(t), w_1(t)\big) \\
&\ge g\big(t, z_1(t), w_1(t)\big)
\end{aligned}
$$

for $t \in J$. This shows that (y_1, u_1), (z_1, w_1) are lower and upper solutions of (1), respectively.

Assume that for some $k > 1$,

$$
\begin{aligned}
y_0(t) \le y_1(t) \le \cdots \le y_k(t) \le z_k(t) \le \cdots \le z_1(t) \le z_0(t)\,, \\
u_0(t) \le u_1(t) \le \cdots \le u_k(t) \le w_k(t) \le \cdots \le w_1(t) \le w_0(t)\,,
\end{aligned}
$$

$t \in J$, and let (y_k, u_k), (z_k, w_k) be lower and upper solutions of problem (1), respectively.

Note that, by (ii), (iv) and (3), we have

$$
W_2(t,k) \le W_2\big(t, z_k(t), z_k(t), w_k(t), w_k(t)\big) - g_y\big(t, z_k(t), w_k(t)\big) < 0\,, \qquad t \in J
$$

Moreover, $W_1(t,k) \ge 0$ and $V_2(t,k) \ge 0$ by assumption (iii). It proves that system (4) for $n - k$ has a unique solution, so the elements y_{k+1}, u_{k+1} are well defined. By similar argument, we see that z_{k+1}, w_{k+1} are well defined, too.

We shall prove that

$$y_k(t) \le y_{k+1}(t) \le z_{k+1}(t) \le z_k(t), \qquad u_k(t) \le u_{k+1}(t) \le w_{k+1}(t) \le w_k(t) \quad (6)$$

for $t \in J$. Let $p = y_k - y_{k+1}$, $q = u_k - u_{k+1}$, so $p(0) = 0$. Thus we have

$$\begin{aligned} p'(t) &\le f(t, y_k(t), u_k(t)) - f(t, y_k(t), u_k(t)) \\ &\quad - V_1(t,k)[y_{k+1}(t) - y_k(t)] - V_2(t,k)[u_{k+1}(t) - u_k(t)] \\ &= V_1(t,k)p(t) + V_2(t,k)q(t), \end{aligned}$$

$$\begin{aligned} 0 &\le g(t, y_k(t), u_k(t)) - g(t, y_k(t), u_k(t)) \\ &\quad - W_1(t,k)[y_{k+1}(t) - y_k(t)] - W_2(t,k)[u_{k+1}(t) - u_k(t)] \\ &= W_1(t,k)p(t) + W_2(t,k)q(t). \end{aligned}$$

This and Lemma 2 yield $y_k(t) \le y_{k+1}(t)$ and $u_k(t) \le u_{k+1}(t)$, $t \in J$. Similarly, we can prove that $z_{k+1}(t) \le z_k(t)$, $w_{k+1}(t) \le w_k(t)$, $t \in J$.

If we put $p = y_{k+1} - z_{k+1}$, $q = u_{k+1} - w_{k+1}$, then $p(0) = 0$, and

$$\begin{aligned} p'(t) &= f(t, y_k(t), u_k(t)) - f(t, z_k(t), w_k(t)) \\ &\quad + V_1(t,k)[y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)] \\ &\quad + V_2(t,k)[u_{k+1}(t) - u_k(t) - w_{k+1}(t) + w_k(t)] \\ &\le -V_1(t, y_k(t), z_k(t), u_k(t), u_k(t))[z_k(t) - y_k(t)] \\ &\quad - V_2(t, z_k(t), z_k(t), u_k(t), w_k(t))[w_k(t) - u_k(t)] \\ &\quad + V_1(t,k)[p(t) + z_k(t) - y_k(t)] + V_2(t,k)[q(t) + w_k(t) - u_k(t)] \\ &\le V_1(t,k)p(t) + V_2(t,k)q(t), \end{aligned}$$

$$\begin{aligned} 0 &= g(t, y_k(t), u_k(t)) - g(t, z_k(t), w_k(t)) \\ &\quad + W_1(t,k)[y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)] \\ &\quad + W_2(t,k)[u_{k+1}(t) - u_k(t) - w_{k+1}(t) + w_k(t)] \\ &\le -W_1(t, y_k(t), z_k(t), u_k(t), u_k(t))[z_k(t) - y_k(t)] \\ &\quad - W_2(t, z_k(t), z_k(t), u_k(t), w_k(t))[w_k(t) - u_k(t)] \\ &\quad + W_1(t,k)[p(t) + z_k(t) - y_k(t)] + W_2(t,k)[q(t) + w_k(t) - u_k(t)] \\ &\le W_1(t,k)p(t) + W_2(t,k)q(t) \end{aligned}$$

by assumption (ii), Lemma 1 and Remark 1. Now Lemma 2 yields $y_{k+1}(t) \le z_{k+1}(t)$ and $u_{k+1}(t) \le w_{k+1}(t)$, $t \in J$, showing that (6) holds. Hence, by the method of mathematical induction, we have

$$\begin{gathered} y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t), \\ u_0(t) \le u_1(t) \le \cdots \le u_n(t) \le w_n(t) \le \cdots \le w_1(t) \le w_0(t) \\ \text{for all } n \quad \text{and} \quad t \in J. \end{gathered}$$

This and Dini's theorem yields that the sequences $\{y_n, u_n\}$, $\{z_n, w_n\}$ converge uniformly and monotonically to the corresponding solutions of problem (1). Since $g_y(t,u,v) < 0$ on Ω, all assumptions of Theorem 1 hold with

$$\tilde{K} = \max_{(t,u,v)\in\Omega} |f_x(t,u,v)|, \qquad \tilde{L} = \max_{(t,u,v)\in\Omega} |f_y(t,u,v)|,$$
$$\tilde{k} = \max_{(t,u,v)\in\Omega} |g_x(t,u,v)|, \qquad \tilde{l} = \min_{(t,u,v)\in\Omega} |g_y(t,u,v)|.$$

Theorem 1 and Remark 2 say that problem (1) has at most one solution (x,y) in Δ. Hence the sequences $\{y_n, u_n\}$, $\{z_n, w_n\}$ converge to the unique solution (x,y) of (1).

It remains to show quadratic convergence. To this end we let

$$p_{n+1}(t) = x(t) - y_{n+1}(t) \ge 0, \quad q_{n+1}(t) = z_{n+1}(t) - x(t) \ge 0, \qquad t \in J,$$
$$\bar{p}_{n+1}(t) = y(t) - u_{n+1}(t) \ge 0, \quad \bar{q}_{n+1}(t) = w_{n+1}(t) - y(t) \ge 0, \qquad t \in J,$$

so $p_{n+1}(0) = q_{n+1}(0) = 0$ for $n \ge 0$. Using assumptions and the mean value theorem, we obtain

$$\begin{aligned}
p'_{n+1}(t) &= f(t,x(t),y(t)) - f(t,y_n(t),y(t)) + f(t,y_n(t),y(t)) - f(t,y_n(t),u_n(t)) \\
&\qquad + V_1(t,n)[p_{n+1}(t) - p_n(t)] + V_2(t,n)[q_{n+1}(t) - q_n(t)] \\
&= f_x(t,\xi_1(t),y(t))p_n(t) + f_y(t,y_n(t),\delta_1(t))\bar{p}_n(t) \\
&\qquad + V_1(t,n)[p_{n+1}(t) - p_n(t)] + V_2(t,n)[\bar{p}_{n+1}(t) - \bar{p}_n(t)] \\
&\le \Big[F_x(t,x(t),y(t)) - F_x(t,y_n(t),y(t)) + F_x(t,y_n(t),y(t)) \\
&\qquad - F_x(t,y_n(t),u_n(t)) + P_x(t,y_n(t),y(t)) - P_x(t,z_n(t),y(t)) \\
&\qquad + P_x(t,z_n(t),y(t)) - P_x(t,z_n(t),w_n(t))\Big]p_n(t) \\
&\qquad + \Big[F_y(t,y_n(t),y(t)) - F_y(t,y_n(t),u_n(t)) + P_y(t,y_n(t),u_n(t)) \\
&\qquad - P_y(t,z_n(t),u_n(t)) + P_y(t,z_n(t),u_n(t)) - P_y(t,z_n(t),w_n(t))\Big]\bar{p}_n(t) \\
&\qquad + V_1(t,n)p_{n+1}(t) + V_2(t,n)\bar{p}_{n+1}(t) \\
&= \Big\{F_{xx}(t,\xi_2(t),y(t))p_n(t) + F_{xy}(t,y_n(t),\delta_2(t))\bar{p}_n(t) \\
&\qquad + P_{xx}(t,\xi_3(t),y(t))\,[y_n(t) - z_n(t)] - P_{xy}(t,z_n(t),\delta_3(t))\bar{q}_n(t)\Big\}p_n(t) \\
&\qquad + \Big\{F_{yy}(t,y_n(t),\delta_4(t))\bar{p}_n(t) + P_{yx}(t,\xi_4(t),u_n(t))\,[y_n(t) - z_n(t)] \\
&\qquad + P_{yy}(t,z_n(t),\delta_5(t))\,[u_n(t) - w_n(t)]\Big\}\bar{p}_n(t) \\
&\qquad + V_1(t,n)p_{n+1}(t) + V_2(t,n)\bar{p}_{n+1}(t) \\
&\le M_1 p_{n+1}(t) + M_2\bar{p}_{n+1}(t) + B_1 p_n^2(t) + B_2\bar{p}_n^2(t) + B_3 q_n^2(t) + B_4\bar{q}_n^2(t)
\end{aligned}$$

where $y_n(t) < \xi_1(t), \xi_2(t) < x(t)$, $y_n(t) < \xi_3(t), \xi_4(t) < z_n(t)$, $u_n(t) < \delta_1(t), \delta_2(t), \delta_4(t) < y(t)$, $y(t) < \delta_3(t) < w_n(t)$, $u_n(t) < \delta_5(t) < w_n(t)$ on J, and

$$B_1 = A_1 + A_5 + \frac{1}{2}(A_2 + 3A_4), \qquad B_2 = A_3 + A_5 + \frac{1}{2}(A_2 + 3A_6),$$
$$B_3 = \frac{1}{2}(A_4 + A_5), \qquad B_4 = \frac{1}{2}(A_5 + A_6),$$
$$|f_x(t,x,y)| \le M_1, \quad |f_y(t,x,y)| \le M_2, \qquad (t,x,y) \in \Omega,$$

$$|F_{xx}(t,x,y)| \le A_1, \quad |F_{xy}(t,x,y)| \le A_2, \quad |F_{yy}(t,x,y)| \le A_3, \qquad (t,x,y) \in \Omega,$$
$$|P_{xx}(t,x,y)| \le A_4, \quad |P_{xy}(t,x,y)| \le A_5, \quad |P_{yy}(t,x,y)| \le A_6, \qquad (t,x,y) \in \Omega.$$

Moreover

$$\begin{aligned}
0 &= g\big(t, x(t), y(t)\big) - g\big(t, y_n(t), y(t)\big) + g\big(t, y_n(t), y(t)\big) - g\big(t, y_n(t), u_n(t)\big) \\
&\qquad + W_1(t,n)\big[p_{n+1}(t) - p_n(t)\big] + W_2(t,n)\big[\bar{p}_{n+1}(t) - \bar{p}_n(t)\big] \\
&= g_x\big(t, \xi_5(t), y(t)\big)p_n(t) + g_y\big(t, y_n(t), \delta_6(t)\big)\bar{p}_n(t) \\
&\qquad + W_1(t,n)\big[p_{n+1}(t) - p_n(t)\big] + W_2(t,n)\big[\bar{p}_{n+1}(t) - \bar{p}_n(t)\big] \\
&\le \Big[G_{xx}\big(t, \xi_6(t), y(t)\big)p_n(t) + G_{xy}\big(t, y_n(t), \delta_7(t)\big)\bar{p}_n(t) \\
&\qquad + Q_{xx}\big(t, \xi_7(t), y(t)\big)\big(y_n(t) - z_n(t)\big) - Q_{xy}\big(t, z_n(t), \delta_8(t)\big)\bar{q}_n(t)\Big]p_n(t) \\
&\qquad + \Big[G_{yy}\big(t, y_n(t), \delta_9(t)\big)\bar{p}_n(t) + Q_{yx}\big(t, \xi_8(t), u_n(t)\big)\big(y_n(t) - z_n(t)\big) \\
&\qquad + Q_{yy}\big(t, z_n(t), \delta_{10}(t)\big)\big(u_n(t) - w_n(t)\big)\Big]\bar{p}_n(t) \\
&\qquad + W_1(t,n)p_{n+1}(t) + W_2(t,n)\bar{p}_{n+1}(t) \\
&\le N_1 p_{n+1}(t) - N_2\bar{p}_{n+1}(t) + D_1 p_n^2(t) + D_2\bar{p}_n^2(t) + D_3 q_n^2(t) + D_4\bar{q}_n^2(t)
\end{aligned}$$

where $y_n(t) < \xi_5(t), \xi_6(t) < x(t)$, $y_n(t) < \xi_7(t), \xi_8(t) < z_n(t)$, $u_n(t) < \delta_6(t), \delta_7(t), \delta_9(t) < y(t)$, $y(t) < \delta_8(t) < w_n(t)$, $u_n(t) < \delta_{10}(t) < w_n(t)$ on J, and

$$D_1 = C_1 + C_5 + \frac{1}{2}(C_2 + 3C_4), \qquad D_2 = C_3 + C_5 + \frac{1}{2}(C_2 + 3C_6),$$
$$D_3 = \frac{1}{2}(C_4 + C_5), \qquad D_4 = \frac{1}{2}(C_5 + C_6),$$
$$|g_x(t,x,y)| \le N_1, \quad g_y(t,x,y) \le -N_2 < 0, \qquad (t,x,y) \in \Omega,$$

$$|G_{xx}(t,x,y)| \le C_1, \quad |G_{xy}(t,x,y)| \le C_2, \quad |G_{yy}(t,x,y)| \le C_3, \qquad (t,x,y) \in \Omega,$$
$$|Q_{xx}(t,x,y)| \le C_4, \quad |Q_{xy}(t,x,y)| \le C_5, \quad |Q_{yy}(t,x,y)| \le C_6, \qquad (t,x,y) \in \Omega.$$

Hence

$$p'_{n+1}(t) \le M_1 p_{n+1}(t) + M_2 \bar{p}_{n+1}(t) + L_1(t)\,,$$
$$\bar{p}_{n+1}(t) \le \frac{1}{N_2}\left[N_1 p_{n+1}(t) + L_2(t)\right],$$

where

$$L_1(t) = B_1 p_n^2(t) + B_2 \bar{p}_n^2(t) + B_3 q_n^2(t) + B_4 \bar{q}_n^2(t)\,,$$
$$L_2(t) = D_1 p_n^2(t) + D_2 \bar{p}_n^2(t) + D_3 q_n^2(t) + D_4 \bar{q}_n^2(t)\,.$$

Thus

$$p'_{n+1}(t) \le \bar{M} p_{n+1}(t) + \frac{M_2}{N_2} L_2(t) + L_1(t) \qquad \text{with} \quad \bar{M} = M_1 + \frac{M_2 N_1}{N_2}\,.$$

Now, the differential inequality implies

$$0 \le p_{n+1}(t) \le \int_0^t \left[\frac{M_2}{N_2} L_2(s) + L_1(s)\right] \mathrm{e}^{\bar{M}(t-s)}\, \mathrm{d}s \le S \max_{s\in J}\left[\frac{M_2}{N_2} L_2(s) + L_1(s)\right],$$

and hence

$$\max_{t\in J} |x(t) - y_{n+1}(t)| \le a_1 \max_{t\in J} |x(t) - y_n(t)|^2 + a_2 \max_{t\in J} |y(t) - u_n(t)|^2 + a_3 \max_{t\in J} |x(t) - z_n(t)|^2 + a_4 \max_{t\in J} |y(t) - w_n(t)|^2$$

and

$$\max_{t\in J} |y(t) - u_{n+1}(t)| \le b_1 \max_{t\in J} |x(t) - y_n(t)|^2 + b_2 \max_{t\in J} |y(t) - u_n(t)|^2 + b_3 \max_{t\in J} |x(t) - z_n(t)|^2 + b_4 \max_{t\in J} |y(t) - w_n(t)|^2$$

for

$$a_i = S\left[\frac{M_2}{N_2} D_i + B_i\right], \quad b_i = \frac{1}{N_2}\left[N_1 a_i + D_i\right], \qquad i = 1,2,3,4\,,$$
$$\text{with} \quad S = \frac{1}{\bar{M}} \mathrm{e}^{\bar{M} b}\,.$$

In a similar way, using assumptions and the mean value theorem, we can obtain

$$q'_{n+1}(t) = f\big(t, z_n(t), w_n(t)\big) - f\big(t, x(t), w_n(t)\big) + f\big(t, x(t), w_n(t)\big) - f\big(t, x(t), y(t)\big) + V_1(t,n)\left[q_{n+1}(t) - q_n(t)\right] + V_2(t,n)\left[\bar{q}_{n+1}(t) - \bar{q}_n(t)\right]$$

$$
\begin{aligned}
&= f_x\big(t, \xi_9(t), w_n(t)\big) q_n(t) + f_y\big(t, x(t), \delta_{11}(t)\big) \bar{q}_n(t) \\
&\quad - V_1(t,n) q_n(t) + V_1(t,n) q_{n+1}(t) - V_2(t,n) \bar{q}_n(t) + V_2(t,n) \bar{q}_{n+1}(t) \\
&\le \Big[F_{xx}\big(t, \xi_{10}(t), w_n(t)\big)\big(z_n(t) - y_n(t)\big) \\
&\quad + F_{xy}\big(t, y_n(t), \delta_{12}(t)\big)\big(w_n(t) - u_n(t)\big) - P_{xx}\big(t, \xi_{11}(t), w_n(t)\big) q_n(t)\Big] q_n(t) \\
&\quad + \Big[F_{yy}\big(t, y_n(t), \delta_{13}(t)\big)\big(w_n(t) - u_n(t)\big) + F_{yx}\big(t, \xi_{12}(t), w_n(t)\big) p_n(t) \\
&\quad - P_{yx}\big(t, \xi_{13}(t), y(t)\big) q_n(t) - P_{yy}\big(t, z_n(t), \delta_{14}(t)\big) \bar{q}_n(t)\Big] \bar{q}_n(t) \\
&\quad + V_1(t,n) q_{n+1}(t) + V_2(t,n) \bar{q}_{n+1}(t) \\
&\le M_1 q_{n+1}(t) + M_2 \bar{q}_{n+1}(t) + E_1 p_n^2(t) + E_2 \bar{p}_n^2(t) + E_3 q_n^2(t) + E_4 \bar{q}_n^2(t)\,,
\end{aligned}
$$

where

$$
E_1 = \frac{1}{2}(A_1 + A_2)\,, \qquad E_2 = \frac{1}{2}(A_2 + A_3)\,,
$$
$$
E_3 = A_2 + A_4 + \frac{1}{2}(3A_1 + A_5)\,, \qquad E_4 = A_2 + A_6 + \frac{1}{2}(3A_3 + A_5)\,,
$$

and

$$
\left.
\begin{aligned}
y_n(t) < \xi_{12}(t) < x(t)\,, \quad y_n(t) < \xi_{10}(t) < z_n(t)\,, \\
x(t) < \xi_9(t), \xi_{11}(t), \xi_{13}(t) < z_n(t)\,, \\
y(t) < \delta_{11}(t), \delta_{14}(t) < w_n(t)\,, \\
u_n(t) < \delta_{12}(t), \delta_{13}(t) < w_n(t)
\end{aligned}
\right\}
\quad \text{for} \quad t \in J\,.
$$

Moreover

$$
\begin{aligned}
0 &= g\big(t, z_n(t), w_n(t)\big) - g\big(t, x(t), w_n(t)\big) + g\big(t, x(t), w_n(t)\big) - g\big(t, x(t), y(t)\big) \\
&\quad + W_1(t,n)\big[z_{n+1}(t) - z_n(t)\big] + W_2(t,n)\big[w_{n+1}(t) - w_n(t)\big] \\
&= g_x\big(t, \xi_{14}(t), w_n(t)\big) q_n(t) + g_y\big(t, x(t), \delta_{15}(t)\big) \bar{q}_n(t) \\
&\quad + W_1(t,n) q_{n+1} - W_1(t,n) q_n(t) + W_2(t,n) \bar{q}_{n+1}(t) - W_2(t,n) \bar{q}_n(t) \\
&\le \Big[G_{xx}\big(t, \xi_{15}(t), w_n(t)\big)\big(z_n(t) - y_n(t)\big) + G_{xy}\big(t, y_n(t), \delta_{16}(t)\big)\big(w_n(t) - u_n(t)\big) \\
&\quad - Q_{xx}\big(t, \xi_{16}(t), w_n(t)\big) q_n(t)\Big] q_n(t) \\
&\quad + \Big[G_{yx}\big(t, \xi_{17}(t), w_n(t)\big) p_n(t) + G_{yy}\big(t, y_n(t), \delta_{17}(t)\big)\big(w_n(t) - u_n(t)\big) \\
&\quad - Q_{yx}\big(t, \xi_{18}(t), y(t)\big) q_n(t) - Q_{yy}\big(t, z_n(t), \delta_{18}(t)\big) \bar{q}_n(t)\Big] \bar{q}_n(t) \\
&\quad + W_1(t,n) q_{n+1}(t) + W_2(t,n) \bar{q}_{n+1}(t) \\
&\le N_1 q_{n+1}(t) - N_2 \bar{q}_{n+1}(t) + F_1 p_n^2(t) + F_2 \bar{p}_n^2(t) + F_3 q_n^2(t) + F_4 \bar{q}_n^2(t)\,,
\end{aligned}
$$

where

$$F_1 = \frac{1}{2}(C_1 + C_2), \qquad F_2 = \frac{1}{2}(C_2 + C_3),$$
$$F_3 = C_2 + C_4 + \frac{1}{2}(3C_1 + C_5), \qquad F_4 = C_2 + C_6 + \frac{1}{2}(3C_3 + C_5),$$

and

$$\left.\begin{aligned} y_n(t) < \xi_{17}(t) < x(t), \quad y_n(t) < \xi_{15}(t) < z_n(t), \\ x(t) < \xi_{14}(t), \xi_{16}(t), \xi_{18}(t) < z_n(t), \\ y(t) < \delta_{15}(t), \delta_{18}(t) < w_n(t), \\ u_n(t) < \delta_{16}(t), \delta_{17}(t) < w_n(t) \end{aligned}\right. \qquad \text{for} \quad t \in J.$$

This implies

$$\max_{t\in J} |x(t) - z_{n+1}(t)| \le c_1 \max_{t\in J} |x(t) - y_n(t)|^2 + c_2 \max_{t\in J} |y(t) - u_n(t)|^2 + c_3 \max_{t\in J} |x(t) - z_n(t)|^2 + c_4 \max_{t\in J} |y(t) - w_n(t)|^2$$

and

$$\max_{t\in J} |y(t) - w_{n+1}(t)| \le d_1 \max_{t\in J} |x(t) - y_n(t)|^2 + d_2 \max_{t\in J} |y(t) - u_n(t)|^2 + d_3 \max_{t\in J} |x(t) - z_n(t)|^2 + d_4 \max_{t\in J} |y(t) - w_n(t)|^2$$

with

$$c_i = S\left[\frac{M_2}{N_2} F_i + E_i\right], \quad d_i = \frac{1}{N_2}[N_1 c_i + F_i], \qquad i = 1, 2, 3, 4,$$

which yields the quadratic convergence and the proof is therefore complete. □

Remark 3. Theorem 2 holds if $f = F + P$, $g = G + Q$ for some Ω-convex or Ω-concave functions (assumption (ii)). Note that this representation holds always.

For example, if

$$\min_{(t,x,y)\in\Omega} f_{xx}(t,x,y) = A, \qquad \min_{(t,x,y)\in\Omega} f_{xy}(t,x,y) = B, \qquad \min_{(t,x,y)\in\Omega} f_{yy}(t,x,y) = -2C$$

for some constants $A, B, C \ge 0$, then it is enough to choose $F = f - P$ with $P(t,x,y) = -Cy^2$.

EXAMPLE. Let us consider the nonlinear differential-algebraic problem of the form

$$\begin{aligned} &x'(t) = x^2(t) + \frac{1}{3}\left[1 + \sin y(t) + t^2\right] \equiv f(t,x,y), \qquad t \in J = [0,1], \\ &x(0) = 0, \\ &\quad 0 = \frac{1}{5}x(t) - y(t) - y^3(t) \equiv g(t,x,y), \qquad t \in J. \end{aligned} \tag{7}$$

Note that

$$\frac{1}{3}t^2 \le f(t,x,y) \le 1 + x^2(t), \qquad t \in J.$$

This shows that (y_0, u_0), (z_0, w_0) are lower and upper solutions of (7) with

$$y_0(t) = \frac{1}{9}t^3, \qquad z_0(t) = \tan t,$$
$$u_0(t) = 0, \qquad w_0(t) = t.$$

If we take

$$F(t,x,y) = f(t,x,y) + y^2, \qquad P(t,x,y) = -y^2,$$
$$G(t,x,y) = \frac{1}{5}x, \qquad Q(t,x,y) = -y - y^3,$$

then F, G are Ω-convex, P, Q are Ω-concave, $f = F + P$, $g = G + Q$ with

$$\Omega = \left\{(t,u,v) : \ \tfrac{1}{9}t^3 \le u \le \tan t, \ 0 \le v \le t, \ t \in [0,1]\right\}.$$

Furthermore, for $(t,x,y), (t,\bar{x},\bar{y}) \in \Omega$ we have

$$W_1(t,x,\bar{x},y,\bar{y}) = \frac{1}{5} > 0, \qquad V_2(t,x,\bar{x},y,\bar{y}) = \frac{1}{3}\cos y > 0,$$
$$g_y(t,x,y) = -1 - 3y^2 < 0.$$

Note that all assumptions of Theorem 1 hold for $(t,x,y), (t,\bar{x},\bar{y}) \in \Omega$ with

$$\tilde{K} = 2\tan 1, \quad \tilde{L} = \frac{1}{3}, \quad \tilde{k} = \frac{1}{5}, \quad \tilde{l} = 1.$$

It proves that the assertion of Theorem 2 holds for problem (7).

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Department of Differential Equations
Faculty of Applied Physics and Mathematics
Technical University of Gdańsk
ul. G. Narutowicza 11/12
PL-80-952 Gdańsk
POLAND

E-mail: tjank@mifgate.mif.pg.gda.pl