Mathematica Slovaca

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Mathematica Slovaca, Vol. 52 (2002), No. 1, 109--126

Persistent URL: http://dml.cz/dmlcz/136857

Math. Slovaca, **52** (2002), No. 1, 109–126

Mathematica Slovaca
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Slovak Academy of Sciences

RESULTS FOR AN OPTIMAL CONTROL PROBLEM WITH A SEMILINEAR STATE EQUATION WITH CONSTRAINED CONTROL

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(*Communicated by Michal Fečkan*)

ABSTRACT. This paper deals with a control problem governed by a semilinear state equation dependent on a small parameter $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, with a constrained control variable $u(t) \in C$, where $C \subset U$ is a closed, convex and bounded set containing the origin. It is proved that for a small ε the associated Hamilton-Jacobi equation has a unique strict solution; consequently, the control problem can be solved by employing a dynamic programming method.

Introduction and statement of results

This paper is concerned with the following problem (P):

(P) Minimize the functional

$$\mathrm{J}(u,x) = \int_0^T \Big[\frac{1}{2}\|u(t)\|_U^2 + g\big(y(t)\big)\Big] \, \mathrm{d}t + \phi_0\big(y(T)\big)$$

over the controls $u \in L^2(0,T;U)$, where y is subject to the state equation

$$\begin{aligned} y' &= Ay + f(\varepsilon, y) + Bu \qquad \text{on} \quad [0,T], \\ y(0) &= x, \quad x \in H, \qquad u(t) \in \mathrm{C}. \end{aligned}$$

Here U i H are real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_U$, $\langle\cdot,\cdot\rangle_H$ respectively.

Functions g and ϕ_0 are smooth and convex. f is a smooth function from H to H which goes uniformly to zero when $\varepsilon \to 0$. The operator

2000 Mathematics Subject Classification: Primary 93C99, 49L20.
Keywords: control problem, semilinear state equation, constrained control, Hamilton-Jacobi equation.

$A\colon D(A) \subset H \to H$ is the infinitenzimal generator of a $\mathbf{C}_0$-semigroup, $B \in L(U,H)$ and the set $C \subset U$ is a closed, convex and bounded set containing the origin.

We treat this problem by the dynamic programming method studying the corresponding Hamilton-Jacobi equation

$$\begin{aligned} \phi'_t(t,x) + F\big(\phi'_x(t,x)\big) - \big\langle Ax + f(\varepsilon,x),\, \phi'_x(t,x)\big\rangle &= g(x) \\ \text{for all} \quad (t,x) \in [0,T] \times D(A)\,, \\ \phi(0,x) &= \phi_0(x)\,, \end{aligned} \qquad (\mathrm{HJ})_\varepsilon$$

where

$$F(z) = \frac{1}{2}\Big[\|B^*z\|^2 - \|B^*z + P_c(-B^*z)\|^2\Big]\,, \tag{1}$$

and P_c denotes the projection from U to C.

In the case where $\varepsilon = 0$ and $H = \mathbb{R}^n$, equation $(\mathrm{HJ})_0$ has been studied by many authors under various assumptions on F, see [8], [9] et al. When H is the infinite dimensional, in [5], the existence and uniqueness of equation $(\mathrm{HJ})_0$ is proved in the case where F is a continuous Frechet differentiable and F' is bounded. These assumptions are suggested by problem (P) ($\varepsilon = 0$) with constrained controls $u(t) \in C$.

If $\varepsilon \neq 0$, then J is generally not convex and the method used in [10], without the constrained control, gives only the existence of the local solution of the Hamilton-Jacobi equation

$$\begin{aligned} \phi'_t(t,x) + \frac{1}{2}|\phi'_x(t,x)|^2 - \big\langle Ax + f(\varepsilon,x), \phi'_x(t,x)\big\rangle &= g(x) \\ \text{for all} \quad (t,x) \in [0,\delta] \times D(A)\,, \\ \phi(0,x) &= \phi_0(x)\,. \end{aligned} \qquad (\mathrm{HJ})_*$$

For smooth and convex functions g, ϕ_0 and f the existence and uniqueness of a global regular solution of equation $(\mathrm{HJ})_*$ was proved in [2].

The aim of this paper is to find a global strict solution of the Hamilton-Jacobi equation $(\mathrm{HJ})_\varepsilon$ for $|\varepsilon|$ sufficiently small, and then to show that this solution is equal to the value function. Subsequently, the dynamic programming method is used to prove the existence and uniqueness of the optimal control in the feedback form.

First we used a successive approximative method to construct the sequence $\{\phi^n\} \in B\big([0,T]; C^2(\Sigma_R) \cap K_R\big)$ and to show that this sequence converges to $\{\phi_{\lambda m}\} \in B\big([0,T]; C^2(\Sigma_R) \cap K_R\big)$ which satisfies the approximate Hamilton-Jacobi equation $(\mathrm{HJ})_{\lambda m}$. Then, when $m \to \infty$ and $\lambda \to 0$, we get the function ϕ_ε which is a strict solution of Hamilton-Jacobi equation $(\mathrm{HJ})_\varepsilon$. Section 1 contains the list of necessary notations and assumptions; in Section 2, it is described how to get the strict solution of Hamilton-Jacobi equation $(\mathrm{HJ})_\varepsilon$. Section 3 is devoted to the study how to get the solution of the optimal control problem, the feedback formula and finally one example is presented.

1. Notation

We begin with specifying the notation which will be used through this paper. Let X and Y be two Hilbert spaces. If $f\colon X \to Y$, by $f^{(k)}$ we shall denote the Frechet derivative of order $k \geq 1$. The following spaces of mappings f from X to Y will be used.

$C^k_\infty(X,Y)$ is the space of all k-time Frechet differentiable mappings $f\colon X \to Y$ such that f^j, $j = 0,1,2,\ldots,k$, are continuous and bounded on X. The space $C^k_\infty(X,Y)$ is endowed with the norm

$$|f|_{C^k_\infty(X,Y)} = \sum_{j=0}^{k} |f|_j ,$$

where $|f|_j = \sup\{|f^{(j)}(x)| : x \in X\}$.

$C^k_{\infty,\mathrm{Lip}}(X,Y)$ is the space of all $f \in C^k_\infty(X,Y)$ such that

$$\|f\|_k = \sup\left\{ \frac{|f^k(x) - f^k(y)|_Y}{|x-y|_X} : x \neq y ,\ x,y \in X \right\} < \infty .$$

$C^k_{\infty,\mathrm{Lip}}(X,Y)$ is the Banach space with the norm

$$|f|_{C^k_{\infty,\mathrm{Lip}}(X,Y)} = \sum_{j=0}^{k} |f|_j + \|f\|_k .$$

For every $r \in \mathbb{R}$, $r > 0$, we denote by Σ_r the closed ball $\{x \in X : |x| \leq r\}$ and by $C^k(\Sigma_r, Y)$ the space of all mappings $f\colon \Sigma_r \to Y$ which are Frechet differentiable up to the order k on Σ_r such that $f^{(j)}$, $j = 0,1,2,\ldots,k$, are continuous and bounded on Σ_r.

We shall denote by $C^k_{\mathrm{Lip}}(\Sigma_r, Y)$ the space of all $f \in C^k(\Sigma_r, Y)$ such that $f^{(k)}$ is Lipshitz on Σ_r, i.e.

$$\|f\|_{k,r} = \sup\left\{ \frac{|f^k(x) - f^k(y)|_Y}{|x-y|_X} : x \neq y ,\ x,y \in \Sigma_r \right\} < \infty .$$

The spaces $C^k(\Sigma_r, Y)$ and $C^k_{\mathrm{Lip}}(\Sigma_r, Y)$ are endowed with the norm

$$|f|_{C^k(\Sigma_r,Y)} = \sum_{j=0}^{k} |f|_{j,r}\,,$$

$$|f|_{C^k_{\mathrm{Lip}}(\Sigma_r,Y)} = \sum_{j=0}^{k} |f|_{j,r} + \|f\|_{k,r}\,,$$

where

$$|f|_{j,r} = \sup\{|f^{(j)}(x)| :\ x \in \Sigma_r\}\,.$$

$C^k(X,Y)$ is the space of all mappings $f\colon X \to Y$ which are k-time Frechet differentiable and their restrictions to every Σ_r belong to $C^k(\Sigma_r, Y)$. The space $C^k_{\mathrm{Lip}}(X,Y)$ is defined as the set of all $f \in C^k(X,Y)$ such that the restrictions of f at every Σ_r, $r > 0$, belong to $C^k_{\mathrm{Lip}}(\Sigma_r, Y)$. If $Y = \mathbb{R}$, we shall often write $C^k_\infty(X)$ and $C^k_{\infty,\mathrm{Lip}}(X)$ instead of $C^k_\infty(X,\mathbb{R})$ and $C^k_{\infty,\mathrm{Lip}}(X,\mathbb{R})$ respectively.

Now, let $\phi\colon [0,T] \times H \to \mathbb{R}$ and $k \geq 1$. We say that $\phi \in B([0,T], C^k(H))$ if ϕ satisfies the following conditions

(i) $\sup\limits_{0\le t\le T} |\phi(t,\cdot)|_{j,r} < +\infty$ for $j = 0,1,2,\dots,k$ and for all $r > 0$,

(ii) $\phi\colon [0,T] \times H \to \mathbb{R}$ is continuous,

(iii) $\phi_x\colon [0,T] \times H \to H$ is continuous,

(iv) $\frac{\partial^2\phi}{\partial x^2}$ is strongly continuous, that is the map $(t,x) \mapsto \frac{\partial^2\phi}{\partial x^2} y\colon [0,T]\times H \to H$ is continuous for all $y \in H$.

Further, we denote by K the subset of $C^1(H)$

$$K = \{\phi \in C^1(H) :\ \phi \text{ is convex and } \phi'(0) = 0\}\,,$$

and by K_r the subset of $C^1(\Sigma_R)$

$$K_r = \{\phi \in C^1(\Sigma_r) :\ \phi \text{ is convex and } \phi'(0) = 0\}\,.$$

It is easy to verify that function $F\colon H \to \mathbb{R}$ defined by (1) has the following properties:

a) F is Frechet differentiable and $F'(z) = -B(P_c(-B^* z))$,
b) $F(0) = 0$, F is convex,
c) $F \in C^1_{\infty,\mathrm{Lip}}(H)$, $|F'| \leq M$ fore some M.

We will make the regularization of the function $\phi \in K$, defined by

$$\phi_\lambda(x) = \inf_{y\in H}\left\{\phi(y) + \lambda F^*\Big(\frac{x-y}{\lambda}\Big)\right\}, \qquad \lambda > 0\,, \tag{2}$$

where F^* is the conjugate function of F i.e.

$$F^*(x) = \sup_y \{ \langle x, y \rangle - F(y) \} .$$

The property of F^* which will be needed later is

$$F^*(x) + F(y) \geq \langle x, y \rangle_H \tag{3}$$

(it follows from the definition of F^*) with equality holding if and only if $x = F'(y)$. If the $F(x) = \frac{1}{2}\|x\|^2$ $(C = U = H)$, the function ϕ_λ is reduced to the well-known regularization of the convex function.

Let $\phi \in K$ and $\lambda' = \|F'\phi'\|^{-1}$, for all the $\lambda \in \langle 0, \lambda' \rangle$ set

$$J_\lambda(x) = (1 + \lambda F'\phi')^{-1}(x) .$$

It is a well defined operator and it has the following properties:

a) $\|J_\lambda(x)\|_H \leq \|x\|_H + \lambda M$,

b) $\|J_\lambda\|_0 \leq \big(1 - \lambda\|F'\phi'\|\big)^{-1}$,

c) $\lim_{\lambda \to 0} \|J_\lambda(x) - x\|_H = 0$ uniformly.

Now, we collect the properties of ϕ_λ:

A) $\phi_\lambda(x) = \phi\big(J_\lambda(x)\big) + \lambda F^*\Big(\frac{x - J_\lambda(x)}{\lambda}\Big)$,

B) $\phi_\lambda(x) = \phi\big(J_\lambda(x)\big) + \lambda F^* F'\phi'\big(J_\lambda(x)\big)$,

C) $\lim_{\lambda \to 0} \phi_\lambda(x) = \phi(x)$ uniformly on bounded set,

D) $\lim_{\lambda \to 0} \frac{\phi(x) - \phi_\lambda(x)}{\lambda} = F\big(\phi'(x)\big)$ uniformly,

E) $\phi'_\lambda(x) = \phi'\big(J_\lambda(x)\big)$.

These properties were proved in [5].

2. Global existence and uniqueness of the solution for $(\mathrm{HJ})_\varepsilon$ equation

We want to get the strict solution of the Hamilton-Jacobi equation $(\mathrm{HJ})_\varepsilon$ for a sufficiently small $\varepsilon > 0$.

We say that the function $\phi \in B\big([0,T]; C^1_{\mathrm{Lip}}(\Sigma_r) \cap K_r\big)$ is a *strict solution* of $(\mathrm{HJ})_\varepsilon$ if $\phi(\cdot, x) \in C^1\big([0,T]\big)$ for all $x \in D(A)$ and satisfies the equation $(\mathrm{HJ})_\varepsilon$. Consequently, we consider the approximate Hamilton-Jacobi equation. Namely, the property (D) allows us to replace the bad term $F(\phi'_x)$ in $(\mathrm{HJ})_\varepsilon$ by $\big(\phi(x) - \phi_\lambda(x)\big)/\lambda$, and now the approximation $(\mathrm{HJ})_\lambda$ equation is

$$\begin{aligned} &\phi'_t(t,x) + \frac{\phi(t,x) - \phi_\lambda(t,x)}{\lambda} - \big\langle Ax + f(\varepsilon, x), \phi'_x(t,x) \big\rangle = g(x) \\ &\qquad\qquad \text{for all} \quad (t,x) \in [0,T] \times D(A), \\ &\phi(0,x) = \phi_0(x), \end{aligned} \tag{HJ_λ}$$

where $\phi_\lambda(t,x)$ is defined by (2).

Further, in order to get a strict solution of equation $(\mathrm{HJ})_\varepsilon$ we shall consider such mappings F which satisfy the following assumptions: there exists a set $\{F_m\} \subset C^2_\infty(H)$ such that

I. F_m is convex, $F_m(0)=0$, $|F'_m|_0 < M'$ for some M',
II. $F_m \to F$ uniformly on the bounded set,
III. $F'_m \to F'$ uniformly on the bounded set.

Then for the F_m and $\phi \in C^2_\infty(H) \cap K$ the function ϕ_λ has the following property:

$$\phi''_\lambda(x) = \phi''\big(J_{\lambda m}(x)\big) \cdot \Big(1 + \lambda F''_m \phi'\big(J_{\lambda m}(x)\big) \cdot \phi''\big(J_{\lambda m}(x)\big)\Big)^{-1} \qquad \text{for all} \quad x \in H\,, \quad \lambda \in \langle 0, \lambda''\rangle \tag{G}$$

where $\lambda'' = \min\{\lambda',\ \|F'_m\|^{-1}\|\phi'\|^{-1}\}$, and $J_{\lambda m}(x) = (1+\lambda F'_m\phi')^{-1}(x)$.

Now we consider the following approximate equation $(\mathrm{HJ})_{\lambda m}$

$$\begin{gathered}\phi'_t(t,x) + \frac{\phi(t,x) - \phi^m_\lambda(t,x)}{\lambda} - \big\langle Ax + f(\varepsilon,x),\ \phi'_x(t,x)\big\rangle = g(x)\,,\\ \phi(0,x) = \phi_0(x)\,,\end{gathered} \tag{$\mathrm{HJ}_{\lambda m}$}$$

where

$$\phi^m_\lambda(t,x) = \sup_{y\in H}\Big\{\phi(t,y) + \lambda F^*_m\Big(\frac{x-y}{\lambda}\Big)\Big\}\,, \qquad \lambda > 0\,.$$

Also, we need the following estimates:

LEMMA 1. *Let $\phi, \overline{\phi} \in C^1_{\mathrm{Lip}}(\Sigma_R) \cap K_R$, then*

$$\begin{aligned} |\phi^m_\lambda|_{i,R} &\le |\phi|_{i,R}\,, & i&=0,1\,,\\ \|\phi^m_\lambda\|_{1,R} &\le \|\phi\|_{1,R}\,, & &\\ |\phi^m_\lambda - \overline{\phi}^m_\lambda|_{i,R} &\le |\phi - \overline{\phi}|_{i,R}\,, & i&=0,1\,.\end{aligned}$$

It was proved in [1], [5], [11].

If ϕ is from $C^2(\Sigma_R) \cap K_R$, then $\|\phi\|_{1,R} = |\phi|_{2,R}$ and it follows that $|\phi^m_\lambda|_{2,R} \le |\phi|_{2,R}$. First, we prove, by using the method of a successive approximation, the existence of a solution $\phi_{\lambda m}$ of the integral form of the approximation equation $(\mathrm{HJ})_{\lambda m}$. Then we show that $\{\phi_{\lambda m}\}$ converges to a strict solution ϕ^ε of the $(\mathrm{HJ})_\varepsilon$ when $m \to \infty$ and $\lambda \to 0$.

Our hypotheses are the following:

(i) $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup in H and there exists $\omega \in \mathbb{R}$ such that $\langle Ax, x\rangle \le \omega|x|^2$ for all $x \in D(A)$.

(ii) $f(\varepsilon,\cdot) \in C^2(H,H)$ for all $\varepsilon \in \mathbb{R}$; $f(\varepsilon,0) = 0$, $\frac{\partial f(\varepsilon,0)}{\partial x} = 0$ for all $\varepsilon \in \mathbb{R}$; $f(\varepsilon,x) \to 0$ when $\varepsilon \to 0$ uniformly on $C^2(H,H)$.

(iii) $g, \phi_0 \in C^2(H) \cap K$ and $\langle g''(x)z, z\rangle \geq \mu|z|^2$ for all $x, z \in H$, and for $\mu > 0$ fixed; that is, g is strictly convex on H.

(iv) $C \subset U$ is a closed, convex and bounded set containing the origin.

We assume that in (i), $\omega < 0$, that is, A is strictly dissipative operator. The proof for the general case is the same; we have only to arrange the constants.

Using the characteristic method if ϕ satisfies $(\mathrm{HJ})_{\lambda m}$, then it satisfies the integral equation

$$\phi(t,x) = \mathrm{e}^{-t/\lambda}\phi_0\big(\xi_\varepsilon(0,t,x)\big) + \int_0^t \mathrm{e}^{-(t-s)/\lambda}\Big[g\big(\xi_\varepsilon(s,t,x)\big) + \frac{1}{\lambda}\phi_\lambda^m\big(s,\xi_\varepsilon(s,t,x)\big)\Big]\,\mathrm{d}s. \tag{$\mathrm{HJI}_{\lambda m}$}$$

Conversely, if ϕ satisfies $(\mathrm{HJI})_{\lambda m}$ and $x \in D(A)$, then ϕ satisfies $(\mathrm{HJ})_{\lambda m}$. In $(\mathrm{HJI})_{\lambda m}$, the function $\xi_\varepsilon(s,t,x)$ is the solution of the Cauchy problem

$$\begin{aligned} &\xi_\varepsilon'(s) = -A\xi_\varepsilon(s) - f\big(\varepsilon,\xi_\varepsilon(s)\big) && \text{for all } \ s \in [0,t\rangle\,, \\ &\xi_\varepsilon(t) = x\,, && x \in H\,,\ \ t \in [0,T]\,, \end{aligned} \tag{4}$$

i.e. ξ_ε is the family of characteristic curves for $(\mathrm{HJ})_{\lambda m}$. Some fundamental estimates for them are put together.

PROPOSITION 1. *Let* (i) *and* (ii) *be true. We fix $R > 0$ and let $t \in [0,T]$, $x \in H$ and $|x| \leq R$. There exists $\varepsilon_1(R) > 0$ such that if $|\varepsilon| \leq \varepsilon_1(R)$, then the Cauchy problem* (4) *has a unique mild solution $\xi_\varepsilon(\cdot,t,x)$ on $[0,T]$, $\xi_\varepsilon(s,t,\cdot) \in C^2(H,H)$ for all $s,t \in [0,T]$, $s < t$, and the following estimates hold*

$$|\xi_\varepsilon(s,t,x)| \leq |x|\,\mathrm{e}^{\omega(t-s)/2}\,,$$

$$|\xi_\varepsilon(s,t,\cdot)|_{1,R} \leq \mathrm{e}^{\omega(t-s)/2}\,,$$

$$\|\xi_{\varepsilon_{xx}}(s,t,x)(z,z)\|_H \leq |f(\varepsilon,\cdot)|_{2,R}\int_s^t \|\xi_{\varepsilon_x}(\sigma,t,x)\|_H^2\,\mathrm{e}^{\omega(\sigma-s)/2}\,\mathrm{d}\sigma\,.$$

This proposition was proved in [2].

Fixed $R > 0$, $m \in \mathbb{N}$ and $\lambda \in \langle 0,\lambda'\rangle$, where

$$\lambda' = \min\Big\{\frac{R}{2M}, \|F_m'\phi'\|^{-1}\Big\}\,,$$

the mapping

$$\begin{aligned} &(\Gamma_{\lambda m}\phi)(t,x) \\ &= \mathrm{e}^{-t/\lambda}\phi_0\big(\xi_\varepsilon(0,t,x)\big) + \int_0^t \mathrm{e}^{-(t-s)/\lambda}\Big[g\big(\xi_\varepsilon(s,t,x)\big) + \frac{1}{\lambda}\phi_\lambda^m\big(s,\xi_\varepsilon(s,t,x)\big)\Big]\,\mathrm{d}s \end{aligned}$$

is well defined from $B([0,T];C^2(\Sigma_R)\cap K_R)$ to $B([0,T];C^2(\Sigma_R))$. Furthermore, for functions $\phi \in B([0,T];C^2(\Sigma_R)\cap K_R)$ with

$$\sup_{0\le t\le T} |\phi(t,\cdot)|_{1,R} < c \qquad (c\in\mathbb{R},\ \ c>0)$$

there exist $\varepsilon(R,c) \in \langle 0,\varepsilon_1(R)\rangle$ such that for $|\varepsilon| < \varepsilon(R,c)$ we have

$$|f(\varepsilon,\cdot)|_{2,R} \le \frac{\mu}{|\phi_0|_{1,R} + |g|_{1,R} + c}.$$

Then the mapping $\Gamma_{\lambda m}$ is a convexity-preserving map, i.e.

$$\Gamma_{\lambda m}\phi \in B([0,T];C^2(\Sigma_R)\cap K_R).$$

To prove this we need only to prove that $\langle(\Gamma_{\lambda m}\phi)_{xx}(t,x)z,\ z\rangle \ge 0$. It was done in [2].

LEMMA 2. *Let* (i) – (iii) *hold true. Fix $R>0$ and set $L = L(R) = |\phi_0|_{C^2_{(\Sigma_R)}} + T|g|_{C^2_{(\Sigma_R)}}$. Then for each $\lambda \in \langle 0,\lambda''\rangle$, where*

$$\lambda'' = \min\left\{\frac{R}{2M}, \left(\sup\|F_m'\|\right)^{-1} L(R)^{-1}\right\},$$

and for each $m\in\mathbb{N}$ and $|\varepsilon| \le \varepsilon_2(R)$, where $\varepsilon_2(R) \in \langle 0,\varepsilon(R,L)\rangle$ ($\varepsilon(R,L) = \varepsilon(R,c)$, $c = L$), there exists a function $\phi_{\lambda m} \in B([0,T];C^2(\Sigma_R)\cap K_R)$ such that

(1) $\phi_{\lambda m}(t,x) = (\Gamma_{\lambda m}\phi_{\lambda m})(t,x)$ *on* $[0,T]\times\Sigma_R$,

(2) $\phi_{\lambda m}$ *is strict solution of* $(\mathrm{HJ})_{\lambda m}$,

(3) $\sup_{0\le t\le T} |\phi_{\lambda m}(t,\cdot)|_{C^2(\Sigma_R)} \le L(R)$.

P r o o f. These statements are the consequences of [2; Lemma 4.3]. Namely, in that lemma, for $|\varepsilon| < \varepsilon(R,L)|$, the sequence

$$\{\phi^n\} \subset B([0,T];C^2(\Sigma_R)\cap K_R)$$

was constructed by successive approximation method: $\phi^0 = \Gamma_{\lambda m}(0)$; $\phi^n = \Gamma_{\lambda m}(\phi^{n-1})$. Then for $|\varepsilon| < \varepsilon_2(R)$ the following was proved:

I. $\sup_{0\le t\le T} |\phi^n(t,\cdot)|_{C^2(\Sigma_R)} \le L(R)$ for all $n\in\mathbb{N}$.

II. $|\phi^n(t,\cdot) - \phi^{n-1}(t,\cdot)|_{C^2(\Sigma_R)} \le \frac{(L(R)+1)^{n+1}t^n}{\lambda^n n!}$.

III. $\Gamma_{\lambda m}$ is continuous on $B([0,T];C^1(\Sigma_R))$.

Now it follows from II that there exists $\phi_{\lambda m} \in B([0,T];C^2(\Sigma_R)\cap K_R)$ such that $\phi^n \to \phi_{\lambda m}$ in $B([0,T];C^2(\Sigma_R))$, and by III,

$$\phi_{\lambda m}(t,x) = \left(\Gamma_{\lambda m}\phi_{\lambda m}\right) \qquad \text{on} \quad [0,T]\times\Sigma_R. \tag{5}$$

Thereby for any $x\in D(A)$ the function $\phi_{\lambda m}(\cdot,x) \in C^1([0,T])$ and $\phi_{\lambda m}$ satisfy equation $(\mathrm{HJ})_{\lambda m}$. So it is a strict solution. Then using Ascoli-Arzela theorem like [1; pp. 40-41], and replacing ϕ_x with ϕ_{xx}, we may conclude that $\phi_{\lambda m} \in C^2(H)$. Further, because the left hand side in I is independent of m and λ, we have (3).

□

LEMMA 3. (Convergence of $\phi_{\lambda m} \to \phi_\lambda$, $m \to \infty$) *Let* (i) – (iii) *hold true. For a fixed $R > 0$ let $\varepsilon_2(R) > 0$ be as in Lemma 2. Let $\phi_{\lambda m} \in B([0,T]; C^2(\Sigma_R) \cap K_R)$ be a strict solution of* $(\mathrm{HJ})_{\lambda m}$ *for $\lambda \in \langle 0, \lambda'' \rangle$ and $m \in \mathbb{N}$. Then there exists $\phi_\lambda \in B([0,T]; C^1_{\mathrm{Lip}}(\Sigma_R) \cap K_R)$ such that $\phi_{\lambda m} \to \phi_\lambda$ in $B([0,T]; C^1(\Sigma_R))$ when $m \to \infty$. Further, ϕ_λ is the solution of the integral equation*

$$\begin{aligned}&\phi_\lambda(t,x)\\&= \mathrm{e}^{-t/\lambda}\,\phi_0\big(\xi_\varepsilon(0,t,x)\big) + \int_0^t \mathrm{e}^{-(t-s)/\lambda}\Big[g\big(\xi_\varepsilon(s,t,x)\big) + \frac{1}{\lambda}(\phi_\lambda)_\lambda\big(s,\xi_\varepsilon(s,t,x)\big)\Big]\,\mathrm{d}s\,,\end{aligned}$$

where $(\phi_\lambda)_\lambda$ if given by (2), *and ϕ_λ has the following properties:*

1') *ϕ_λ is a strict solution of* $(\mathrm{HJ})_\lambda$,
2') *$\phi_\lambda(t,\cdot) \in C^2(\Sigma_R) \cap K_R$ for all $t \in [0,T]$,*
3') $\sup\limits_{0 \le t \le T} |\phi_\lambda(t,\cdot)|_{C^2(\Sigma_R)} \le L(R)$.

P r o o f. Let $\phi_{\lambda m}$ be the solution of (5) and let $\phi_{\lambda n}$ be the solution of (5) with m replaced by n. Then we have

$$\begin{aligned}&\phi_{\lambda m}(t,x) - \phi_{\lambda n}(t,x)\\&\qquad= \frac{1}{\lambda}\int_0^t \mathrm{e}^{-(t-s)/\lambda}\Big[(\phi_{\lambda m})_{\lambda m}\big(s,\xi_\varepsilon(s,t,x)\big) - (\phi_{\lambda n})_{\lambda n}\big(s,\xi_\varepsilon(s,t,x)\big)\Big]\,\mathrm{d}s\,.\end{aligned}$$

For $\lambda \in \langle 0, \lambda'' \rangle$ we have

$$\begin{aligned}&(\phi_{\lambda m})_{\lambda m}\big(s,\xi_\varepsilon(s,t,x)\big)\\&\qquad= \phi_{\lambda m}\Big(s, J_{\lambda m}(\xi_\varepsilon(s,t,x)) - \lambda F_m^* F_m'(\phi_{\lambda m})_x'\big(s,\xi_\varepsilon(s,t,x)\big)\Big)\end{aligned}$$

and analogously for $\phi_{\lambda n}$. So

$$\begin{aligned}&\big|(\phi_{\lambda m})_{\lambda m}\big(s,\xi_\varepsilon(s,t,x)\big) - (\phi_{\lambda n})_{\lambda n}\big(s,\xi_\varepsilon(s,t,x)\big)\big|\\&\le \big|\phi_{\lambda m}\big(s, J_{\lambda m}(\xi_\varepsilon(s,t,x)\big) - \phi_{\lambda n}\big(s, J_{\lambda n}(\xi_\varepsilon(s,t,x)\big)\big|\\&\qquad + \lambda\big|F_n^* F_n'(\phi_{\lambda n})_x'\big(s,\xi_\varepsilon(s,t,x)\big) - F_m^* F_m'(\phi_{\lambda m})_x'\big(s,\xi_\varepsilon(s,t,x)\big)\big|\,.\end{aligned} \tag{6}$$

From the relation $\frac{x - J_{\lambda m}(x)}{\lambda} = F_m'(\phi_{\lambda m})_x'\big(t, J_{\lambda m}(x)\big)$, analogously for F_n', and the fact that $\{F_n'\}$ converges to F' uniformly on the bounded set, we get

$$\|J_{\lambda m}(x) - J_{\lambda n}(x)\|_H \le \varepsilon(n,m) + c|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{0,R}\,, \tag{7}$$

where $\varepsilon(n,m) \to 0$ when $m,n \to \infty$ and c denote different constants independent of m and n. Then for the first term on the right-hand side in (6) we have

$$\begin{aligned}&\big|\phi_{\lambda m}\big(s, J_{\lambda m}(\xi_\varepsilon(s,t,x))\big) - \phi_{\lambda n}\big(s, J_{\lambda n}(\xi_\varepsilon(s,t,x))\big)\big|\\&\quad\le \|\phi_{\lambda m}\| \cdot \big\|J_{\lambda m}\big(\xi_\varepsilon(s,t,x)\big) - J_{\lambda n}\big(\xi_\varepsilon(s,t,x)\big)\big\|_H + |\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{0,R}\\&\quad\le L(R)\varepsilon(n,m) + c|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{C^1(\Sigma_R)}\,.\end{aligned}$$

From equality (3), with $x = F'(y)$, for the second term on the right-hand side in (6) we get

$$\begin{aligned}&\big|F_n^* F_n'(\phi_{\lambda n})_x'(s,\xi_\varepsilon(s,t,x)) - F_m^* F_m'(\phi_{\lambda m})_x'(s,\xi_\varepsilon(s,t,x))\big|\\ &\le |F_n' - F_m'|\cdot L(R) + |F_n'||\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{1,R}\cdot L(R)\\ &\qquad + 2|F_n'||\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{1,R} + |F_m - F_n|_0\,.\end{aligned}$$

So we have that

$$\begin{aligned}&\big|(\phi_{\lambda m})_{\lambda m}(s,\xi_\varepsilon(s,t,x)) - (\phi_{\lambda n})_{\lambda n}(s,\xi_\varepsilon(s,t,x))\big|\\ &\le \varepsilon(n,m) + c|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{1,R} + \varepsilon(n,m)\lambda\cdot L(R)\\ &\qquad + (2\lambda M' + 1)|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{1,R}\\ &\le \tilde c\varepsilon(n,m) + c|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{C^1(\Sigma_R)}\,.\end{aligned} \tag{8}$$

Further

$$\begin{aligned}&(\phi_{\lambda m})_x'(t,x) - (\phi_{\lambda n})_x'(t,x)\\ &\quad = \frac{1}{\lambda}\int_0^t e^{-\frac{(t-s)}{\lambda}}\Big[(\phi_{\lambda m})_{\lambda m x}'(s,\xi_\varepsilon(s,t,x)) - (\phi_{\lambda n})_{\lambda n x}'(s,\xi_\varepsilon(s,t,x))\Big]\,\mathrm{d}s\,.\end{aligned}$$

From the property (E) and Lemma 1 we have

$$\begin{aligned}&\big|(\phi_{\lambda m})_{\lambda m x}'(s,\xi_\varepsilon(s,t,x)) - (\phi_{\lambda n})_{\lambda n x}'(s,\xi_\varepsilon(s,t,x))\big|\\ &\quad\le \big|(\phi_{\lambda m})_x'(s,J_{\lambda m}(\xi_\varepsilon(s,t,x))) - (\phi_{\lambda n})_x'(s,J_{\lambda n}(\xi_\varepsilon(s,t,x)))\big|\\ &\quad\le L(R)\varepsilon(n,m) + c|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{C^1(\Sigma_R)}\,.\end{aligned} \tag{9}$$

Now we get

$$\begin{aligned}&|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{C^1(\Sigma_R)}\\ &\le \frac{1}{\lambda}\int_0^t e^{-\frac{(t-s)}{\lambda}}\Big[\varepsilon(n,m) + c|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{C^1(\Sigma_R)}\Big]\,\mathrm{d}s\\ &\qquad + \frac{1}{\lambda}\int_0^t e^{-\frac{(t-s)}{\lambda}}\, e^{-\frac{\omega(t-s)}{2}}\Big[\varepsilon(n,m) + c_1|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{C^1(\Sigma_R)}\Big]\,\mathrm{d}s\,.\end{aligned}$$

Then

$$|\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{C^1(\Sigma_R)} \le \varepsilon(n,m) + c\int_0^t |\phi_{\lambda m}(s,\cdot) - \phi_{\lambda n}(s,\cdot)|_{C^1(\Sigma_R)}\,\mathrm{d}s\,,$$

and from the Gronwall's inequality we get

$$|\phi_{\lambda m}(s,\cdot)-\phi_{\lambda n}(s,\cdot)|_{C^1(\Sigma_R)}\le \varepsilon(n,m)\cdot \mathrm{e}^{ct}\,,\qquad c>0\,,\ \ 0\le t\le T\,,$$

where $\varepsilon(n,m)\to 0$ when $n,m\to\infty$. Therefore, $\{\phi_{\lambda m}\}_{m=0}^{\infty}$ is the Cauchy sequence in $B([0,T];C^1(\Sigma_R))$ and then there exists the function

$$\phi_\lambda\in B([0,T];C^1(\Sigma_R))$$

such that $\phi_{\lambda m}\to\phi_\lambda$ in $B([0,T];C^1(\Sigma_R))$ when $m\to\infty$. From (7) it follows that $J_{\lambda m}=(1+\lambda F_m^*\phi')^{-1}\to(1+\lambda F^*\phi')^{-1}$, and from (8), $(\phi_{\lambda m})_{\lambda m}\to(\phi_\lambda)_\lambda$, so that ϕ_λ satisfies

$$\begin{aligned}&\phi_\lambda(t,x)\\&=\mathrm{e}^{-\frac{t}{\lambda}}\phi_0(\xi_\varepsilon(0,t,x))+\frac{1}{\lambda}\int_0^t \mathrm{e}^{-\frac{(t-s)}{\lambda}}\Big[g(\xi_\varepsilon(s,t,x))+\frac{1}{\lambda}(\phi_\lambda)_\lambda(s,\xi_\varepsilon(s,t,x))\Big]\,\mathrm{d}s\,.\end{aligned}$$

For $x\in D(A)$ the function $\phi_\lambda(\cdot,x)\in C^1([0,T])$ and it satisfies equation $(\mathrm{HJ})_\lambda$. Then ϕ_λ is a strict solution of $(\mathrm{HJ})_\lambda$. Since the function $\phi_{\lambda m}$ satisfies (3) and the Ascolli-Arzela theorem, it follows that $\phi_{\lambda m}\in C^2(\Sigma_R)$ and 3') holds. □

THEOREM 1. *Let* (i) – (iii) *hold true. Let* $R>0$ *be fixed, and let* $\lambda\in\langle 0,\lambda''\rangle$ *and* $\varepsilon_2(R)$ *be as in Lemma* 2. *Let* $\phi_\lambda\in B([0,T];C^2(\Sigma_r)\cap K_R)$ *be a strict solution of equation* $(\mathrm{HJ})_\lambda$. *Then there exists* $\phi^\varepsilon\in B([0,T];C^1_{\mathrm{Lip}}(\Sigma_r)\cap K_R)$ *such that* $\phi_\lambda\to\phi^\varepsilon$ *in* $B([0,T];C^1(\Sigma_r)\cap K_R)$ *when* $\lambda\to 0$. *Function* ϕ^ε *has the following properties:*

1'') ϕ^ε *is a strict solution of* $(\mathrm{HJ})_\varepsilon$,
2'') $\phi^\varepsilon(t,\cdot)\in C^2(\Sigma_R)\cap K_R$ *for all* $t\in[0,T]$,
3'') $\sup\limits_{0\le t\le T}|\phi^\varepsilon(t,\cdot)|_{C^2(\Sigma_R)}\le L(R)$.

Proof. First for $\lambda,\mu\in\langle 0,\lambda''\rangle$ we define

$$R_{\mu\phi_\mu(t,\cdot)}(x)=\frac{1}{\mu}(\phi_\mu(t,x)-(\phi_\mu)_\mu(t,x))-F(\phi'_{\mu_x}(t,x))\,,$$

$$R_{\lambda\phi_\mu(t,\cdot)}(x)=\frac{1}{\lambda}(\phi_\mu(t,x)-(\phi_\mu)_\lambda(t,x))-F(\phi'_{\mu_x}(t,x))\,.$$

It is easy to prove that the following estimates hold

$$\begin{aligned}&|R_{\mu\phi_\mu(t,\cdot)}|_{0,R}\le\mu M^2L(R)(L(R)+1)\,,\qquad |R_{\mu\phi_\mu(t,\cdot)}|_{1,R}\le 2M^2L(R)\,,\\&|R_{\lambda\phi_\mu(t,\cdot)}|_{0,R}\le\lambda M^2L(R)(L(R)+1)\,,\qquad |R_{\lambda\phi_\mu(t,\cdot)}|_{1,R}\le 2M^2L(R)\,.\end{aligned}\tag{10}$$

Further, the function ϕ_μ satisfies the equation

$$\begin{aligned}&\phi_\mu(t,x)+\frac{1}{\lambda}(\phi_\mu(t,x)-(\phi_\mu)_\lambda(t,x))-\langle Ax+f(\varepsilon,x),\phi'_{\mu_x}(t,x)\rangle\\&=g(x)-R_{\mu\phi_\mu(t,\cdot)}(x)+R_{\lambda\phi_\mu(t,\cdot)}(x)\,,\\&\qquad\qquad\phi_\mu(0,x)=\phi_0(x)\,.\end{aligned}$$

Now, using integral form we get

$$
\begin{aligned}
&|\phi_\mu(t,\cdot)-\phi_\lambda(t,\cdot)|_{C^1(\Sigma_R)}\\
&\leq \frac{1}{\lambda}\int_0^t e^{-\frac{(t-s)}{\lambda}}\,|(\phi_\mu)_\lambda(s,\cdot)-(\phi_\lambda)_\lambda(s,\cdot)|_{0,R}\,\mathrm{d}s\\
&\qquad+\int_0^t e^{-\frac{(t-s)}{\lambda}}\left[|R_{\lambda\phi_\mu(t,\cdot)}|_{0,R}+|R_{\mu\phi_\mu(t,\cdot)}|_{0,R}\right]\mathrm{d}s\\
&\qquad+\frac{1}{\lambda}\int_0^t e^{-\frac{(t-s)}{\lambda}}\,|(\phi_\mu)_\lambda(s,\cdot)-(\phi_\lambda)_\lambda(s,\cdot)|_{1,R}|\xi_\varepsilon(s,t,\cdot)|_{0,R}\,\mathrm{d}s\\
&\qquad+\int_0^t e^{-\frac{(t-s)}{\lambda}}\left[|R_{\lambda\phi_\mu(t,\cdot)}|_{1,R}+|R_{\mu\phi_\mu(t,\cdot)}|_{1,R}\right]|\xi_\varepsilon(s,t,\cdot)|_{1,R}\,\mathrm{d}s\,.
\end{aligned}
$$

Therefore, from estimates (10) it follows

$$
|\phi_\mu(t,\cdot)-\phi_\lambda(t,\cdot)|_{C^1(\Sigma_R)}\leq c_1\int_0^t |\phi_\mu(s,\cdot)-\phi_\lambda(s,\cdot)|_{C^1(\Sigma_R)}\,\mathrm{d}s+c_2(\lambda+\mu)\,.
$$

By applying the Gronwall inequality we get

$$
|\phi_\mu(t,\cdot)-\phi_\lambda(t,\cdot)|_{C^1(\Sigma_R)}\leq c_2(\lambda+\mu)\,e^{c_1 t}\,,
$$

i.e. $\{\phi_\lambda\}$ is a Cauchy sequence in $B\big([0,T];C^1(\Sigma_r)\big)$. Then there exists the function $\phi^\varepsilon\in B\big([0,T];C^1(\Sigma_r)\big)$ such that $\phi_\lambda\to\phi^\varepsilon$ in $B\big([0,T];C^1(\Sigma_r)\big)$ when $\lambda\to 0$. To show 1'') we note that if $x\in D(A)$, then $\phi_\lambda\in C^1\big([0,T]\big)$ and

$$
\phi'_{\lambda_t}(t,x)=g(x)-R_{\lambda_{\phi\lambda(t,\cdot)}}(x)-F\big(\phi'_{\lambda_x}(t,x)\big)+\big\langle Ax+f(\varepsilon,x),\phi'_{\lambda_x}(t,x)\big\rangle\,,
$$

so $\phi'_{\lambda_t}(\cdot,x)\to\phi^{\varepsilon\prime}_t(\cdot,x)$ in $C\big([0,T]\big)$, then ϕ^ε is a strict solution of $(\mathrm{HJ})_\varepsilon$. Since functions ϕ_λ satisfy 2') and 3'), then from the Ascoli-Arzela theorem it follows that $\phi^\varepsilon\in C^2(\Sigma_r)$ and it satisfies 3''). To prove the uniqueness, see [1]. $\square$

3. Solution of the control problem

In this section we apply the results of Theorem 1 to solve the control problem (P). First we prove the fundamental identity.

LEMMA 4. *Let* (i) – (ii) *hold true. Let* $r \in \mathbb{R}$, $r > 0$, *be fixed. Let*

$$|x| \leq \frac{r}{4}, \qquad |Bu|_{L^2(0,T;H)} < M_1, \qquad R = r + \sqrt{T} M_1, \qquad \bar{\varepsilon} \leq \varepsilon_2(R) < \varepsilon_1(R).$$

Let y *be a mild solution of the state equation; the following fundamental identity holds*

$$\begin{aligned} \phi^\varepsilon(T-t,x) + \int_t^T \Big[F\big(\phi_x^{\varepsilon\prime}(T-s,y(s))\big) \\ + \langle \phi_x^\varepsilon(T-s,y(s)), Bu(s)\rangle + \tfrac{1}{2}\|u(s)\|_U^2 \Big] \, \mathrm{d}s \\ = \int_0^T \Big[g(y(s)) + \tfrac{1}{2}\|u(s)\|_U^2 \Big] \, \mathrm{d}s + \phi_0(y(T)) \qquad \text{for all} \quad (t,x) \in [0,T] \times \Sigma_R, \end{aligned} \tag{11}$$

where ϕ^ε *is a strict solution of equation* $(\mathrm{HJ})_\varepsilon$.

Proof. For $n > \omega$ set $x_n = n(n-A)^{-1}x$, $u_n(s) = n(n-A)^{-1}Bu(s)$ and let y_n be the solution of the following problem

$$\begin{aligned} y_n'(s) &= Ay_n(s) + f\big(\varepsilon, y_n(s)\big) + u_n(s), \qquad s \in [t,T], \\ y_n(t) &= x_n. \end{aligned}$$

Since $x_n \in D(A)$ and $u_n \in L^2(0,T : D(A))$, then $y_n \in C^1([0,T];H)$ and $y_n \to y$ in $C([t,T];H)$. Further, for all $x \in \Sigma_r$ the function $\phi^\varepsilon(\cdot,x) \in C^1([0,T])$, so the function $s \to \phi^\varepsilon(T-s, y_n(s)) \in C^1([t,T])$ and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\phi^\varepsilon\big(T-s,y_n(s)\big) \\ &\quad = -\phi_t^{\varepsilon\prime}\big(T-s,y_n(s)\big) + \big\langle \phi_x^{\varepsilon\prime}\big(T-s,y_n(s)\big), y_n'(s)\big\rangle \\ &\quad = F\big(\phi_x^{\varepsilon\prime}(T-s,y_n(s))\big) + \big\langle \big(\phi_x^{\varepsilon\prime}\big(T-s,y_n(s)\big), u_n(s) - g\big(y_n(s)\big)\big\rangle. \end{aligned}$$

Now, if we integrate this, we get

$$\phi^\varepsilon(T-t,y_n(t)) + \int_t^T \Big[F\big(\phi_x^{\varepsilon\prime}(T-s,y_n(s))\big) + \big\langle\big(\phi_x^{\varepsilon\prime}(T-s,y_n(s)\big),u_n(s)\big\rangle + \tfrac{1}{2}\|u_n(s)\|_U^2\Big]\,\mathrm{d}s$$

$$= \int_t^T \Big[g\big(y_n(s)\big) + \tfrac{1}{2}\|u_n(s)\|_U^2\Big]\,\mathrm{d}s + \phi_0\big(y_n(T)\big).$$

Hence, letting $n\to\infty$ we get (11).

From the fundamental identity (11) it follows that

$$\phi^\varepsilon(T-t,x) \le \int_t^T \Big[g\big(y(s)\big) + \tfrac{1}{2}\|u(s)\|_U^2\Big]\,\mathrm{d}s + \phi_0\big(y(T)\big) \tag{12}$$

for all (y,u) which satisfy the state equation. Namely, from the identity (11) it follows

$$\phi^\varepsilon(T-t,x) + \int_t^T \Big[F\big(\phi_x^{\varepsilon\prime}(T-s,y(s))\big) + F^*\big(-Bu(s)\big) + \big\langle\big(\phi_x^{\varepsilon\prime}(T-s,y(s)\big),Bu_n(s)\big)\big\rangle\Big]\,\mathrm{d}s$$

$$+ \int_t^T \Big[-F^*\big(-Bu(s)\big) + \tfrac{1}{2}\|u(s)\|_U^2\Big]\,\mathrm{d}s$$

$$= \int_t^T \Big[g\big(y(s)\big) + \tfrac{1}{2}\|u(s)\|_U^2\Big]\,\mathrm{d}s.$$

On the left-hand side, the first integral is greater then or equal to zero. It follows from (3). It remains to prove that the second integral on the left-hand side is also greater then or equal to zero. For $u\in C$ there holds

$$\|P_c(-B^*v) + B^*v\|_H^2 \le \|u + B^*v\|_H^2 \qquad \text{for all} \quad v\in U.$$

Then we have

$$\langle u,-B^*u\rangle - F(v) = \langle u,-B^*u\rangle - \frac{1}{2}\big[\|B^*v\|_H^2 - \|B^*v + P_c(-B^*v)\|_H^2\big] \le \frac{1}{2}\|u\|_U^2.$$

Hence $F^*\langle Bu\rangle = \sup\limits_v\big\{\langle u,-B^*v\rangle - F(v)\big\} \le \frac{1}{2}\|u\|_U^2$, so we prove the inequality (12). □

THEOREM 2. *Let $r_0 > 0$ be fixed, $|x| \le \frac{r_0}{4}$ and $\varepsilon_3(r_0)$ be such that $\varepsilon_3(r_0) < \varepsilon_2(R)$ $(R = r_0 + M_1\sqrt{T})$. Let ϕ^ε, for $|\varepsilon| < \varepsilon_3(r_0)$, be a strict solution of* $(\mathrm{HJ})_\varepsilon$ *and let w be the value function*

$$w(t,x) = \inf \int_t^T \left[\frac{1}{2}\|u(s)\|_U^2 + g\big(y(s)\big)\right] \mathrm{d}s + \phi_0\big(y(T)\big)\,,$$

where the infimum is taken over all (y,u) which satisfy the state equation in the mild sense. Then

(i) $\phi^\varepsilon(T-t,x) = w(t,x)$.

Moreover, if $(y_\varepsilon, u_\varepsilon)$ is the optimal pair of (P), *then we have*

(ii) $u_\varepsilon(s) = P_c\big(-B^*\phi_x^{\varepsilon'}(T-s, y_\varepsilon(s))\big)$.

P r o o f. Let ϕ^ε be a strict solution of $(\mathrm{HJ})_\varepsilon$ and let u_ε be defined as in (ii), where y_ε is a mild solution of

$$\begin{aligned} &y_\varepsilon'(s) = Ay_\varepsilon(s) + f\big(\varepsilon, y_\varepsilon(s)\big) - F'\big(\phi_x^{\varepsilon'}(T-s, y_\varepsilon(s))\big)\,, \\ &y_\varepsilon(t) = x \end{aligned} \tag{13}$$

$\big(F'\big(\phi_x^{\varepsilon'}(T-s,\cdot)\big)$ is a locally Lipshitz monotone operator on H), then (13) has a unique mild solution

$$y_\varepsilon \in C\big([t,T],H\big) \qquad \text{with} \quad |y_\varepsilon(s)| \le R \ \text{for all} \ s \in [t,T]\,.$$

Since $F'\big(\phi_x^{\varepsilon'}(T-s, y_\varepsilon(s))\big) = -Bu_\varepsilon(s)$, then from (3) it follows that

$$F'\big(\phi_x^{\varepsilon'}(T-s, y_\varepsilon(s))\big) + F'\big(-Bu_\varepsilon(s)\big) + \big\langle \phi_x^{\varepsilon'}(T-s, y_\varepsilon(s)), Bu_\varepsilon(s)\big\rangle = 0$$

and $F^*\big(-Bu_\varepsilon(s)\big) = \frac{1}{2}\|u_\varepsilon(s)\|_U^2$. From the fundamental identity it follows that

$$\begin{aligned} \phi^\varepsilon(T-t,x) &= \int_t^T \left[\frac{1}{2}\|u_\varepsilon(s)\|_U^2 + g\big(y_\varepsilon(s)\big)\right] \mathrm{d}s + \phi_0\big(y_\varepsilon(T)\big) \\ &\ge \inf \left\{ \begin{matrix} \displaystyle\int_t^T \left[\frac{1}{2}\|u(s)\|_U^2 + g\big(y(s)\big)\right] \mathrm{d}s + \phi_0\big(y(T)\big)\,, & s \in [t,T] \\ y_t'(s) = Ay(s) + f\big(\varepsilon, y(s)\big) + Bu(s)\,, \quad y(t) = x\,, & u \in C \end{matrix} \right\} \end{aligned} \tag{14}$$

i.e. $\phi^\varepsilon(T-t,x) \ge w(t,x)$. Now, from (12) we have

$$\begin{aligned} \phi^\varepsilon(T-t,x) &= \int_t^T \left[\frac{1}{2}\|u_\varepsilon(s)\|_U^2 + g\big(y_\varepsilon(s)\big)\right] \mathrm{d}s + \phi_0\big(y_\varepsilon(T)\big) \\ &\le \inf \left\{ \int_t^T \left[\frac{1}{2}\|u(s)\|_U^2 + g\big(y(s)\big)\right] \mathrm{d}s + \phi_0\big(y(T)\big) \right\}, \end{aligned}$$

where the infimum is taken over all (y, u) which satisfies the state equation (14). Then we get

$$\phi^{\varepsilon}(T-t, x) = w(t, x).$$

Moreover $(y_\varepsilon, u_\varepsilon)$ is a unique optimal pair of the problem (P). The optimal control u_ε is given by the feedback formula

$$u_\varepsilon(s) = P_c\big(-B^*\phi_x^{\varepsilon'}(T-s, y_\varepsilon(s))\big),$$

where $y_\varepsilon(s)$ is the solution of the closed-loop equation (13) for $t = 0$. □

EXAMPLE. Let $H = U = L^2(0,1)$ and $B = I$. First, we get an example for the subset $C \subset L^2(0,1)$. It is the same as in [5],

$$C = \{u \in L^2(0,1) : \ \|u\| \le R\},$$

where $R > 0$ is a given constant and the projection P_c is given by

$$P_c(u) = \begin{cases} u & \text{if } \|u\| \le R, \\ R\frac{u}{\|u\|} & \text{if } \|u\| > R. \end{cases}$$

Therefore, the function F defined in (1) is

$$F(z) = \begin{cases} \frac{1}{2}\|z\|^2 & \text{if } \|z\| \le R, \\ R\|z\| - \frac{1}{2}R^2 & \text{if } \|z\| > R. \end{cases}$$

If we set

$$\gamma(t) = \begin{cases} \frac{1}{2}t & 0 \le t \le M^2, \\ M\sqrt{t} - \frac{1}{2}M^2, & M^2 < t, \end{cases}$$

we get

$$F(z) = \gamma(\|z\|^2).$$

We take the function $j \in C_0^\infty(\mathbb{R})$ such that $j \ge 0$, $\operatorname{supp} j \subset \langle -1, 1\rangle$, $\int\limits_{\mathbb{R}} j(t)\ \mathrm{d}t = 1$. Set $j_m(t) = \frac{1}{m} j\big(\frac{1}{m}\big)$ and $\gamma_m(t) = \int\limits_{\mathbb{R}} j_m(t-s)\gamma(s)\ \mathrm{d}s$; then we define

$$F_m(z) = \gamma_m(\|z\|^2) - \gamma_m(0) \qquad (\gamma(t) = -\gamma(-t),\ \ t < 0).$$

The functions F_m satisfy the properties I – III on page 114. Now we consider the problem:

(P_1) Minimize the cost

$$J(y_0, u) = \int\limits_0^T \Big[\frac{1}{2}|y(s)|^2 + \frac{1}{2}|u(s)|^2\Big]\ \mathrm{d}s + \frac{1}{2}|y(T)|^2$$

over the all controls $u \in L^2(0,T : L^2(0,1)) = L^2(Q)$ ($Q = [0,T]\times\langle 0,1\rangle$), where y is the mild solution of

$$y'_t(t,x) = \Delta_x y(t,x) + \varepsilon f\big(y(t,x)\big) + u(t,x)\,, \qquad t\in[0,T]\,,\ \ x\in\langle 0,1\rangle\,,$$
$$y(0,x) = y_0(x) \in L^2(0,1)\,, \qquad y(t,0) = y(t,1) = 0\,,$$
$$u(t) \in C\,.$$

We denote by A the operator $\Delta_x = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with $D(A) = H^2(0,1)\cap H_0^1(0,1)$ and let $f \in C^2(\mathbb{R})$, then for $z \in L^2(0,1)$ the function $(f\circ z)(x) = f\big(z(x)\big)$ is from $C^2(H)$. The assumptions (i)–(iv) are verified, particularly, $(Ay,y) \le -c_0|y|^2$ for all $y \in D(A)$, where $c_0 > 0$. Then according to Theorem 1 and Theorem 2 there exists a unique optimal pair $(y_\varepsilon, u_\varepsilon)$ for the problem (P_1) and the following feedback formula holds

$$u_\varepsilon(s) = P_c\big(-\phi_x^\varepsilon(T-s, y_\varepsilon(s))\big)\,,$$

where ϕ^ε is the strict solution of $(\mathrm{HJ})_\varepsilon$ equations.

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Received October 20, 1999
Revised September 21, 2000

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