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# REGULAR REPRESENTATIONS OF SEMISIMPLE $MV$-ALGEBRAS BY CONTINUOUS REAL FUNCTIONS

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ABSTRACT. In this paper we define the notion of the regular representation of an $MV$-algebra $\mathcal{A}$ by continuous real functions and we prove the existence of such representation in the case when $\mathcal{A}$ is semisimple.

## 1. Introduction

A homomorphism $\varphi$ of an $MV$-algebra $A_1$ into an $MV$-algebra $A_2$ is said to be *regular* if, whenever $x \in A_1$, $X \subseteq A_1$ and $x = \sup X$, then $\varphi(x) = \sup \varphi(X)$, and if the corresponding dual condition is also satisfied.

Analogously we define a regular homomorphism of a lattice ordered group $G_1$ into a lattice ordered group $G_2$.

This terminology is in accordance with [10], where it was used for the case of Boolean algebras.

For a topological space $M$ we denote by $F_{\mathrm{cb}}(M)$ the set of all continuous bounded real functions on $M$. Under the addition and the partial order defined componentwise, $F_{\mathrm{cb}}(M)$ is an abelian lattice ordered group with the strong unit $u_M$, where $u_M(t) = 1$ for each $t \in M$.

By applying the notation from [2] we can construct the $MV$-algebra

$$\Gamma(F_{\mathrm{cb}}(M), u_M).$$

We denote this $MV$-algebra by $A_1(M)$.

The underlying set of $A_1(M)$ is the set of all $f \in F_{\mathrm{cb}}(M)$ such that $0 \leqq f(t) \leqq 1$ for each $t \in M$.

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Let $A$ be an $MV$-algebra. If there exists a topological space $M$ and an isomorphism $\varphi$ of $A$ into $A_1(M)$, then we say that $\varphi$ is a *representation of $A$ by continuous real functions.*

It is obvious that for each topological space $M$, $A_1(M)$ is a semisimple $MV$-algebra. Hence an $MV$-algebra $A$ has a representation by continuous functions only if $A$ is semisimple.

Let $A$ be a semisimple $MV$-algebra; we denote by $A^D$ and $A^d$ the *Dedekind completion* or the *divisible hull* of $A$, respectively. Then $A$ is a subalgebra of $A^{dD}$. (For the terminology, cf. Section 2 below.)

If $X_1$, $X_2$ and $Y$ are sets with $X_1 \subseteq X_2$ and $\varphi$ is a mapping of $X_2$ into $Y$, then we denote by $\varphi|_{X_1}$ the corresponding restriction to the set $X_1$.

In the present paper we prove:

($\alpha$) Let $A$ be a semisimple $MV$-algebra. There exists a compact Hausdorff topological space $M$ such that:

- (i) the space $M$ is extremal (i.e., the closure of each open subset of $M$ is open);
- (ii) there exists an isomorphism $\varphi_1$ of the $MV$-algebra $A^{dD}$ onto the $MV$-algebra $A_1(M)$;
- (iii) the mapping $\varphi = \varphi_1|_A$ is a regular representation of $A$ by continuous real functions.

For proving ($\alpha$), we apply a well-known result on vector lattices (namely, [11; Theorem V. 3.1]).

Again, let $M$ be a topological space. Assume that $A_2$ is a subalgebra of $A_1(M)$. An isomorphism $\psi$ of an $MV$-algebra $A$ into $A_2$ is said to be *separating* if, whenever $t_1$ and $t_2$ are distinct elements of $M$, then there exists $x \in A$ such that

$$\varphi(x)(t_1) > 0 \qquad \text{and} \qquad \varphi(x)(t_2) = 0 .$$

The following result is contained in the monograph [2; Section 3.6]:

($\beta$) For any $MV$-algebra $A$ the following conditions are equivalent:

- (i) $A$ is semisimple;
- (ii) there exists a compact Hausdorff space $M$ and a separating isomorphism of $A$ into $A_1(M)$.

For proving ($\beta$), deep results on free $MV$-algebras have been used.

We remark that if $M$, $A_1(M)$ and $A_2$ are as above and if $\psi$ is a separating isomorphism of $A$ into $A_2$, then $\psi$ need not be regular (cf. the example in Section 2 below).

Further, we remark that a related result is proved in [4; Theorem 2.5] (in this theorem it is assumed that the $MV$-algebra under consideration is semisimple and divisible).

## 2. Preliminaries and auxiliary results

For lattice ordered groups we apply the notation and terminology as in [1] and [3].

We recall briefly some relevant notions from the theory of $MV$-algebras.

An *$MV$-algebra* $\mathcal{A}$ is defined to be a nonempty set $A$ with binary operations $\oplus$, $*$, a unary operation $\neg$ and unary operations $0$, $1$ on $A$ such that the conditions (M1) – (M8) from [5] are satisfied; cf. also [6]. (For a formally different but equivalent definition cf., e.g., [9].)

If no misunderstanding can occur, then we write $A$ instead of $\mathcal{A}$.

We will apply in an essential way the following results $(*)$ and $(**)$ (cf. [9]):

$(*)$ Let $G$ be an abelian ordered group with a strong unit $u$. Let $A$ be the interval $[0,u]$ of $G$. For $a,b \in A$ we put

$$a \oplus b = (a+b) \wedge u\,, \qquad \neg a = u - a\,,$$
$$a * b = \neg(\neg a \oplus \neg b)\,, \qquad 1 = u\,.$$

Then the algebraic system $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ is an $MV$-algebra.

The $MV$-algebra from $(*)$ will be denoted by $\Gamma(G,u)$.

$(**)$ For each $MV$-algebra $\mathcal{A}$ there exists an abelian lattice ordered group $G$ with a strong unit $u$ such that $\mathcal{A} = \Gamma(G,u)$.

In view of $(**)$ the lattice operations $\vee$ and $\wedge$ are defined on $A$; the corresponding lattice will be denoted by $\ell(A)$.

The $MV$-algebra $A$ is said to be *complete* if the lattice $\ell(A)$ is complete.

Let $A$ and $G$ be as above. $A$ is called *semisimple* if the lattice ordered group $G$ is archimedean. (Other equivalent definitions have been used in the literature; also, instead of "semisimple" the term "archimedean" has been applied. Cf., e.g., [7].)

Let $A$ be an $MV$-algebra, $a \in A$, $a_i = a$ $(i = 1,2,\dots,n)$. We denote

$$a_1 \oplus a_2 \oplus \dots \oplus a_n = n \cdot a\,;$$

as usual, we write

$$a_1 + a_2 + \dots + a_n = na\,,$$

where $+$ is the group operation in $G$.

The $MV$-algebra $A$ will be called *divisible* if for each $b \in A$ with $b \neq 0$ and each positive integer $n$ there exists $a \in A$ such that

$(\mathrm{i}_1)$ $n \cdot a = b$,
$(\mathrm{ii}_1)$ $a < 2 \cdot a < 3 \cdot a < \dots < (n-1) \cdot a < b$.

Now let $H$ be an archimedean lattice ordered group. The Dedekind completion of $H$ will be denoted by $H^D$. For the definition and the properties of the divisible hull $H^d$ of $H$ cf., e.g. [8].

In [6], it has been proved that if $A = \Gamma(G,u)$, then $A$ is complete if and only if the lattice ordered group $G$ is complete. This is the reason for defining, for each semisimple $MV$-algebra $A$, the Dedekind completion $A^D$ of $A$ by putting

$$A^D = \Gamma(G^D, u). \tag{1}$$

Further, from the construction of $G^d$ (cf., e.g., [8]) it follows that $u$ is a strong unit of the lattice ordered group $G^d$ and that $G^d$ is archimedean. From the conditions $(\mathrm{i}_1)$ and $(\mathrm{ii}_1)$ we easily obtain that $A$ is divisible if and only if $G$ is divisible. In view of this fact we define

$$A^d = \Gamma(G^d, u). \tag{1'}$$

**2.1. LEMMA.** *Let $G_2$ be an abelian lattice ordered group with a strong unit $u$. Further, suppose that $G_1$ is a lattice ordered group which is regularly embedded into $G_2$ and that $u$ belongs to $G_1$. Put*

$$A_1 = \Gamma(G_1, u), \qquad A_2 = \Gamma(G_2, u).$$

*Then $A_1$ is regularly embedded into $A_2$.*

Proof. This is an immediate consequence of $(*)$. □

**2.2. LEMMA.** *Let $A$ be a semisimple $MV$-algebra. Then $A$ is regularly embedded into $A^D$.*

Proof. This is a consequence of Lemma 2.1, of relation (1) and of the well-known fact that $G$ is regularly embedded into $G^D$. □

**2.3. LEMMA.** *Let $A$ be a semisimple $MV$-algebra with $A = \Gamma(G,u)$. Then*

(i) *$A^d$ is a divisible $MV$-algebra;*
(ii) *$A$ is regularly embedded into $A^d$.*

Proof. The assertion (i) is an immediate consequence of the fact that $G^d$ is divisible. Further, $G$ is regularly embedded into $G^d$ (cf. [8]). Hence, in view of 2.1, $A$ is regularly embedded into $A^d$. □

From 2.2 and 2.3 we conclude:

**2.4. LEMMA.** *Let $A$ be a semisimple $MV$-algebra. Then $A$ is regularly embedded into $A^{dD}$.*

2.5. EXAMPLE. Let $\mathbb{R}$ be the set of all reals with the usual topology. Let $G_1$ be the set of all bounded real functions on $\mathbb{R}$; assume that the operation $+$ and the partial order on $G_1$ are defined componentwise. Finally, let $u \in G_1$ such that $u(t) = 1$ for each $t \in \mathbb{R}$. Then we can construct the $MV$-algebra $\Gamma(G_1, u) = A_1$.

Under the notation as in Section 1, let

$$A_2 = \Gamma(F_{\mathrm{cb}}(\mathbb{R}), u).$$

Then the identity mapping on $A_2$ is a separating isomorphism of $A_2$ into $A_1$, but this isomorphism fails to be regular.

2.6. EXAMPLE. As above, let $A = \Gamma(G, u)$. By the definition of a divisible $MV$-algebra we applied the conditions $(\mathrm{i}_1)$ and $(\mathrm{ii}_1)$; we already remarked above that $A$ is divisible if and only if $G$ is divisible. On the other hand, if $A$ is assumed to satisfy only the condition $(\mathrm{i}_1)$, then $G$ need not be divisible. In fact, let $G = \mathbb{Z}$ ($=$ the set of all integers, the operation $+$ and the linear order being defined in the natural way). Put $u = 1$. Then $(\mathrm{i}_1)$ holds, but $G$ fails to be divisible.

## 3. Proof of $(\alpha)$

Assume that $A$ is a semisimple $MV$-algebra. There exists an archimedean lattice ordered group $G$ with a strong unit $u$ such that

$$A = \Gamma(G, u).$$

Denote

$$G_1 = G^{dD}. \tag{2}$$

Then $G_1$ is an archimedean lattice ordered group and $u$ is a strong unit of $G_1$. Put

$$A_1 = \Gamma(G_1, u).$$

In view of 2.4, $A$ is regularly embedded into $A_1$.

From (2) and from [8] we conclude that $G_1$ is a vector lattice. Since $G_1$ has a strong unit, by applying [11; Theorem V. 3.1] we get that there exists a Hausdorff compact space $M$ such that $M$ is extremal and that there exists an isomorphism $\varphi_0$ of $G_1$ onto the vector lattice $F_{\mathrm{cb}}(M)$.

Put $\varphi_1 = \varphi_0|_{A_1}$. Thus $\varphi_1$ is an isomorphism of the $MV$-algebra $A_1$ onto the $MV$-algebra $A_1(M)$ (we apply the notation as in Section 1).

Denote $\varphi = \varphi_1|_A$. Hence $\varphi$ is an isomorphism of $A$ into the $MV$-algebra $A_1(M)$.

We already mentioned that $A$ is regularly embedded into $A_1$. Therefore the isomorphism $\varphi$ is regular. This completes the proof of $(\alpha)$.

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