Thiruvaiyaru V. Panchapagesan
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WEAK COMPACTNESS OF UNCONDITIONALLY CONVERGENT OPERATORS ON $C_0(T)$

T. V. PANCHAPAGESAN

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ABSTRACT. Let T be a locally compact Hausdorff space and let $C_0(T)$ be the Banach space of all complex valued continuous functions vanishing at infinity in T, provided with the supremum norm. Let X be a locally convex Hausdorff space (briefly, an lcHs) which is quasicomplete. By using Rosenthal's lemma and the locally convex space analogue of the Bartle-Dunford-Schwartz representation theorem it is shown that every X-valued unconditionally convergent operator on $C_0(T)$ is weakly compact. Then it is deduced that every continuous linear map $u: C_0(T) \to X$ is weakly compact if $c_0 \not\subset X$.

1. Introduction

Let T be a locally compact Hausdorff space and $C_0(T)$ the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with the supremum norm. Let $M(T)$ be the Banach dual of $C_0(T)$.

If X is a Banach space and K is a compact Hausdorff space, then P e ł c z y ń - s k i [10] showed that every X-valued unconditionally convergent operator on $C(K)$ is weakly compact. This result was extended in [8; Theorem 12] to unconditionally convergent continuous linear maps on $C_0(T)$ with values in a locally convex Hausdorff space (briefly, an lcHs) which is quasicomplete.

In [1] Rosenthal's lemma and the Bartle-Dunford-Schwartz representation theorem are used to obtain the above mentioned theorem of P e ł c z y ń s k i. See [1; Theorem VI.2.15, Corollary VI.2.17]. Since the Bartle-Dunford-Schwartz representation theorem has been generalized in [8] to quasicomplete lcHs-valued

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continuous linear mappings on $C_0(T)$, the following question arises: Is it possible to give an alternative proof of [8; Theorem 12] using Rosenthal's lemma and the generalized Bartle-Dunford-Schwartz representation theorem? The aim of the present note is to answer the question in the affirmative.

2. Preliminaries

In this section we fix notation and terminology. For the convenience of the reader we also give some definitions and results from the literature.

In the sequel, T will denote a locally compact Hausdorff space and $C_0(T)$ the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with the supremum norm $\|f\|_T = \sup_{t\in T}|f(t)|$.

DEFINITION 1. Let $\mathcal{B}(T)$ be the σ-algebra of Borel sets of T. A complex measure μ on $\mathcal{B}(T)$ is said to be *Borel-regular* (resp. *Borel-outer regular*) if, given $E \in \mathcal{B}(T)$ and $\varepsilon > 0$, there exist a compact K and an open set U in T with $K \subset E \subset U$ (resp. an open set U in T with $E \subset U$) such that $|\mu(B)| < \varepsilon$ for every $B \in \mathcal{B}(T)$ with $B \subset U \setminus K$ (resp. $B \subset U \setminus E$).

$M(T)$ is the Banach dual of $C_0(T)$ and is the Banach space of all bounded complex Radon measures on T. Consequently, $M(T)$ is identified with the Banach space of all regular (bounded) complex Borel measures μ on $\mathcal{B}(T)$ with norm $\|\mu\| = \mathrm{var}(\mu, \mathcal{B}(T))(T)$ where the variation of μ is taken with respect to $\mathcal{B}(T)$. We denote $\mathrm{var}(\mu, \mathcal{B}(T))(E)$ by $|\mu|(E)$, for $E \in \mathcal{B}(T)$.

A *vector measure* is an additive set function defined on a ring of sets with values in an lcHs. In the sequel X denotes an lcHs with topology τ. Γ is the set of all τ-continuous seminorms on X. The dual of X is denoted by X^*.

The strong topology $\beta(X^*, X)$ of X^* is the locally convex topology induced by the seminorms $\{p_B : B \text{ bounded in } X\}$, where $p_B(x^*) = \sup_{x\in B}|x^*(x)|$. X^{**} denotes the dual of $\big(X^*, \beta(X^*, X)\big)$ and is endowed with the locally convex topology τ_e of uniform convergence on equicontinuous subsets of X^*. Note that $\big(X^*, \beta(X^*, X)\big)$ and (X^{**}, τ_e) are lcHs.

It is well known that the canonical injection $J\colon X \to X^{**}$, given by $\langle Jx, x^*\rangle = \langle x, x^*\rangle$ for all $x \in X$ and $x^* \in X^*$, is linear. On identifying X with $JX \subset X^{**}$, one has $\tau_e|_{JX} = \tau_e|_X = \tau$.

Let $\mathcal{E} = \{A \subset X^* : A \text{ is equicontinuous}\}$. Then the family of seminorms $\Gamma_{\mathcal{E}} = \{p_A : A \in \mathcal{E}\}$ induces the topology τ of X and the topology τ_e of X^{**}, where $p_A(x) = \sup_{x^*\in A}|x^*(x)|$ for $x \in X$ and $p_A(x^{**}) = \sup_{x^*\in A}|x^{**}(x^*)|$ for $x^{**} \in X^{**}$.

DEFINITION 2. A linear map $u\colon C_0(T) \to X$ is called a *weakly compact operator* on $C_0(T)$ if $\{uf : \|f\|_T \le 1\}$ is relatively weakly compact in X.

The following result is the same as [8; Lemma 2], where the hypothesis of quasicompleteness of X is redundant.

PROPOSITION 1. *Let X be an lcHs and let $u\colon C_0(T) \to X$ be a continuous linear map. Then u^*A is bounded in $M(T)$ for each $A \in \mathcal{E}$.*

The following result ([3; Corollary 9.3.2] of Edwards which is essentially due to [4; Lemma 1] of Grothendieck) plays a key role in Section 3.

PROPOSITION 2. *Let E and F be lcHs with F quasicomplete and let $u\colon E \to F$ be linear and continuous. Then the following assertions are equivalent:*

(i) $u^{**}(E^{**}) \subset F$.

(ii) *u maps bounded subsets of E into relatively weakly compact subsets of F.*

(iii) *$u^*(A)$ is relatively $\sigma(E^*, E^{**})$-compact for each equicontinuous subset A of F^*.*

The following result is due to [4; Theorem 2] of Grothendieck, and is needed in Section 3.

PROPOSITION 3. *A bounded set A in $M(T)$ is relatively weakly compact if and only if, for each disjoint sequence $(U_n)_1^\infty$ of open sets in T,*

$$\sup_{\mu \in A} |\mu(U_n)| \to 0 \qquad as \quad n \to \infty.$$

For each τ-continuous seminorm p on X, let $p(x) = \|x\|_p$, $x \in X$, and let $X_p = (X, \|\cdot\|_p)$ be the associated seminormed space. The completion of the quotient normed space $X_p/p^{-1}(0)$ is denoted by $\widetilde{X}_p$. Let $\Pi_p\colon X_p \to X_p/p^{-1}(0) \subset \widetilde{X}_p$ be the canonical quotient map.

Let $\mathcal{S}$ be a σ-algebra of subsets of a non empty set Ω. An X-valued vector measure m on $\mathcal{S}$ is said to be *bounded* if $\{m(E) : E \in \mathcal{S}\}$ is bounded in X.

For the theory of integration of bounded $\mathcal{S}$-measurable scalar functions with respect to a bounded quasicomplete lcHs-valued vector measure defined on the σ-algebra $\mathcal{S}$, the reader may refer to [7] or [8]. We need the following results from [7; Lemma 6] and [8; Proposition 7].

PROPOSITION 4. *Let X be a quasicomplete lcHs and let $\mathcal{S}$ be a σ-algebra of subsets of Ω. Then:*

(i) *If f is a bounded $\mathcal{S}$-measurable scalar function and m is an X-valued bounded vector measure on $\mathcal{S}$, then f is m-integrable in Ω and*

$$x^*\Bigl(\int_\Omega f \,\mathrm{d}m\Bigr) = \int_\Omega f \,\mathrm{d}(x^* \circ m) \qquad \textit{for each} \quad x^* \in X^*.$$

(ii) (*Lebesgue bounded convergence theorem*) *If m is an X-valued σ-additive vector measure on $\mathcal{S}$ and (f_n) is a bounded sequence of $\mathcal{S}$-measurable scalar functions with $\lim_n f_n(w) = f(w)$ for each $w \in \Omega$, then f is m-integrable in each E in $\mathcal{S}$ and*

$$\int_E f \, \mathrm{d}m = \lim_n \int_E f_n \, \mathrm{d}m \qquad \textit{for each} \quad E \in \mathcal{S}.$$

The following result is due to the first part of [8; Theorem 1] and it is the locally convex space analogue of the Bartle-Dunford-Schwartz theorem (see [1; Theorem VI.2.1]) for continuous linear maps on $C_0(T)$. It plays a vital role in Section 3.

PROPOSITION 5 (GENERALIZED BARTLE-DUNFORD-SCHWARTZ REPRESENTATION THEOREM). *Let X be an lcHs and let $u\colon C_0(T) \to X$ be a continuous linear map. Then there exists a unique X^{**}-valued vector measure m on $\mathcal{B}(T)$ satisfying the following properties:*

(i) *$(x^* \circ m) \in M(T)$ for each $x^* \in X^*$ and consequently, $m\colon \mathcal{B}(T) \to X^{**}$ is σ-additive in $\sigma(X^{**}, X^*)$-topology.*

(ii) *The mapping $x^* \mapsto x^* \circ m$ of X^* into $M(T)$ is weak*-weak* continuous. Moreover, $u^* x^* = x^* \circ m$, $x^* \in X^*$.*

(iii) *$x^* u f = \int_T f \, \mathrm{d}(x^* \circ m)$ for each $f \in C_0(T)$ and $x^* \in X^*$.*

(iv) *The range of m is τ_e-bounded in X^{**}.*

(v) *$m(E) = u^{**}(\chi_E)$ for $E \in \mathcal{B}(T)$.*

(vi) *Moreover, if X is quasicomplete, then by* (iii) *and* (iv) *and by Proposition* 4(i), *$uf = \int_T f \, \mathrm{d}m$ for $f \in C_0(T)$.*

DEFINITION 3. Let $u\colon C_0(T) \to X$ be a continuous linear map. Then the vector measure m, as given in Proposition 5, is called the *representing measure* of u.

DEFINITION 4. Let X and Y be quasicomplete lcHs and let $u\colon X \to Y$ be a continuous linear map. Then u is called an *unconditionally convergent operator* if for every unconditonally weakly Cauchy series $\sum_1^\infty x_n$ in X (in the sense that $(x_n)_1^\infty \subset X$ with $\sum_1^\infty |x^*(x_n)| < \infty$ for all $x^* \in X^*$), the series $\sum_1^\infty u(x_n)$ is unconditionally convergent in Y.

3. Main theorem

The aim of the present section is to use Rosenthal's lemma ([1; p. 18]) and Proposition 5 to provide an alternative proof of [8; Theorem 12].

LEMMA 1. *Let X be a quasicomplete lcHs and let $u\colon C_0(T) \to X$ be a non weakly compact continuous linear map. Then $C_0(T)$ contains a subspace Y isometrically isomorphic with c_0. Moreover, there exists an equicontinuous set A in X^* such that $\Pi_{p_A} \circ u$ is a topological isomorphism of Y into $\widetilde{X}_{p_A}$ (see the notation given after Proposition 3).*

Proof. Since u is not weakly compact, by Proposition 2 there exists an equicontinuous subset A of X^* such that u^*A is not relatively weakly compact in $M(T)$. By Proposition 1, u^*A is bounded in $M(T)$ and hence, by Propositions 3 and 5(ii) there exist a disjoint sequence $(U_n)_1^\infty$ of open sets in T and an $\varepsilon > 0$ such that $\sup\limits_{x^*\in A} |(x^* \circ m)(U_n)| > 2\varepsilon$ for each n. Consequently, there exists $x_n^* \in A$ such that $|(x_n^* \circ m)(U_n)| > 2\varepsilon$ for each n. Since $x_n^* \circ m$ is Borel regular by Proposition 5(i), there exists a compact $K_n \subset U_n$ such that $|(x_n^* \circ m)(K_n)| > 2\varepsilon$.

Let $D(K_n) = \{U : U \text{ open}, K_n \subset U \subset U_n\}$ and for $U, V \in D(K_n)$, let $U \geq V$ if $U \subset V$. Then by the Borel outer regularity of $x_n^* \circ m$ in the set K_n, there exists $\widehat{U}_n \in D(K_n)$ such that $|(x_n^* \circ m)(U \setminus K_n)| < \varepsilon$ for all $U \geq \widehat{U}_n$, $U \in D(K_n)$. Then by [5; Theorem 50.D] of Halmos, we can choose an open Baire set V_n such that $K_n \subset V_n \subset \widehat{U}_n$ so that $|(x_n^* \circ m)(V_n \setminus K_n)| < \varepsilon$. Consequently, $|(x_n^* \circ m)(V_n)| \geq |(x_n^* \circ m)(K_n)| - |(x_n^* \circ m)(V_n \setminus K_n)| > \varepsilon$. Thus $|(x_n^* \circ m)(V_n)| > \varepsilon$ for each n.

CLAIM 1. *Suppose V is an open Baire set in T with $|(x^* \circ m)(V)| > \varepsilon$. Let $C_c(T)$ be the set of all continuous complex functions with compact support in T. Then there exists $f \in C_c(T)$ with support contained in V such that $0 \leq f \leq \chi_V$, $\|f\|_T = 1$ and $\left|\int_T f \,\mathrm{d}(x^*\circ m)\right| > \varepsilon$. Consequently, there exists a sequence $(f_n)_1^\infty \subset C_c(T)$ such that* $\operatorname{supp} f_n \subset V_n$, $\|f_n\|_T = 1$, $0 \leq f_n \leq \chi_{V_n}$ *and $\left|\int_T f_n \,\mathrm{d}(x_n^*\circ m)\right| > \varepsilon$ for all n.*

In fact, by [2; §14] of Dinculeanu, V is a countable union of compact G_δ's and hence, there exists a sequence $(C_k)_{k=1}^\infty$ of compact G_δ's such that $C_k \nearrow V$. Now by Urysohn's lemma there exists $h_k \in C_c(T)$ with $\operatorname{supp} h_k \subset V$ such that $0 \leq h_k \leq \chi_V$ with $h_k(t) = 1$ for all $t \in C_k$. Let $g_p = \max\limits_{1\leq k\leq p} h_k$. Then $(g_p)_{p=1}^\infty \subset C_c(T)$ and $g_p \nearrow \chi_V$. Then by the Lebesgue dominated convergence theorem there exists $p_0 \in \mathbb{N}$ such that $\left|\int_T g_{p_0} \,\mathrm{d}(x^*\circ m)\right| > \varepsilon$. Taking $f = g_{p_0}$, the first part of the claim is established. The second part is immediate from the first as $|(x_n^* \circ m)(V_n)| > \varepsilon$ for all n.

By Propositions 1 and 5(ii), $(x_n^* \circ m)_{n=1}^{\infty}$ is uniformly bounded in $M(T)$. Then by Rosenthal's lemma (see [1; p. 18], which holds for complex measures too) we can assume that the sequences (x_n^*) and (V_n) have been chosen such that

$$|(x_n^* \circ m)(V_n)| > \varepsilon \quad \text{and} \qquad |x_n^* \circ m|\Big(\bigcup_{p\neq n} V_p\Big) < \frac{\varepsilon}{2} \qquad \text{for all} \quad n. \qquad (1)$$

Then by Claim 1 and Proposition 5(iii) we have

$$|x_n^* u f_n| = \Big|\int_T f_n \, \mathrm{d}(x_n^* \circ m)\Big| > \varepsilon \qquad \text{for all} \quad n. \qquad (2)$$

Moreover, $\|f_n\|_T = 1$ and $\operatorname{supp} f_n \subset V_n$ for all n.

Let $Y = \Big\{ \sum_{n=1}^{\infty} \alpha_n f_n : (\alpha_n)_{n=1}^{\infty} \in c_0 \Big\}$ be provided with the supremum norm. Then $Y \subset C_0(T)$. As $\|f_n\|_T = 1$ for all n and as (f_n) have disjoint supports, Y is isometrically isomorphic with c_0. Moreover, if $f = \sum_{n-1}^{\infty} \alpha_n f_n$ for some sequence $(\alpha_n)_{n=1}^{\infty} \in c_0$, then by (1) and (2) we have for each n

$$\begin{aligned}
|x_n^* u(f)| &= \Big|\int_T f \, \mathrm{d}(x_n^* \circ m)\Big| \\
&= \Big|\alpha_n \int_T f_n \, \mathrm{d}(x_n^* \circ m) + \int_{\bigcup_{p\neq n} V_p} f \, \mathrm{d}(x_n^* \circ m)\Big| \\
&\geq |\alpha_n|\varepsilon - \int_{\bigcup_{p\neq n} V_p} |f| \, \mathrm{d}(|x_n^* \circ m|) \\
&\geq |\alpha_n|\varepsilon - |x_n^* \circ m|\Big(\bigcup_{p\neq n} V_p\Big)\|f\|_T \\
&> |\alpha_n|\varepsilon - \frac{\varepsilon}{2}\|f\|_T .
\end{aligned}$$

But $\|f\|_T = \sup_n |\alpha_n|$ and hence

$$\begin{aligned}
\|(\Pi_{p_A} \circ u)(f)\|_{p_A} = p_A(uf) &= \sup_{x^* \in A} |(x^* u)(f)| \\
&\geq \sup_n |(x_n^* u)(f)| \geq \varepsilon\|f\|_T - \Big(\frac{\varepsilon}{2}\Big)\|f\|_T = \Big(\frac{\varepsilon}{2}\Big) \|f\|_T .
\end{aligned}$$

where $\Pi_{p_A} : X_{p_A} \to X_{p_A}/p_A^{-1}(0) \subset \widetilde{X}_{p_A}$ is the canonical quotient map.

Hence $(\Pi_{p_A} \circ u)|_Y$ is a topological isomorphism of Y onto a subspace of $\widetilde{X}_{p_A}$ and this completes the proof of the lemma. □

COROLLARY 1. *If X is a Banach space and $c_0 \not\subset X$, then every continuous linear map $u\colon C_0(T) \to X$ is weakly compact.*

Remark 1. In the proof of [1; Theorem VI.2.15] there is no hint nor reference as to the construction of the sequence (f_n) in $C(\Omega)$ with the desired properties. One has to invoke [5; Theorem 50.D] and the fact that every open Baire set is a countable union of compact G_δ's. (See the proof of Claim 1 above.)

The following result is due to Grothendieck [4]. An alternative measure theoretic proof is given in [8]. For the sake of completeness we include the proof as in [8].

LEMMA 2. *$C_0(T)$ has the strict Dunford-Pettis property (briefly, (SDP)-property). That is, for each weakly compact operator $u\colon C_0(T) \to X$, X a quasicomplete lcHs, u transforms each weak Cauchy sequence in $C_0(T)$ into a convergent sequence in X.*

Proof. If (f_n) is weakly Cauchy in $C_0(T)$, then it is a norm bounded sequence converging pointwise to some function f in T and clearly, f is also bounded and Borel measurable. By Proposition 2 and by the fact that $m(E) = u^{**}(\chi_E)$ for $E \in \mathcal{B}(T)$ (see Proposition 5(v)), the representing measure m has range in X and consequently, by Proposition 5(i) and by the Orlicz-Pettis theorem for lcHs (see [6]) m is σ-additive in τ. Then by (vi) of Proposition 5 and (ii) of Proposition 4 we have

$$\lim_n u f_n = \lim_n \int_T f_n \, \mathrm{d}m = \int_T f \, \mathrm{d}m \in X .$$

Hence the result holds. This completes the proof of the lemma. □

From Lemmas 1 and 2 we shall now deduce the main theorem (which is the same as [8; Theorem 12]).

THEOREM 1. ([8; Theorem 12]) *Let $u\colon C_0(T) \to X$ be a continuous linear map and let X be a quasicomplete lcHs. Then the following are equivalent:*

(i) *u is unconditionally convergent.*
(ii) *u is weakly compact.*
(iii) *u maps sequences that tend to zero weakly into sequences convergent to zero.*
(iv) *u maps weak Cauchy sequences into τ-Cauchy sequences.*
(v) *If (f_n) is a bounded sequence in $C_0(T)$ with $f_n \cdot f_l = 0$ for $n \neq l$, then $\lim_n u(f_n) = 0$.*

Proof. Let m be the representing measure of u.

(i) $\Longrightarrow$ (ii):
If $u\colon C_0(T) \to X$ is not weakly compact, then by Lemma 1 there exist an equicontinuous subset A of X^* and a subspace Y of $C_0(T)$ isometrically isomorphic with c_0 such that $(\Pi_{p_A} \circ u)|_Y$ is a topological isomorphism onto a subspace of $\widetilde{X}_{p_A}$. By (i), $\Pi_{p_A} \circ u$ maps weakly unconditionally Cauchy series in Y into unconditionally convergent series in $\widetilde{X}_{p_A}$. This is impossible as c_0 contains plenty of nonconvergent weakly unconditionally Cauchy series. Hence u is weakly compact.

(ii) $\Longrightarrow$ (i) by Lemma 2 and the Orlicz-Pettis theorem.

(ii) $\Longrightarrow$ (iv) by Lemma 2.

(iv) $\Longrightarrow$ (iii):
Let $(f_n)_1^\infty$ be weakly convergent to zero in $C_0(T)$. Then it is norm bounded and converges to zero pointwise. Then by Proposition 5(i) and by the Lebesgue bounded convergence theorem, $\lim_n \int_T f_n \, \mathrm{d}(x^*\circ m) = 0$ for each $x^* \in X^*$. Then by Proposition 5(iii), $\lim_n x^*u(f_n) = 0$ for each $x^* \in X^*$. Since by hypothesis (iv), there exists $\omega \in X$ such that $\lim_n u(f_n) = \omega$, by the Hahn-Banach theorem $\omega = 0$ and hence (iii) holds.

(iii) $\Longrightarrow$ (v):
Such a norm bounded sequence (f_n) converges to zero pointwise and hence by Proposition 5(i) and by the Lebesgue bounded convergence theorem, $\lim_n \int_T f_n \, \mathrm{d}(x^*\circ m) = 0$ for each $x^* \in X^*$. Then by Proposition 5(iii), $(u(f_n))$ is weakly convergent to zero. Then by hypothesis (iii), $\lim_n u(f_n) = 0$ in τ.

(v) $\Longrightarrow$ (ii):
If u is not weakly compact, then as in the proof of Lemma 1 we have an equicontinuous subset A of X^*, an $\varepsilon > 0$, a sequence $(f_n)_1^\infty \subset C_0(T)$ with disjoint supports such that $\|f_n\|_T = 1$ for all n and a sequence $(x_n^*)_1^\infty$ in A with $\left|\int_T f_n \, \mathrm{d}(x_n^*\circ m)\right| > \varepsilon$ for all n. Then by Proposition 5(iii), $\|u(f_n)\|_{p_A} \geq |x_n^*(uf_n)| = \left|\int_T f_n \, \mathrm{d}(x_n^*\circ m)\right| > \varepsilon$ for all n. This contradicts (v) and hence u is weakly compact. This completes the proof of the theorem. □

Now we deduce the first part of [11; Theorem 5.3] of Thomas as a corollary of the above theorem.

COROLLARY 2. (First part of [11; Theorem 5.3]) *Let X be a quasicomplete lcHs with $c_0 \not\subset X$. Then every continuous linear map $u\colon C_0(T) \to X$ is weakly compact.* (*Then by Propositions* 2 *and* 5 *and the Orlicz-Pettis theorem, the representing measure m of u has range in X and is σ-additive in τ.*)

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C_0(T)$ such that

$$\sum_{n=1}^{\infty} \left| \int_T f_n \, \mathrm{d}\mu \right| < \infty$$

for each $\mu \in M(T)$. Let $u\colon C_0(T) \to X$ be a continuous linear map with the representing measure m. Then by Proposition 5(i), $x^* \circ m \in M(T)$ for each $x^* \in X^*$ and hence by the hypothesis on $(f_n)_{n=1}^{\infty}$ and by Proposition 5(iii) we have

$$\sum_{n=1}^{\infty} |x^*(uf_n)| = \sum_{n=1}^{\infty} \left| \int_T f_n \, \mathrm{d}(x^* \circ m) \right| < \infty$$

for each $x^* \in X^*$. Since $c_0 \not\subset X$, by [12; Theorem 4] of Tumarkin it follows that $\sum_{n=1}^{\infty} u(f_n)$ converges unconditionally in X (in τ). Thus u is an unconditionally convergent operator, and hence by (i) $\implies$ (ii) of Theorem 1, u is weakly compact. □

Remark 2. The above corollary is deduced from Lemma 1 via (i) $\implies$ (ii) of Theorem 1, while its Banach space analogue is immediate from Lemma 1. Note that the strict Dunford-Pettis property of $C_0(T)$ is not used in proving the corollary.

REFERENCES

[1] DIESTEL, J.—UHL, J. J.: *Vector Measures.* Survey No. 15, Amer. Math. Soc, Providence, R.I., 1977.

[2] DINCULEANU, N.: *Vector Measures*, Pergamon Press, New York, 1967.

[3] EDWARDS, R. E.: *Functional Analysis. Theory and Applications*, Holt Rinehart and Winston, New York-Chicago-San Francisco-Toronto-London, 1965.

[4] GROTHENDIECK, A.: *Sur les applications linéares faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. **5** (1953), 129–173.

[5] HALMOS, P. R.: *Measure Theory*, Van Nostrand, New York, 1950.

[6] MCARTHUR, C. W.: *On a theorem of Orlicz and Pettis*, Pacific J. Math. **22** (1967), 297 302.

[7] PANCHAPAGESAN, T. V.: *Applications of a theorem of Grothendieck to vector measures*, J. Math. Anal. Appl. **214** (1997), 89–101.

[8] PANCHAPAGESAN, T. V.: *Characterizations of weakly compact operators on $C_0(T)$*, Trans. Amer. Math. Soc. **350** (1998), 4849–4867.

[9] PEŁCZYŃSKI, A.: *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209 228.

[10] PEŁCZYŃSKI, A.: *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Polish Acad. Sci. Math. **10** (1962), 641–648.

[11] THOMAS, E.: *L'integration par rapport a une mesure de Radon vectorielle*, Ann. Inst. Fourier (Grenoble) **20** (1970), 55 191.

[12] TUMARKIN, JU. B.: *On locally convex spaces with basis*, Dokl. Akad. Nauk. SSSR **11** (1970), 1672–1675.

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Departamento de Matemáticas
Facultad de Ciencias
Universidad de los Andes
Mérida, 5101
VENEZUELA
E-mail: panchapa@ciens.ula.ve