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AFFINE COMPLETENESS OF DE MORGAN ALGEBRAS

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ABSTRACT. An algebra is called affine complete if all its congruence compatible functions are polynomial functions. Boolean algebras are affine complete by a well-known result of G. Grätzer. Various generalizations of this result have been obtained. Among them, a characterization of affine complete and locally affine complete Kleene algebras and a description of local polynomial functions of Kleene algebras was given by M. Haviar, K. Kaarli and M. Ploščica. In this paper we present a generalization of these results to de Morgan algebras.

1. Introduction

Let $\mathbf{A}$ be a universal algebra. A function $f\colon A^n \to A$ is called *compatible* if, for any congruence ρ of $\mathbf{A}$, $(a_i, b_i) \in \rho$, $i = 1, \dots, n$, implies

$$\big(f(a_1, \dots, a_n), f(b_1, \dots, b_n)\big) \in \rho .$$

LEMMA 1.1. *Let $\mathbf{A}$ be an algebra and e be a unary idempotent compatible function on $\mathbf{A}$ such that $C = e(A)$ is a subuniverse of some reduct of $\mathbf{A}$. Then, if f is an n-ary compatible function of that subreduct, then the function*

$$g(x_1, \dots, x_n) = f\big(e(x_1), \dots, e(x_n)\big)$$

is a compatible function on $\mathbf{A}$ and extends f.

Proof. The proof can be found in [8]. □

An algebra $\mathbf{A}$ is called *affine complete* if every compatible function on $\mathbf{A}$ is a polynomial. Further, an algebra $\mathbf{A}$ is said to be *locally affine complete*, if for

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every $n \geq 1$, every n-ary compatible function on $\mathbf{A}$ can be interpolated on any finite subset $F \subseteq A^n$ by a polynomial of $\mathbf{A}$.

Originally, the problem of characterization of affine complete algebras was formulated in [2]. For various varieties of algebras affine completeness has already been investigated. In [3] affine completeness of a class of algebras containing Kleene algebras was studied. In particular, it was shown there that a finite Kleene algebra is affine complete if and only if it is a Boolean algebra. In [4] a characterization of affine completeness and local affine completeness for Kleene algebras in general was given. In [6] an alternative approach to Kleene algebra together with illustrating examples was presented. These papers are partly based on ideas developed in [5] and [9] where affine completeness of Stone algebra and distributive lattices was studied. The aim of this paper is to generalize the results to de Morgan algebras.

A *distributive Ockham algebra* is an algebra $\langle L; \vee, \wedge, ^*, 0, 1\rangle$, wher $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and * is a unary operation such that $0^* = 1$, $1^* = 0$ and for all $x, y \in L$,

$$(x \wedge y)^* = x^* \vee y^* , \tag{1}$$

$$(x \vee y)^* = x^* \wedge y^* . \tag{2}$$

The *variety of de Morgan algebras* is the subvariety of the variety of Ockham algebras defined by the following additional identity:

$$x^{**} = x . \tag{3}$$

Every Kleene algebra is a de Morgan algebra. In fact the variety of Kleene algebras can be defined in the variety of de Morgan algebras by the following identity:

$$(x \wedge x^*) \vee (y \vee y^*) = y \vee y^* . \tag{4}$$

A simplest de Morgan algebra which is not Kleene is

$\mathbf{M_4} = \{0, a_1, a_2, 1 : \ 0 < a_1 < 1 , \ 0 < a_2 < 1 , \ a_1 \vee a_2 = 1, \ a_1 \wedge a_2 = 0\}$ with $a_i^* = a_i$, $i = 1, 2$ (see Figure 1).

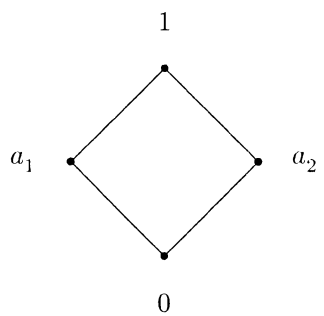

FIGURE 1. 4-element de Morgan algebra

It is known that the variety of de Morgan algebras is generated by $\mathbf{M_4}$. Moreover, $\mathbf{M_4}$ and its subalgebras (two copies of the three-element Kleene algebra $\mathbf{K_3}$ and the two-element Boolean algebra $\mathbf{B_2}$) are the only subdirectly irreducible de Morgan algebras. In fact all of them are simple. Every de Morgan algebra is isomorphic to a subdirect product of subdirectly irreducible de Morgan algebras. Thus, given a de Morgan algebra $\mathbf{M}$, we may write $\mathbf{M} \leq_{\text{s.d.}} \prod_{i\in I} \mathbf{A}_i$, where $\mathbf{A}_i \in \{\mathbf{B_2}, \mathbf{K_3}, \mathbf{M_4}\}$. We denote by $\pi_i \colon M \to A_i$ the projection map to the ith subdirect factor of $\mathbf{M}$.

In what follows we often make use of the following ideas. Given a de Morgan algebra $\mathbf{M}$, we assume that it is embedded in $\mathbf{M}_4^I$ for some index set I. We write the elements of $\mathbf{M}$ in the form $x = (x_i)_{i\in I}$. It is easy to see that if $f \colon M^n \to M$ is a compatible function of $\mathbf{M}$ and $\mathbf{x}, \mathbf{y} \in M^n$, then $x_i = y_i$ implies $f(\mathbf{x})_i = f(\mathbf{y})_i$. This means that every compatible function f of $\mathbf{M}$ determines the coordinate functions f_i of $\pi_i(\mathbf{M})$ such that $f_i(x_i) = f(\mathbf{x})_i$ for all $\mathbf{x} \in M^n$. Obviously, the family $(f_i)_{i\in I}$ completely determines f, so we may identify f with this family.

Let $\mathbf{S}$ be a join semilattice. A *filter* of a semilattice $\mathbf{S}$ is a nonempty subset $F \subseteq S$ such that for all $x \in F$ and $y \in S$, $y \geq x$ implies $y \in F$. A filter F of a semilattice $\mathbf{S}$ is said to be *principal* if it is of the form $\uparrow a = \{x \in S : x \geq a\}$ for some $a \in S$. We say that a filter F is an *almost principal* filter if its intersection with every principal filter of $\mathbf{S}$ is a principal filter of $\mathbf{S}$. Any almost principal filter F of a semilattice $\mathbf{S}$ defines a function $f_F \colon S \to S$ such that $\uparrow f_F(a) = \uparrow a \cap F$ for every $a \in S$.

For every de Morgan algebra $\mathbf{M}$ we denote $M^\vee = \{x \vee x^* : x \in M\}$. It is easy to prove that $M^\vee$ is a filter of the join semilattice $\mathbf{M}$. Obviously, if $\mathbf{M} \leq \mathbf{M}_4^I$, then $s \in M^\vee$ if and only if $s_i \in M_4^\vee$ for every $i \in I$. Note, that if $\mathbf{M}$ is a Kleene algebra, then $\mathbf{M}^\vee$ is a filter of the distributive lattice $\mathbf{M}$.

The following results on affine completeness of Kleene algebras were proved by M. Haviar, K. Kaarli and M. Ploščica in [4]:

THEOREM 1.2.

1. *A function on a Kleene algebra* $\mathbf{K}$ *is a local polynomial function if and only if it preserves the congruences of* $\mathbf{K}$ *and the so called uncertainty order* $\sqsubseteq$ *of* K, *defined by*

$$x \sqsubseteq y \iff x \wedge s \leq y \leq x \vee s^* \qquad \textit{for some} \quad s \in K^\vee .$$

2. *A Kleene algebra* $\mathbf{K}$ *is locally affine complete if and only if the lattice* $\mathbf{K}^\vee$ *does not contain nontrivial Boolean intervals.*

3. *A Kleene algebra* $\mathbf{K}$ *is affine complete if and only if it satisfies the following two conditions:*
   (a) *the lattice* $\mathbf{K}^{\vee}$ *does not contain nontrivial Boolean intervals;*
   (b) *for every almost principal filter* F *in the lattice* $\mathbf{K}^{\vee}$*, there exists* $b \in K$ *such that* $F = \uparrow b \cap K^{\vee}$.

Considering affine completeness problems for de Morgan algebras we start with a characterization of local polynomial functions. Similarly to the case of Kleene algebras, local polynomial functions of a de Morgan algebra are compatible functions preserving a certain binary relation. The difference is that in the case of de Morgan algebras there are several relations which have equal right to play the role of the relation $\sqsubseteq$ of Kleene algebras. This is because $\mathbf{M_4}$ has a nontrivial automorphism. Next we characterize locally affine complete de Morgan algebras. The proof is based on the same ideas which were used in the case of Kleene algebras, i.e. it is based on the following results for lattices due to Dorninger and Eigenthaler [1]:

THEOREM 1.3. *A distributive lattice is locally affine complete if and only if it does not contain nontrivial Boolean intervals.*

THEOREM 1.4. *A function on a distributive lattice is a local polynomial if and only if it is compatible and order preserving.*

Finally, we will characterize affine complete de Morgan algebras. Since in the case of a de Morgan algebra $\mathbf{M}$ the set $M^{\vee}$ is not closed under meets, in general, the techniques of Kleene algebras do not work here. We noticed that instead it is possible to use the following result due to Kaarli, Márki and Schmidt [7]:

LEMMA 1.5. *Let* f *be a unary function on a join semilattice* $\mathbf{S}$ *such that* $f(x) \geq x$. *Then* f *is compatible if and only if it is of the form* $f = f_F$ *for some almost principal filter* F *of* $\mathbf{S}$.

We will also use the following description of compatible unary functions on semilattices (see [8]):

LEMMA 1.6. *Let* $\mathbf{S}$ *be a join semilattice. Then*

1. *A unary function* f *on* $\mathbf{S}$ *is compatible with* $\operatorname{Con} S$ *if and only if it is compatible with all principal congruences* $\operatorname{Cg}(a,b)$ *of* $\mathbf{S}$ *such that* $a < b$.
2. *Let* $a, b \in S$, $a < b$. *Then*

$$\operatorname{Cg}(a,b) = \{(c,d) \in S^2 : \ c = d \ \text{ or } \ c,d \geq a \ \text{ and } \ b \vee c = b \vee d\}.$$

2. Local polynomials of de Morgan algebras

First we consider a canonical form for n-ary polynomials of de Morgan algebras. Let $\underline{n} = \{1, \dots, n\}$ and consider a pair of subsets $\alpha_1, \alpha_2 \subseteq \underline{n}$. To every such pair $\alpha = (\alpha_1, \alpha_2)$ we assign the n-ary term

$$C_\alpha(x_1, \dots, x_n) = \Big(\bigvee_{i \in \alpha_1} x_i \Big) \vee \Big(\bigvee_{i \in \alpha_2} x_i^* \Big).$$

It follows easily from the axioms of de Morgan algebras that every n-ary polynomial function on a de Morgan algebra $\mathbf{M}$ can be represented as a meet of polynomials $K_\alpha \vee C_\alpha(x_1, \dots, x_n)$ where $K_\alpha \in M$ are constants.

Now we introduce a binary relation

$$\Psi = \big\{(0,0), (0,a_2), (a_1,0), (a_1,a_1), (a_1,a_2), (a_1,1), (a_2,a_2), (1,a_2), (1,1)\big\}$$

on M_4 which has an important role in our work. It is easy to check that Ψ is a subuniverse in $\mathbf{M_4} \times \mathbf{M_4}$.

Our first goal is to describe polynomial functions of the algebra $\mathbf{M_4}$. Let f be an n-ary function on $\mathbf{M_4}$ and let $\mathbf{u}, \mathbf{v} \in M_4^n$, $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$. For every pair of subsets $\alpha = (\alpha_1, \alpha_2)$ we put

$$K_\alpha = \big(f(\mathbf{u}) \vee f(\mathbf{b})\big) \wedge \big(f(\mathbf{v}) \vee f(\mathbf{c})\big),$$

where

$$b_i = \begin{cases} 0 & \text{if } i \in \alpha_1 \setminus \alpha_2, \\ 1 & \text{if } i \in \alpha_2 \setminus \alpha_1, \\ u_i & \text{if } i \in \alpha_1 \cap \alpha_2, \\ v_i & \text{otherwise,} \end{cases} \qquad c_i = \begin{cases} 0 & \text{if } i \in \alpha_1 \setminus \alpha_2, \\ 1 & \text{if } i \in \alpha_2 \setminus \alpha_1, \\ v_i & \text{if } i \in \alpha_1 \cap \alpha_2, \\ u_i & \text{otherwise.} \end{cases}$$

LEMMA 2.1. *Let f be an n-ary function on $\mathbf{M_4}$ which preserves the relation Ψ. Then for every two-element subset $\{\mathbf{u}, \mathbf{v}\} \subseteq M_4^n$ the function f coincides on $\{\mathbf{u}, \mathbf{v}\}$ with the polynomial function*

$$p(\mathbf{x}) = \bigwedge_\alpha \big(C_\alpha(\mathbf{x}) \vee K_\alpha\big).$$

Proof. Suppose that f is an n-ary function of $\mathbf{M_4}$ which preserves the relation Ψ and let $\mathbf{u}, \mathbf{v} \in M_4^n$. Since the constants K_α are symmetric with respect to $\mathbf{u}$ and $\mathbf{v}$, we only need to show that $f(\mathbf{u}) = p(\mathbf{u})$. Now we have to consider the following four cases:

1. $f(\mathbf{u}) = 0$.

We will show that in this case there exist α and β such that

$$C_\alpha(\mathbf{u}) \vee K_\alpha \le a_1 \qquad \text{and} \qquad C_\beta(\mathbf{u}) \vee K_\beta \le a_2 .$$

Then obviously $p(\mathbf{u}) = 0$.

Let $\alpha = (\alpha_1, \alpha_2)$ where

$$\alpha_1 = \{i:\ u_i \le a_1\} \qquad \text{and} \qquad \alpha_2 = \{i:\ u_i \ge a_1\}.$$

Then $C_\alpha(\mathbf{u}) \le a_1$ and

$$K_\alpha = \big(f(\mathbf{u}) \vee f(\mathbf{b})\big) \wedge \big(f(\mathbf{v}) \vee f(\mathbf{c})\big) \le f(\mathbf{u}) \vee f(\mathbf{b}),$$

where

$$b_i = \begin{cases} 0 & \text{if } u_i = 0, \\ 1 & \text{if } u_i = 1, \\ a_1 & \text{if } u_i = a_1, \\ v_i & \text{if } u_i = a_2. \end{cases}$$

Now $(\mathbf{b}, \mathbf{u}) \in \Psi$ implies $\big(f(\mathbf{b}), f(\mathbf{u})\big) \in \Psi$ and hence $f(\mathbf{b}) \le a_1$.

Let $\beta = (\beta_1, \beta_2)$ where

$$\beta_1 = \{i:\ u_i \le a_2\} \qquad \text{and} \qquad \beta_2 = \{i:\ u_i \ge a_2\}.$$

Then $C_\beta(\mathbf{u}) \le a_2$ and

$$K_\beta = \big(f(\mathbf{u}) \vee f(\mathbf{b})\big) \wedge \big(f(\mathbf{v}) \vee f(\mathbf{c})\big) \le f(\mathbf{u}) \vee f(\mathbf{b}),$$

where

$$b_i = \begin{cases} 0 & \text{if } u_i = 0, \\ 1 & \text{if } u_i = 1, \\ a_2 & \text{if } u_i = a_2, \\ v_i & \text{if } u_i = a_1. \end{cases}$$

Now $(\mathbf{u}, \mathbf{b}) \in \Psi$ implies $\big(f(\mathbf{u}), f(\mathbf{b})\big) \in \Psi$ and hence $f(\mathbf{b}) \le a_2$.

2. $f(\mathbf{u}) = a_1$.

First we will show that $C_\alpha(\mathbf{u}) \vee K_\alpha \ge a_1$. Assume that $C_\alpha(\mathbf{u}) \not\ge a_1$. If $u_i = 0$, then $i \in \underline{n} \setminus \alpha_2$ and hence $c_i = 0$. If $u_i = a_1$, then $i \in \underline{n} \setminus (\alpha_1 \cup \alpha_2)$ and hence $c_i = a_1$. If $u_i = a_2$, then c_i can be arbitrary, and if $u_i = 1$, then $i \in \underline{n} \setminus \alpha_1$ and $c_i = 1$. Thus $(\mathbf{c}, \mathbf{u}) \in \Psi$. Since f preserves the relation Ψ we have $\big(f(\mathbf{c}), f(\mathbf{u})\big) \in \Psi$ and $f(\mathbf{c}) = a_1$. Hence

$$K_\alpha = \big(f(\mathbf{u}) \vee f(\mathbf{b})\big) \wedge \big(f(\mathbf{v}) \vee f(\mathbf{c})\big) \ge a_1$$

and $C_\alpha(\mathbf{u}) \vee K_\alpha \ge a_1$.

Now we are going to show that there exists β such that

$$C_\beta(\mathbf{u}) \vee K_\beta = a_1.$$

Then obviously $p(\mathbf{u}) = a_1$. Let $\beta = (\beta_1, \beta_2)$ where

$$\beta_1 - \{i:\ u_i < a_1\} \qquad \text{and} \qquad \beta_2 \quad \{i:\ u_i \quad a_1\}.$$

Then $C_\beta(\mathbf{u}) \le a_1$ and

$$b_i = \begin{cases} 0 & \text{if } u_i = 0\,, \\ 1 & \text{if } u_i = 1\,, \\ a_1 & \text{if } u_i = a_1\,, \\ v_i & \text{if } u_i = a_2\,. \end{cases}$$

Since $(\mathbf{b},\mathbf{u}) \in \Psi$ implies $(f(\mathbf{b}), f(\mathbf{u})) \in \Psi$, we have $f(\mathbf{b}) = a_1$. Thus $K_\beta \le a_1$ and $C_\beta(\mathbf{u}) \vee K_\beta \le a_1$. Above we have showed that $C_\beta(\mathbf{u}) \vee K_\beta \ge a_1$, thus we are done.

3. $f(\mathbf{u}) = a_2$.

The proof of this case is similar to the proof of the preceding one.

4. $f(\mathbf{u}) = 1$.

It is obviously enough to show that for every pair of subsets $\alpha = (\alpha_1, \alpha_2)$ we have $C_\alpha(\mathbf{u}) \vee K_\alpha = 1$. Assume that $C_\alpha(\mathbf{u}) < 1$. If $u_i = 0$, then $i \in \underline{n} \setminus \alpha_2$ and hence $c_i = 0$. If $u_i = 1$, then $i \in \underline{n} \setminus \alpha_1$ and hence $c_i = 1$. If $u_i = a_1$ for some $i \in \alpha_1 \cup \alpha_2$, then $C_\alpha(\mathbf{u}) \ge a_1$ and there is no $j \in \alpha_1 \cup \alpha_2$ such that $u_j = a_2$. Thus $(\mathbf{u},\mathbf{c}) \in \Psi$ and $f(\mathbf{c}) \in \{a_2, 1\}$. Hence $K_\alpha \ge a_2$ and $C_\alpha(\mathbf{u}) \vee K_\alpha = 1$. If $u_i = a_2$ for some $i \in \alpha_1 \cup \alpha_2$, then $C_\alpha(\mathbf{u}) \ge a_2$ and there is no $j \in \alpha_1 \cup \alpha_2$ such that $u_j = a_1$. Thus $(\mathbf{c},\mathbf{u}) \in \Psi$ and $f(\mathbf{c}) \in \{a_1, 1\}$. Hence $K_\alpha \ge a_1$ and $C_\alpha(\mathbf{u}) \vee K_\alpha = 1$. If there is no $i \in \alpha_1 \cup \alpha_2$ such that $u_i \in \{a_1, a_2\}$, then $\mathbf{c} = \mathbf{u}$ and $C_\alpha(\mathbf{u}) \vee K_\alpha = 1$. □

Using Lemma 2.1 it is easy to describe local polynomial functions of a de Morgan algebra in general.

THEOREM 2.2. *An n-ary function on a de Morgan algebra $\mathbf{M} \le \mathbf{M}_4^I$ is a local polynomial function if and only if it preserves the congruences and for every $i \in I$ the function $f_i\colon \pi_i(M)^n \to \pi_i(M)$ preserves the relation Ψ.*

Proof. The congruences of a de Morgan algebra $\mathbf{M}$ are the subuniverses of $\mathbf{M}^2$ containing the diagonal, so they are preserved by all local polynomial functions of $\mathbf{M}$. Also the relation Ψ is a diagonal subuniverse of $\mathbf{M}_4^2$, hence it is preserved by all local polynomial functions of $\mathbf{M}_4$. (Obviously, if p is a local polynomial function of $\mathbf{M} \le \mathbf{M}_4^I$, then for every $i \in I$ the function $p_i\colon \pi_i(M)^n \to \pi_i(M)$ is a polynomial function of $\mathbf{M}_4$.)

Suppose that f is an n-ary compatible function of a de Morgan algebra $\mathbf{M} \le \mathbf{M}_4^I$ such that for every $i \in I$ the function $f_i\colon \pi_i(M)^n \to \pi_i(M)$ preserves the relation Ψ. We have to prove that, given any finite subset $X \subseteq M^n$, there exists an n-ary polynomial p of $\mathbf{M}$ such that $f|_X = p|_X$. Since $\mathbf{M}$ has the ternary lattice median term, it suffices to find interpolating polynomials separately for every two-element subset $\{\mathbf{u},\mathbf{v}\} \subseteq M^n$. Let

$$p(\mathbf{x}) = \bigwedge_\alpha \left(C_\alpha(\mathbf{x}) \vee K_\alpha\right),$$

where

$$K_\alpha = \big(f(\mathbf{u}) \vee f(\mathbf{b})\big) \wedge \big(f(\mathbf{v}) \vee f(\mathbf{c})\big)$$

with

$$b_i = \begin{cases} 0 & \text{if } i \in \alpha_1 \setminus \alpha_2 \,, \\ 1 & \text{if } i \in \alpha_2 \setminus \alpha_1 \,, \\ u_i & \text{if } i \in \alpha_1 \cap \alpha_2 \,, \\ v_i & \text{otherwise,} \end{cases} \qquad c_i = \begin{cases} 0 & \text{if } i \in \alpha_1 \setminus \alpha_2 \,, \\ 1 & \text{if } i \in \alpha_2 \setminus \alpha_1 \,, \\ v_i & \text{if } i \in \alpha_1 \cap \alpha_2 \,, \\ u_i & \text{otherwise.} \end{cases}$$

We have to show that $f\big|\{\mathbf{u},\mathbf{v}\} = p\big|\{\mathbf{u},\mathbf{v}\}$. Since f is compatible, it suffices to give the proof just for the case $\mathbf{M} = \mathbf{M_4}$. Thus the result follows from Lemma 2.1. $\square$

The next lemma will prove useful in characterization of locally affine complete de Morgan algebras.

LEMMA 2.3. *For every de Morgan algebra* $\mathbf{M}$, *if* f *is a compatible function on* $\mathbf{M}$ *and* f *preserves* $\uparrow s$ *for some* $s \in M^\vee$, *then* $f|\uparrow s$ *is a compatible function on the lattice* $\uparrow s$.

Proof. Let $s \in M^\vee$ and Φ be a congruence of the lattice $\uparrow s$. We are going to show that there exists a congruence ϕ of the algebra $\mathbf{M}$ such that $\phi|\uparrow s = \Phi$.

Define an equivalence relation ϕ on $\mathbf{M}$ by

$$(x,y) \in \phi \iff (x \vee s,\, y \vee s) \in \Phi \quad \& \quad (x^* \vee s,\, y^* \vee s) \in \Phi\,.$$

First we verify that ϕ is a subalgebra of $\mathbf{M}^2$. Let $(x,y) \in \phi$, then since

$$(x^{**} \vee s,\, y^{**} \vee s) = (x \vee s,\, y \vee s)\,,$$

we have $(x^*, y^*) \in \phi$.

Now we assume that $(x,y),(u,v) \in \phi$ and check that then also $(x \vee u, y \vee v) \in \phi$. By the definition of ϕ we have $(x \vee s, y \vee s) \in \Phi$ and $(u \vee s, v \vee s) \in \Phi$. Now, since Φ is a congruence of $\uparrow s$, we have $\big((x \vee u) \vee s, (y \vee v) \vee s\big) \in \Phi$. Similarly $(x^* \vee s, y^* \vee s) \in \Phi$ and $(u^* \vee s, v^* \vee s) \in \Phi$ imply $\big((x \vee u)^* \vee s, (y \vee v)^* \vee s\big) \in \Phi$. This proves that ϕ is closed with respect to joins. Analogously we can prove that ϕ is closed with respect to meets. Hence ϕ is a congruence of $\mathbf{M}$.

Now obviously $\phi|\uparrow s \subseteq \Phi$. Assume that $x, y \in \uparrow s$ and $(x,y) \in \Phi$. Since $(x \vee s,\, y \vee s) = (x,y)$, we have $(x \vee s,\, y \vee s) \in \Phi$. Note that if $x \in \uparrow s$, then $x^* \vee s = s$. Thus $(x^* \vee s,\, y^* \vee s) = (s,s) \in \Phi$. Hence $(x,y) \in \phi$ and we have $\phi|\uparrow s = \Phi$. $\square$

Next we describe locally affine complete de Morgan algebras.

THEOREM 2.4. *A de Morgan algebra* $\mathbf{M}$ *is locally affine complete if and only if the semilattice* $\mathbf{M}^{\vee}$ *does not contain nontrivial Boolean intervals.*

Proof. Suppose that $\mathbf{M}^{\vee}$ contains a nontrivial Boolean interval $[s,t]$ and consider the lattice $\uparrow s$. By Theorem 1.3, the lattice $\uparrow s$ is not locally affine complete, and thus, by Theorem 1.4, has a compatible function $g(x)$ which does not preserve the order relation. Let $f(x) = g(x \vee s)$. It easily follows from Lemma 1.1 that f is a compatible function on $\mathbf{M}$ which extends g. We assume that $\mathbf{M}$ is embedded in $\mathbf{M}_4^I$. Then there exists $i \in I$ such that g_i does not preserve the order relation. This easily implies that f_i does not preserve the relation Ψ. Thus, by Theorem 2.2, f is not a local polynomial function.

For the converse, suppose that $\mathbf{M}$ has a compatible function f which is not a local polynomial and also assume that $\mathbf{M} \leq \mathbf{M}_4^I$ for some index set I. By Theorem 2.2, there exists $i \in I$ such that f_i does not preserve the relation Ψ. Thus there exist $u_i, v_i \in M$, $u_i \neq v_i$, such that $(u_i, v_i) \in \Psi$ but $\big(f_i(u_i), f_i(v_i)\big) \notin \Psi$. This is possible only if

$$(u_i, v_i) \in \big\{(0,a_2),(1,a_2),(a_1,0),(a_1,1),(a_1,a_2)\big\}$$

and

$$\big(f_i(u_i), f_i(v_i)\big) \in \big\{(0,a_1),(0,1),(a_2,0),(a_2,a_1),(a_2,1),(1,0),(1,a_1)\big\}.$$

In fact we may assume that

$$(u_i, v_i) \in \big\{(1,a_2),(a_1,1)\big\}$$

because if $(u_i, v_i) \in \big\{(0,a_2),(a_1,0)\big\}$, we could replace $f(x)$ by $f(x^*)$, and if $(u_i,v_i) = (a_1,a_2)$, we could replace $f(x)$ by $f(x \vee v)$. Also we may assume that

$$\big(f_i(u_i), f_i(v_i)\big) \in \big\{(0,a_1),(0,1),(a_2,a_1),(a_2,1)\big\}$$

because otherwise we could replace $f(x)$ by $f(x)^*$.

Thus we have to consider the following two cases:

1. $u_i = 1$ and $v_i = a_2$.

Let $s = v \vee v^*$ and $g(x) = f(x) \vee s$. Note that by Lemma 2.3 the restriction of g to $\uparrow s$ is compatible for the lattice $\uparrow s$. We will show that $g|\uparrow s$ does not preserve the order relation on $\uparrow s$. Indeed

$$g_i(1) = f_i(1) \vee s_i = a_2$$

and

$$g_i(s_i) = g_i(a_2) = f_i(a_2) \vee s_i = 1.$$

Consequently $g(s) \not\leq g(1)$.

2. $u_i = a_1$ and $v_i = 1$.
This case can be handled similarly to the preceding one. Let $s = u \vee u^*$ and $g(x) = f(x)^* \vee s$. Then

$$g_i(1) = f_i(1)^* \vee s_i = a_1$$

and

$$g_i(s_i) = g_i(a_1) = f_i(a_1)^* \vee s_i = 1 .$$

Thus $g(s) \nleq g(1)$.

Hence, given a compatible function f of $\mathbf{M}$ which is not a local polynomial, we can construct a compatible function g of some lattice $\uparrow s \subseteq M^\vee$ which is not a local polynomial function of this lattice. By Theorem 1.3 this means that the lattice $\uparrow s$ contains a nontrivial Boolean interval. Thus also the semilattice $\mathbf{M}^\vee$ contains a nontrivial Boolean interval. □

3. Affine completeness

In this section we describe affine complete members in the variety of de Morgan algebras.

Our first goal is to find for the given compatible function f of a de Morgan algebra $\mathbf{M}$ a polynomial of $\mathbf{M}$ which coincides with f on the set $M^\vee$.

Lemma 3.1. *Let f be an n-ary local polynomial function on a de Morgan algebra $\mathbf{M}$ and let*

$$g(x_1,\dots,x_n) = \Bigl(f(1,\dots,1) \wedge \bigwedge_{1\le i\le n} (f(x_1,\dots,x_n) \vee x_i)\Bigr) \vee \bigvee_{1\le i\le n} (f(x_1,\dots,x_i^*,\dots,x_n) \wedge x_i^*) .$$

Then the functions f and g coincide on $M^\vee$.

Proof. We have to show that $f(\mathbf{b}) = g(\mathbf{b})$ for every $\mathbf{b} \in (M^\vee)^n$. Keeping in mind the embedding $\mathbf{M} \le \mathbf{M}_4^I$, it suffices to consider only the case $\mathbf{M} = \mathbf{M}_4$ and $\mathbf{b} \in \{a_1, a_2, 1\}^n$. We also note that since f is a local polynomial function, by Theorem 2.2, it preserves the relation Ψ. Now we have to consider the following four cases:

1. $\mathbf{b} = (1,\dots,1)$.
In this case the equality $f(\mathbf{b}) = g(\mathbf{b})$ is obvious.

2. $\{b_i :\ 1 \le i \le n\} \subseteq \{a_1, 1\}$ and there exists j such that $b_j = a_1$.
Then

$$g(\mathbf{b}) = \Bigl(f(1,\dots,1) \wedge (f(\mathbf{b}) \vee a_1)\Bigr) \vee (f(\mathbf{b}) \wedge a_1) .$$

Since f preserves the relation Ψ, we have $f(\mathbf{b})\,\Psi\, f(1,\dots,1)$. Thus, if $f(\mathbf{b}) = 0$, then $f(1,\dots,1) \in \{0, a_2\}$ and

$$g(\mathbf{b}) = \big(f(1,\dots,1) \wedge a_1\big) \vee 0 = 0 .$$

If $f(\mathbf{b}) = a_1$, then $g(\mathbf{b}) = \big(f(1,\dots,1) \wedge a_1\big) \vee a_1 = a_1$. If $f(\mathbf{b}) = a_2$, then $f(1,\dots,1) = a_2$ and $g(\mathbf{b}) = (a_2 \wedge 1) \vee 0 = a_2$, and if $f(\mathbf{b}) = 1$, then $f(1,\dots,1) \in \{a_2, 1\}$, thus $g(\mathbf{b}) = \big(f(1,\dots,1) \wedge 1\big) \vee a_1 = 1$.

3. $\{b_i :\ 1 \le i \le n\} \subseteq \{a_2, 1\}$ and there exists j such that $b_j = a_2$.

This case can be handled similarly to the preceding one.

4. $\{b_i :\ 1 \le i \le n\} \subseteq \{a_1, a_2, 1\}$ and there exist j and k such that $b_j = a_1$ and $b_k = a_2$.

Then

$$\begin{aligned} g(\mathbf{b}) = \Big(&f(1,\dots,1) \wedge \big(f(\mathbf{b}) \vee a_1\big) \wedge \big(f(\mathbf{b}) \vee a_2\big)\Big) \\ &\vee \big(f(\mathbf{b}) \wedge a_1\big) \vee \big(f(\mathbf{b}) \wedge a_2\big) . \end{aligned}$$

Now, if $f(\mathbf{b}) = 0$, then $g(\mathbf{b}) = \big(f(1,\dots,1) \wedge a_1 \wedge a_2\big) \vee 0 = 0$. If $f(\mathbf{b}) = a_1$, then $g(\mathbf{b}) = \big(f(1,\dots,1) \wedge a_1 \wedge 1\big) \vee a_1 \vee 0 = a_1$. If $f(\mathbf{b}) = a_2$, then $g(\mathbf{b}) = \big(f(1,\dots,1) \wedge 1 \wedge a_2\big) \vee 0 \vee a_2 = a_2$, and if $f(\mathbf{b}) = 1$, then $g(\mathbf{b}) = \big(f(1,\dots,1) \wedge 1 \wedge 1\big) \vee a_1 \vee a_2 = 1$. □

Lemma 3.2. *Let $\mathbf{M}$ be a locally affine complete de Morgan algebra such that for every almost principal filter F of the semilattice $\mathbf{M}^\vee$, there exists $b \in M$ such that $F = \uparrow b \cap M^\vee$, and let f be an n-ary compatible function on the algebra $\mathbf{M}$. Then the function*

$$g^j(x_1,\dots,x_n) = f(x_1,\dots,x_n) \vee x_j , \qquad 1 \le j \le n ,$$

can be interpolated on the set $M^\vee$ by a polynomial function of $\mathbf{M}$.

Proof. First assume that f is a unary function. We are going to prove that the function $f(x) \vee x$ can be interpolated on the set $M^\vee$ by a polynomial function. First we show that the function

$$g(x) = \big(f(x) \vee x\big)\big|_{M^\vee}$$

is a compatible function on the semilattice $\mathbf{M}^\vee$. Furthermore, g has the form $g = g_F$ for some almost principal filter F of $\mathbf{M}^\vee$.

By Lemma 1.6 we only need to show that $g(x)$ is compatible with all principal semilattice congruences $\mathrm{Cg}_{\mathbf{M}^\vee}(c,d)$ of $\mathbf{M}^\vee$ such that $c < d$. Assume that $c, d \in M^\vee$, $c < d$, and $(x,y) \in \mathrm{Cg}_{\mathbf{M}^\vee}(c,d)$. We have to show that also $\big(f(x), f(y)\big) \in \mathrm{Cg}_{\mathbf{M}^\vee}(c,d)$. If $x = y$, then obviously $g(x) = g(y)$. Now we suppose that $x, y \ge c$ and $d \vee x = d \vee y$. Then $g(x) = f(x) \vee x \ge x \ge c$ and $g(y) = f(y) \vee y \ge y \ge c$.

To prove the equality $d \vee g(x) = d \vee g(y)$, we consider the function $f(x) \vee x$ as a compatible function of the de Morgan algebra $\mathbf{M}$ and assume that we have an embedding $\mathbf{M} \leq \mathbf{M}_4^I$. Then we only need to show that

$$f_i(x_i) \vee d_i \vee x_i = f_i(y_i) \vee d_i \vee y_i ,$$

where $c_i, d_i, x_i, y_i \in M_4^\vee$, $c_i \leq d_i$, $x_i, y_i \geq c_i$ and $d_i \vee x_i = d_i \vee y_i$. Now if $d_i = 1$, then the equality is obvious. Thus without loss of generality we may assume that $d_i = a_1$. Since $c_i \leq d_i$, $c_i \in M_4^\vee$ and $x_i, y_i \geq c_i$, we have $x_i, y_i \geq a_1$ and since $d_i \vee x_i = d_i \vee y_i$, we have $x_i = y_i$.

By Lemma 1.5, $g(x)$ has the form $g = g_F$ for some almost principal filter F of the semilattice $\mathbf{M}^\vee$.

Now, by our assumption, there exists $b \in M$ such that $F = \uparrow b \cap M^\vee$. Then the polynomial $b \vee x$ interpolates g on $M^\vee$.

We proceed by induction on n. Assume that the statement of the lemma holds for all $(n-1)$-ary functions and let f be an n-ary function. For every $\mathbf{u} \in M^{n-1}$ we introduce the unary function

$$g_{\mathbf{u}}^j(x) = g^j(\mathbf{u}, x) = f(u_1, \ldots, u_{j-1}, x, u_{j+1}, \ldots, u_{n-1}) \vee x .$$

As we have proved above $g_{\mathbf{u}}^j$ can be interpolated on $M^\vee$ by a polynomial function of the form $r \vee x$, where $r \in M$. Thus there exists a function $h\colon M^{n-1} \to M$ such that

$$\begin{aligned} &g^j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n)|_{M^\vee} \\ &\qquad = \big(h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \vee x_j\big)|_{M^\vee} . \end{aligned} \tag{5}$$

Let

$$\begin{aligned} \tilde{g}(x_1, \ldots, x_n) &= g^j(x_1 \vee x_1^*, \ldots, x_n \vee x_n^*) ; \\ \tilde{h}(x_1, \ldots, x_n) &= h(x_1 \vee x_1^*, \ldots, x_{n-1} \vee x_{n-1}^*) . \end{aligned}$$

Then $g|_{M^\vee} = \tilde{g}|_{M^\vee}$, $h|_{M^\vee} = \tilde{h}|_{M^\vee}$ and by (5)

$$\tilde{g}(x_1, \ldots, x_n) = \tilde{h}(x_1, \ldots, x_{n-1}) \vee x_j$$

for all $x_1, \ldots, x_n \in M$. Obviously $\tilde{g}$ is a compatible function on $\mathbf{M}$. Now we are going to show that the function $\tilde{h}$ is compatible on $\mathbf{M}$, too. Let $\mathbf{u}, \mathbf{v} \in M^{n-1}$ and put $c = \tilde{h}(\mathbf{u}) \wedge \tilde{h}(\mathbf{v})$. Then

$$\tilde{g}(u_1, \ldots, u_{j-1}, c, u_{j+1}, \ldots, u_n) = \tilde{h}(\mathbf{u}) \vee \big(\tilde{h}(\mathbf{u}) \wedge \tilde{h}(\mathbf{v})\big) = \tilde{h}(\mathbf{u})$$

and

$$\tilde{g}(v_1, \ldots, v_{j-1}, c, v_{j+1}, \ldots, v_n) = \tilde{h}(\mathbf{v}) \vee \big(\tilde{h}(\mathbf{u}) \wedge \tilde{h}(\mathbf{v})\big) = \tilde{h}(\mathbf{v}) .$$

Thus the compatibility of $\tilde{g}$ implies that of $\tilde{h}$. Hence we have

$$\begin{aligned} g^j(x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_n)|_{M^\vee} \\ = \big(\tilde{h}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)\vee x_j\big)|_{M^\vee}\,. \end{aligned}$$

The latter formula shows that whenever the function $\tilde{h}$ can be interpolated on $M^\vee$ by a polynomial function, then so can g^j. Since $\mathbf{M}$ is locally affine complete, by Lemma 3.1, $\tilde{h}$ can be represented as

$$\begin{aligned} \tilde{h}(x_1,\ldots,x_{n-1}) = \Big(\tilde{h}(1,\ldots,1)\wedge \bigwedge_{1\le i\le n-1}\big(\tilde{h}(x_1,\ldots,x_{n-1})\vee x_i\big)\Big) \\ \vee \bigvee_{1\le i\le n-1}\big(\tilde{h}(x_1,\ldots,x_i^*,\ldots,x_{n-1})\wedge x_i^*\big)\,. \end{aligned}$$

Now, since

$$\tilde{h}(x_1,\ldots,x_i^*,\ldots,x_{n-1})\wedge x_i^* = \big(\tilde{h}(x_1,\ldots,x_i^*,\ldots,x_{n-1})^*\vee x_i\big)^*\,,$$

the result follows from the induction hypothesis. □

Ideas used in the proof of the following theorem are similar to those applied in [4]–[6]. In the proof of the sufficiency part we use the techniques developed in [8] for Kleene algebras.

THEOREM 3.3. *A de Morgan algebra* $\mathbf{M}$ *is affine complete if and only if it satisfies the following two conditions:*

1. *the semilattice* $\mathbf{M}^\vee$ *does not contain nontrivial Boolean intervals;*
2. *for every almost principal filter* F *of the semilattice* $\mathbf{M}^\vee$, *there exists* $b\in M$ *such that* $F = \uparrow b\cap M^\vee$.

Proof. If $\mathbf{M}$ is affine complete, then it is locally affine complete and by Theorem 2.4 the semilattice $\mathbf{M}^\vee$ does not contain nontrivial Boolean intervals. Let F be an almost principal filter in the semilattice $\mathbf{M}^\vee$. By Lemma 1.5, F defines a compatible function f_F on the semilattice $\mathbf{M}^\vee$. Hence it follows from Lemma 1.1 that the function $g(x) = f_F(x\vee x^*)$ is a compatible function on $\mathbf{M}$. If $\mathbf{M}$ is affine complete, then there must exist constants $k_1,\ldots,k_4\in M$ such that

$$g(x) = (k_1\vee x)\wedge(k_2\vee x^*)\wedge(k_3\vee x\vee x^*)\wedge k_4$$

for every $x\in M$. Since $g(1)=1$, we have $k_2\wedge k_4 = 1$, thus $k_2 = k_4 = 1$. If $x\in M^\vee$, then $x\vee x^* = x$ and $g(x) = f_F(x)$. Therefore

$$f_F(x) = (k_1\wedge k_3)\vee x$$

for every $x\in M^\vee$, implying $F = \uparrow(k_1\wedge k_3)\cap M^\vee$. This proves the necessity part of the theorem.

Now we prove the sufficiency of the two conditions. We assume that $\mathbf{M}$ is embedded in $\mathbf{M}_4^I$. Then for every $i \in I$ we have $\pi_i(\mathbf{M}) \in \{\mathbf{M_4}, \mathbf{K_3}, \mathbf{B_2}\}$. Let f be an n-ary compatible function on $\mathbf{M}$. Our aim is to find a finite set P_2 of polynomial functions on $\mathbf{M}$ such that for every n-tuples $\mathbf{u}, \mathbf{v} \in M_4^n$ there exists $p \in P_2$ such that $p_i(\mathbf{u}) = f_i(\mathbf{u})$ and $p_i(\mathbf{v}) = f_i(\mathbf{v})$ for all $i \in I$ for which it makes sense.

In fact, if we have such a set P_2, then the rest is easy. Using the majority term $m(x, y, z)$ we shall define

$$P_{k+1} = \{m(p_1, p_2, p_3) : \ p_1, p_2, p_3 \in P_k\}, \qquad k \geq 2 .$$

Then every set of polynomials P_k has the property that given an arbitrary k-element subset X of M_4^n, there exists $p \in P_k$ such that $p_i(\mathbf{x}) = f_i(\mathbf{x})$ for every $\mathbf{x} \in X$ and every $i \in I$ for which it makes sense. Since $|M_4^n| = 4^n$, the set P_{4^n} consists of a single polynomial function which must coincide with f.

We start the construction of the polynomials which form the set P_2, as we shall see later. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a triple of disjoint subsets of $\underline{n}$ whose union is $\underline{n}$. With every such triple α we associate a set S_α of all $(b_1, \ldots, b_n) \in M^n$ such that

$$b_i \in \begin{cases} \{0,1\} & \text{if } i \in \alpha_1 , \\ \mathbf{M}^\vee & \text{if } i \in \alpha_2 , \\ \mathbf{M}^\wedge & \text{if } i \in \alpha_3 . \end{cases}$$

We are going to find, for every α, a polynomial p_α such that $f|_{S_\alpha} = p_\alpha|_{S_\alpha}$.

First consider the case $\alpha_1 = \emptyset$. Because the complementation $*$ is an antiisomorphism between the semilattices $\mathbf{M}^\vee$ and $\mathbf{M}^\wedge$, $M^\wedge = \{x \wedge x^* : \ x \in M\}$, we may assume without loss of generality that $\alpha_3 = \emptyset$. Now Lemmas 3.1 and 3.2 imply that a suitable polynomial is:

$$\begin{aligned} p(x_1, \ldots, x_n) = \Big(f(1, \ldots, 1) \wedge & \bigwedge_{1 \leq i < n} (f(x_1, \ldots, x_n) \vee x_i)\Big) \\ & \vee \bigvee_{1 \leq i \leq n} (f(x_1, \ldots, x_i^*, \ldots, x_n) \wedge x_i^*) . \end{aligned}$$

Now consider the subsets S_α corresponding to the triples α with $\alpha_1 \neq \emptyset$. We construct p_α by induction on the size of α_1. Suppose, without loss of generality, that $n \in \alpha_1$ and take polynomials q_0 and q_1 such that

$$\begin{aligned} q_0(\mathbf{b}) = f(\mathbf{b}) \qquad & \text{if} \quad \mathbf{b} \in S_\alpha \quad \text{and} \quad b_n = 0 , \\ q_1(\mathbf{b}) = f(\mathbf{b}) \qquad & \text{if} \quad \mathbf{b} \in S_\alpha \quad \text{and} \quad b_n = 1 . \end{aligned}$$

Such polynomials q_0 and q_1 exist by the induction hypothesis. Define

$$q(x_1, \ldots, x_n) = (q_0(x_1, \ldots, x_n) \wedge x_n) \quad (q_1(x_1, \ldots, \quad) \wedge x \text{ [illegible]} .$$

It is easy to see that $q(\mathbf{b}) = f(\mathbf{b})$ for every $\mathbf{b} \in S_\alpha$.

It remains to show that the set of all polynomials p_α has the property that P_2 must satisfy. We take $\mathbf{u}, \mathbf{v} \in M^n$ and define a triple α as follows:

$$j \in \begin{cases} \alpha_1 & \text{if } u_i, v_i \in \{0,1\}, \\ \alpha_2 & \text{if } u_i, v_i \in \{a_1, a_2, 1\} \text{ and } \{u_i, v_i\} \neq \{1\}, \\ \alpha_3 & \text{if } \{u_i, v_i\} = \{0, a_1\} \text{ or } \{u_i, v_i\} = \{0, a_2\}. \end{cases}$$

Keeping in mind the embedding $\mathbf{M} \leq \mathbf{M}_4^I$, we take $\mathbf{b}, \mathbf{c} \in M^n$ such that $\pi_i(\mathbf{b}) = \mathbf{u}$ and $\pi_i(\mathbf{c}) = \mathbf{v}$. It is easy to see that such n-tuples $\mathbf{b}$ and $\mathbf{c}$ exist in S_α. Now clearly $(p_\alpha)_i(\mathbf{u}) = f_i(\mathbf{u})$ and $(p_\alpha)_i(\mathbf{v}) = f_i(\mathbf{v})$. This proves the theorem. □

4. Examples

EXAMPLE 1. Since every Kleene algebra is a de Morgan algebra, all examples presented in [6] are also suitable for de Morgan algebras.

Also every finite de Morgan algebra which is not a Boolean algebra is not locally affine complete.

EXAMPLE 2. Let $M_1 = [-1,1]^2$. With respect to the natural order relation, M_1 is a bounded distributive lattice. Let $*$ be defined by $(x,y)^* = (-y,-x)$. It is easy to check that

$$\mathbf{M_1} = \langle M_1; \vee, \wedge, {}^*, (-1,-1), (1,1) \rangle$$

is a de Morgan algebra which is not a Kleene algebra. It is also easy to see that

$$M_1^\vee = \{(x,y) \in [-1,\ 1]^2 :\ x \geq -y\}.$$

Obviously the semilattice $\mathbf{M_1^\vee}$ does not contain nontrivial Boolean intervals. Hence, by Theorem 2.4, $\mathbf{M_1}$ is a locally affine complete de Morgan algebra.

Let F be an almost principal filter of the semilattice $\mathbf{M_1^\vee}$. Then

$$F \cap \uparrow(-1,1) = \uparrow(x_0, 1) \qquad \text{and} \qquad F \cap \uparrow(1,-1) = \uparrow(1, y_0)$$

for some $x_0, y_0 \in [-1,1]$. Take $(x_0,1) \wedge (1,y_0) = (x_0,y_0) \in M_1$ and show that $F = M_1^\vee \cap \uparrow(x_0,y_0)$. Clearly $F \subseteq M_1^\vee \cap \uparrow(x_0,y_0)$. Now take any $(x_1,y_1) \in M_1^\vee \cap \uparrow(x_0,y_0)$ and let

$$F \cap \uparrow(x_1,y_1) = \uparrow(x_2,y_2).$$

Then $(x_2,y_2) \geq (x_1,y_1)$. Further, since

$$(x_1,1),(1,y_1) \in F \qquad \text{and} \qquad (x_1,1),(1,y_1) \in \uparrow(x_1,y_1),$$

we have $(x_1,1),(1,y_1) \in F \cap \uparrow(x_1,y_1)$. Thus $(x_1,1),(1,y_1) \geq (x_2,y_2)$ and also $(x_1,y_1) = (x_1,1) \wedge (1,y_1) \geq (x_2,y_2)$. Hence $(x_1,y_1) \in F$. Thus $F = M_1^\vee \cap \uparrow(x_0,y_0)$ and by Theorem 3.3 the algebra $\mathbf{M_1}$ is affine complete.

EXAMPLE 3. Let $\mathbf{M_2}$ be the subalgebra of $\mathbf{M_1}$ with the universe

$$M_1 \setminus \{(-1,1),(1,-1)\}.$$

Now $F = \{(x,y) \in M_2 : (x,y) > (-1,1)\}$ is a proper almost principal filter of the semilattice $\mathbf{M_2^\vee}$ without a smallest element and there is no element $b \in M_2$ such that $F = \uparrow b \cap M_2^\vee$. Thus $\mathbf{M_2}$ is not an affine complete de Morgan algebra. However, it is clearly locally affine complete.

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