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With the deepest admiration to my esteemed friend Kálmán Győry on the occasion of his sixties

COVERING SYSTEMS, KUBERT IDENTITIES AND DIFFERENCE EQUATIONS

ŠTEFAN PORUBSKÝ

(*Communicated by Stanislav Jakubec*)

ABSTRACT. A. S. Fraenkel proved that the following identities involving Bernoulli polynomials

$$B_n(0) = \sum_{i=1}^{m} b_i^{n-1} B_n\left(\frac{a_i}{b_i}\right) \qquad \text{for all} \quad n \geq 0$$

are true if and only if the system of arithmetic congruences $\{a_i \pmod{b_i} : 1 \leq i \leq m\}$ is an exact cover of $\mathbb{Z}$. Generalizations of this result involving other functions and more general covering systems have been successively found by A. S. Fraenkel, J. Beebee, Z.-W. Sun and the author. Z.-W. Sun proved an algebraic characterization of functions capable of identities of this type, and independently J. Beebee observed a connection of these results to the Raabe multiplication formula for Bernoulli polynomials and with the so-called Kubert identities.

In this article, we shall analyze some analytic aspects of connections between finite systems of arithmetical progressions and the generalized Kubert identities

$$\theta(m)f(x) = \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right).$$

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We also show further connections and possibilities how to extend some of the previously proved results by Beebee, Fraenkel and the author to general systems of arithmetic sequences, or how to concretize Sun results to other classical functions. Connection of these results to solutions of some simple difference equation is also shown.

1. Covering by arithmetic progressions

In the early thirties P. Erdős used a system of arithmetic progressions covering the set of all the integers in a disproof of a problem of N. P. Romanoff from the additive number theory. Although some of the major problems posed by P. Erdős at the very beginning are still open, the notion itself undergone a wide diversification (cf. [Poru1981] for more details on results about systems of arithmetical progressions).

Consider a finite collection of residue classes

$$\{a_i \ (\mathrm{mod}\ b_i) : \ 1 \le i \le w\}. \tag{1.1}$$

Using the Iverson notation, let $[x \in a \ (\mathrm{mod}\ b)]$ stand for the indicator of the class a (mod b). Let $\mu_i = \mu(i)$, $i = 1, \dots, w$, be a weight function with complex values. If $\mathfrak{m}(n) = \sum\limits_{i=1}^{w} \mu(i)[n \in a_i \ (\mathrm{mod}\ b_i)]$, then (1.1) is called ([Poru1975]) a $(\mu, \mathfrak{m})$*-cover* and $\mathfrak{m}$ the *covering function* of the (μ-weighted) system (1.1). If $\mu_i = 1$, $i = 1, \dots, w$, and $\mathfrak{m}(n) \ge 1$ for every $n \in \mathbb{Z}$, i.e. if each integer belongs to at least one class, then (1.1) is called a *cover* and if each integer belongs to precisely m ($m \in \mathbb{N}$, the set of positive integers) members of (1.1), it is called an exact m*-cover* ([Poru1976]). An exact m-cover with $m = 1$ will be called an *exact cover*. Exact covers, or, more generally, the exact m-covers with $m > 1$ are the most closest generalizations of the systems of arithmetic progressions representing the complete residue systems

$$0 \ (\mathrm{mod}\ w),\ 1 \ (\mathrm{mod}\ w),\ \dots,\ w-1 \ (\mathrm{mod}\ w). \tag{1.2}$$

Note ([Poru1976]) that there are exact m-covers which cannot be written as a collection of members of m exact covers.

The a_i's are the *offsets*, the b_i's are the *moduli* of the system (1.1). If $0 \le a_i < b_i$, the offsets are called *standardized*. Unless otherwise stated, it wil. be assumed the offsets in any exact cover mentioned have been standardized.

Covering function $\mathfrak{m}$ of a $(\mu, \mathfrak{m})$-cover is a periodic function and its (smallest non-negative) period b_0 is a divisor of the $\mathrm{lcm}\{b_i : \ 1 \le i \le w\}$ of all moduli of (1.1).

Seemingly complicated notion of a $(\mu, \mathfrak{m})$-cover provides advantages in some situations. So for instance, adding the classes $i \pmod{b_0}$ weighted with $-\mathfrak{m}(i)$ for $i = 0, 1, \ldots, b_0 - 1$ to the original system (1.1), we obtain a $(\mu, \mathfrak{m})$-cover with identically vanishing covering function.

In [Poru1974] it was proved that every exact cover can be constructed by a finite number of steps starting with the set of integers, that is with the class $0 \pmod 1$, where at every step

- either a residue class of the system is partition into several classes,
- or a set of classes of the system which together disjointly cover a residue class is replaced by the covered class.

(This process was later rediscovered also in [Kore1984], [Beeb1991], [Beeb1995].) These steps can be characterized by the fact that after every step the resulting system is always an exact cover. In other words, the covering function is an invariant of the whole process. For the case of the general $(\mu, \mathfrak{m})$-covers these *covering function preserving operations*, abbreviated CFP-operations in what follows, can be reformulated more precisely as follows:

Operation S. Let

$$c_j \pmod{d_j}, \qquad j = 1, \ldots, m,$$

be an arbitrary $(\nu, 1)$-cover. This splitting operation S results in a new $(\mu, \mathfrak{m})$-cover

$$\{a_i \pmod{b_i} : i = 1, \ldots, w, \ i \neq i_0\} \cup \{a_{i_0} + c_j b_{i_0} \pmod{d_j b_{i_0}} : j = 1, \ldots, m\}$$

from the given $(\mu, \mathfrak{m})$-cover (1.1). Here a class $a_{i_0} \pmod{n_{i_0}}$, $i_0 \in \{1, 2, \ldots, n\}$, of the original system (1.1) is replaced by m classes

$$\{a_{i_0} + c_j b_{i_0} \pmod{d_j b_{i_0}} : j = 1, \ldots, m\},$$

where the jth class is endowed with the weight $\mu_{i_0}\nu_j$ for $j = 1, \ldots, m$. The classes $a_i \pmod{b_i}$ for $i \neq i_0$ keep the original weights μ_i.

Operation J. If in a $(\mu, \mathfrak{m})$-cover (1.1) there exists a subcollection (note that a repetition of classes is not excluded)

$$a_j \pmod{b_j}, \qquad j \in I_0 \subset \{1, \ldots, w\}$$

such that

$$\sum_{j \in I_0} \mu_j [x \in a_j \pmod{b_j}] = \nu [x \in c \pmod{d}] \qquad \text{for every} \quad x \in \mathbb{Z},$$

for some $c \pmod d$ and $\nu \in \mathbb{C}$, then the system (1.1) is replaced by the new $(\mu, \mathfrak{m})$-cover

$$\{a_i \pmod{b_i} : i = 1, \ldots, n, \ i \notin I_0\} \cup \{c \pmod{d}\},$$

where the classes $a_i \pmod{b_i}$ for $i \notin I_0$ keep the original weights μ_i, but the new class $c \pmod d$ obtains the weight ν.

Operation W. Given a class

$$a_{i_0} \pmod{b_{i_0}}$$

having weight μ_{i_0} of a $(\mu, \mathfrak{m})$-cover (1.1) and a finite collection of numbers $\kappa_i \in \mathbb{C}$, $i = 1, \dots t$, such that $\mu_{i_0} = \sum_{i=1}^{t} \kappa_i$, then this weight splitting operation W results in the system in which the class $a_{i_0} \pmod{b_{i_0}}$ of (1.1) is replaced by its t copies each of them endowed with one of the weights $\kappa_1, \dots, \kappa_t$.

The role of operations S, J and W is demonstrated in the next result:

Theorem 1.1. *Let* (1.1) *be a* $(\mu, \mathfrak{m})$*-cover. Then* (1.1) *can be constructed from the* $(\mathfrak{m}, \mathfrak{m})$*-cover*

$$0 \pmod{b_0},\ 1 \pmod{b_0},\ \dots,\ b_0 - 1 \pmod{b_0} \tag{1.3}$$

using the CFP operations , J and W in finitely many steps.

Proof. If (1.1) is an exact m-cover, then (1.3) is formed by one class $0 \pmod 1$. We shall follow the idea used for exact covers in [Poru1974].

In the first step, we apply operation W in such a way that we replace each class $j \pmod{b_0}$ by its $t = \sum_{i=1}^{w} [j \in a_i \pmod{b_i}]$ copies each with weight $\mu(i)$ provided $[j \in a_i \pmod{b_i}] \neq 0$.

In the second step, using operation S we partition every copy of thus obtained class $j \pmod{b_0}$ of (1.3) into M/b_0 classes $j + sb_0 \pmod M$, where $s = 0, 1, \dots, M/b_0 - 1$ and M is a common multiple of the numbers $b_1, b_2, \dots, b_w$. We obtain a collection of classes

$$0 \pmod M,\ 1 \pmod M,\ \dots,\ M - 1 \pmod M$$

having the original covering function $\mathfrak{m}$ and in which the weight of the class $j \pmod M$ is one of the numbers $\mu(i)$, $i = 1, \dots, w$. (Note, that we arrived at this stage at the collection of classes coinciding with one which can be obtained by the application of operation S to the original $(\mu, \mathfrak{m})$-cover in such a way that each class $a_i \pmod{b_i}$ is replaced by M/b_i classes modulo M.)

To finish the construction, it suffices to apply the operation J to reduce intermediate system of classes modulo M to the classes of the final $(\mu, \mathfrak{m})$-cover (1.1). □

In the course of the proof of Theorem 1.1 we saw that for the process of the generation of a $(\mu, \mathfrak{m})$-cover it is not necessary to have the all residue classes at the disposal. A successful application of operations S, J and W can be generally realized with a thinner set of the available moduli. If $\mathcal{M}$ is a set of positive integers which can serve as a set of moduli of system of residue classes over which the CFP operations are applied, then a successful application of the CFP operations requires that set $\mathcal{M}$ possesses the following two properties:

(a) if $b_1, b_2 \in \mathcal{M}$, then also a common multiple of numbers b_1, b_2 belongs to $\mathcal{M}$,

(b) if $b_1, b_2 \in \mathcal{M}$ and b_1 divides b_2, then also $b_2 b_1^{-1} \in \mathcal{M}$.

Sets of positive integers satisfying (a) and (b) will be called *divisible*. A special case of divisible sets are the so called *divisor closed sets*, which contain the product of two (positive) integers if and only if both factors belong to the set. A set is divisor closed if and only if it contains 1 and all those positive integers which have all prime divisors in a given set of primes. The set of integers of the form $4n+1$, $n = 0, 1, 2, \dots$, is an example of a divisible set which is not divisor closed. More generally, the sets of integers of the form $fn+1$ with $n = 0, 1, 2, \dots$ and fixed $f \in \mathbb{N}$ are always divisible. Note, that it is not generally possible to replace a common multiple of b_1, b_2 by the least common multiple, as it might follow from the proof. The set of numbers $4n+1$, $n = 1, 2, \dots$, shows this, e.g. $\operatorname{lcm}[9, 21] = 63$.

We say that a $(\mu, \mathfrak{m})$-cover (1.1) is $\mathcal{M}$*-admissible*, where $\mathcal{M}$ is a divisible set, if there is a sequence of CFP operations starting with the $(\mathfrak{m}, \mathfrak{m})$-cover (1.3) and resulting in (1.1) and with the all moduli used in the process of the construction described in the proof of Theorem 1.1, belonging to $\mathcal{M}$.

Operations S, J, and W are in a close connection with the step functions defined as follows: Given a directed system of sets $\mathbb{Z}/m\mathbb{Z}$ with $m \in \mathcal{M} \subset \mathbb{Z}$ ordered by divisibility with connecting homomorphisms

$$\mathbf{r}_m \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$$

being reduction mod m if $m \mid n$, a function f defined on a subset of the projective limit

$$Z_{\mathcal{M}} = \varprojlim_{m \in \mathcal{M}} \mathbb{Z}/m\mathbb{Z}$$

is called $\mathcal{M}$*-step function* (or *locally constant*) if there is an $m \in \mathcal{M}$ such that f factors through $\mathbb{Z}/m\mathbb{Z}$. Characteristic functions of residue classes mod m for $m \in \mathcal{M}$ are obviously $\mathcal{M}$-step functions. Related important family of functions are $\mathcal{M}$-compatible functions. We say that the family $\{\varphi_m\}_{m \in \mathcal{M}}$ of functions of $\mathbb{Z}/m\mathbb{Z}$ into an Abelian group, say V, is $\mathcal{M}$*-compatible* if for each $m, n \in \mathcal{M}$ with $n \mid m$ and each $x \in \mathbb{Z}/n\mathbb{Z}$ we have

$$\varphi_n(x) = \sum_{\substack{y \in \mathbb{Z}/m\mathbb{Z} \\ \mathbf{r}_m(y) = x}} \varphi_m(y) \,. \tag{1.4}$$

If $\mathcal{M} = \mathbb{N}$, then $Z_{\mathbb{N}} = \prod_p \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the set of the all integral p-adic numbers. Characteristic functions of residue classes, and consequently covering function of finite systems (1.1) are $\mathbb{N}$-step functions. (For the role of $\mathbb{N}$-compatible functions cf. for instance [Lang1978; p. 33]).

The above considerations immediately imply the following reformulation of the defining property of $\mathcal{M}$-compatible family of functions:

LEMMA 1.2. *Let $\mathcal{M}$ be a divisible set and $\varphi_n(a)$ a family of arithmetical functions defined for $0 \le a < n$, $n \in \mathcal{M}$. Then $\{\varphi_n\}_{n\in\mathcal{M}}$ is an $\mathcal{M}$-compatible family of functions if and only if*

$$\sum_{j=0}^{n-1} \varphi_{nd}(a + jd) = \varphi_d(a)$$

for all $n \in \mathcal{M}$ and couples (a, d) with $0 \le a < d$, $a, d \in \mathcal{M}$.

The role of the $\mathbb{Z}$-admissible functions in the process of proving some identities involving Bernoulli and Euler polynomials was indicated in [Poru1982]. However, not based on this idea, a number of various related identities involving also some other functions (cf. Section 6 for some details) was proved in [Beeb1978], [Beeb1991], [Beeb1992], [Beeb1994], [Por1994a], [Por1994b], [Por1999], mainly motivated by Fraenkel's characterization of exact covers proved in [Frae1973]. For instance, Fraenkel's characterization can be reproved in one direction using the above idea as follows:

$$\begin{aligned}
B_n(x) = B_n\left(\frac{x}{1}\right) &= M^{n-1} \sum_{j=0}^{M-1} B_n\left(\frac{x+j}{M}\right) \\
&= \sum_{i=1}^{w} \left(\frac{M}{b_i}\right)^{n-1} b_i^{n-1} \sum_{j=0}^{M/b_i-1} B_n\left(\frac{x + a_i + jb_i}{\frac{M}{b_i} b_i}\right) \\
&= \sum_{i=1}^{w} b_i^{n-1} \left(\frac{M}{b_i}\right)^{n-1} \sum_{j=0}^{M/b_i-1} B_n\left(\frac{\frac{x+a_i}{b_i} + j}{\frac{M}{b_i}}\right) \\
&= \sum_{i=1}^{w} b_i^{n-1} B_n\left(\frac{x + a_i}{b_i}\right).
\end{aligned}$$

A complete algebraic characterization of function suitable for use in such identities was given by Sun [Sun1989a]. Independently of this development a wide class of functions falling into this category was studied in algebraic number theory which gave the impetus to the investigation of functions satisfying so called Kubert identities (cf. (KI_s) below). In the rest of his paper, we shall give an extension of Sun's result, then we shall deduce various functions satisfying extended Kubert identities from Lerch $\mathfrak{K}$-function, and finally, we show a connection to difference equations.

2. Kubert and generalized Kubert identities

Note that the above projective family $\{\mathbb{Z}/m\mathbb{Z}\}_{m\in\mathcal{M}}$ can be represented also in an isomorphic form as the family

$$\left\{\frac{1}{m}\,\mathbb{Z}/\mathbb{Z}\right\}_{m\in\mathcal{M}}$$

with multiplication by $\frac{m}{n}$ with $m,n\in\mathcal{M}$ as the connecting homomorphism:

$$\frac{m}{n}\colon \frac{1}{m}\,\mathbb{Z}/\mathbb{Z}\to\frac{1}{n}\,\mathbb{Z}/\mathbb{Z}\,.$$

If $x\in\frac{1}{m}\mathbb{Z}/\mathbb{Z}$, then $x=x'/m$ can be viewed as an element of $\mathbb{Q}/\mathbb{Z}$. This observation enables us to reduce the attention either to functions $f(x)$ or functions $f(x',m)=f_m(x')$ with the variable x, or x', resp., varying over $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$. In what follows we shall reduce our attention to (complex) functions $f(x)$, or $f_m(x)$ of argument x varying over $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ instead of functions defined on the projective limit of the above projective system.

Probably the most known prototypes of $\mathbb{N}$-compatible functions are given in terms of Bernoulli polynomials $B_r(x)$ for every fixed $r\geq 0$ and real x

$$\varphi_n(x+a)=n^{r-1}B_r\left(\frac{x+a}{n}\right).$$

Related functions are Euler polynomials

$$\varphi_n(x+a)=(2n+1)^r(-1)^aE_r\left(\frac{x+a}{2n+1}\right)$$

in variable x. They are $\mathcal{M}$-compatible for $\mathcal{M}=\{2n+1:\ n=0,1,2,\dots\}$ and any fixed $r\in\{0,1,2,\dots\}$. The corresponding compatibility relation of Bernoulli polynomials was discovered by Raabe [Raab1848] and it is usually referred to as the (Raabe) multiplication relation.[1] From this reason the compatibility relations are often called multiplication formulas or relations. Both these families of polynomials are examples of a more general class of *Kubert functions*, which are functions f satisfying the so called *Kubert identities*

$$f(x)=m^{s-1}\sum_{k=0}^{m-1}f\left(\frac{x+k}{m}\right)\tag{KI$_s$}$$

for every positive integer m and fixed $s\in\mathbb{C}$.

[1]The corresponding multiplication formula for Euler polynomials goes back to Nielsen [Niel1923], and splits into two parts. For even indices Bernoulli polynomials appear in the corresponding sum.

The compatibility property of Bernoulli polynomials and other functions expressed by Kubert identities plays an important role not only in the algebraic number theory (cf. [Lang1978] for more details), but also in some analytic questions as Frenel-Kluyver integral formulas, etc. An essential part of the huge number of papers devoted to this topics can be found in [DiSS1991].

Kubert identities (KI_s) are capable of further generalizations. For instance, if θ is a given arithmetical function, we say that f fulfils *generalized Kubert identities* (or *generalized θ-Kubert identities* if the role of the function θ will be important for us), provided f satisfies the identity

$$\theta(m)f(x) = \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) \tag{GKI}$$

for every positive integer m. Functions satisfying Kubert identities (KI_s) and (GKI) will be called *Kubert*, or *generalized Kubert functions* (or *generalized θ-Kubert functions*), respectively.

The next result supplements [Walu1991; Theorem 2.1]:

PROPOSITION 2.1. *Let f be a non-zero function and θ a nowhere vanishing arithmetic function. Then f satisfies* (GKI) *if and only if θ is completely multiplicative and if for each prime p we have the identity*

$$\theta(p)f(x) = \sum_{k=0}^{p-1} f\left(\frac{x+k}{p}\right).$$

Proof.

The *only if* part of the proof follows from [Walu1991; Theorem 2.1], where it is proved that θ is completely multiplicative.

The *if* part can be proved in a more general setting. Namely, it can be proved along the same lines as used in the proof of [Beeb1995; Theorem 1] that our hypotheses imply

$$f(x) = \sum_{i=1}^{w} \theta(b_i)^{-1} f\left(\frac{x+a_i}{b_i}\right)$$

for every exact cover (1.1). Since $a_i = i-1$, $b_i = m$ for $i = 1, \dots m$ is an exact cover for every m, the relation (GKI) is a special case of (2.2). □

As we have seen in the proof of Proposition 2.1, the appearance of general $(\mu, \mathfrak{m})$-covers in the Kubert identities leads to the necessity to consider a modified form of (GKI). To this purpose we shall suppose, unless contrary is stated, that the arithmetical function θ is nowhere vanishing, i.e.

$$\theta(m) \neq 0 \qquad \text{for every} \quad m.$$

For the sake of simplicity we shall use the notation

$$\phi(m) = \frac{1}{\theta(m)} .$$

Function ϕ is completely multiplicative if and only if such is the function θ, and then the generalized θ-Kubert identities convert to the form

$$f(x) = \phi(m) \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right). \tag{GKI*}$$

Kubert studied functions f satisfying (KI_s) for an arbitrary but fixed positive integer s and with x varying over $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$. However the shape of (KI_s) also makes sense if x is taken from $\langle 0,1)$, $\langle 0,\infty)$, or $\mathbb{R}$. He proved the following lemma.

Lemma 2.2. *If a function $f\colon (0,1) \to \mathbb{C}$ satisfies* (KI_s) *with $s \neq 1$, then it extends uniquely to a function $\mathbb{R}/\mathbb{Z}$ satisfying* (KI_s).

Thus, if f satisfies (KI_s) for every positive integer m with x varying over $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$, then by periodic continuation the validity of (KI_s) can be extended to $\langle 0,\infty)$ or $\mathbb{R}$. But even the reverse process is also possible (here $\{x\}$ denotes the fractional part of x):

Lemma 2.3. *If f satisfies* (GKI) *for every positive integer m and θ is a nowhere vanishing arithmetic function, then also $f(\{x\})$ satisfies* (GKI).

P r o o f . The identity to be proven says

$$\theta(m) f(\{x\}) = \sum_{a=0}^{m-1} f\left(\left\{\frac{x+a}{m}\right\}\right). \tag{2.3}$$

To see this, let

$$G(x) = \theta(m)^{-1} \sum_{a=0}^{m-1} f\left(\left\{\frac{x+a}{m}\right\}\right), \qquad F(x) = \theta(m)^{-1} \sum_{a=0}^{m-1} f\left(\frac{x+a}{m}\right).$$

Then the proof can be based on the following two facts:

(1) If $0 \le x < 1$, then $(x+i)/m = \{(x+i)/m\}$ for every $i = 0,\dots,m-1$. Therefore, if $0 \le x < 1$,

$$G(x) = F(x) = f(x) = f(\{x\}) .$$

(2) Since $G(x) - f(\{x\})$ is periodic with period 1, (2.3) follows.

□

In [Miln1983] many interesting properties of Kubert functions can be found. Thus for instance, if $\mathrm{Re}(s) > 1$ and f is continuous, then also the extension of Lemma 2.2 is continuous. A powerful way how to construct new Kubert functions from the available ones is given in the next result [Miln1983; p. 287, Lemma 5]:

LEMMA 2.4. *Let $\mathcal{K}_s$ be the complex vector space of all continuous maps $f\colon (0,1) \to \mathbb{C}$ which satisfy* (KI$_s$). *Then*

(i) *The correspondence $f(x) \mapsto \mathrm{d}f(x)/\,\mathrm{d}x$ maps the vector space $\mathcal{K}_s$ bijectively onto $\mathcal{K}_{s-1}$, except when $s = 0$.*

(ii) *Let $f \in \mathcal{K}_s$. If $s \neq -1$, then there is one and only one constant c so that the function $\int f(x)\,\mathrm{d}x + c$ satisfies* (KI$_{s+1}$).

3. Identities involving $\mathcal{M}$-compatible families

In this section, we show how systems (1.2) can be replaced by arbitrary systems (1.1) (cf. proof of Proposition 2.1) in identities of type (GKI) or their generalizations. We shall see that such extensions of Kubert type identities to general $(\mu, \mathfrak{m})$-covers significantly depend upon the ability of the involved functions to conserve (or perhaps absorb) the "changes" caused by application of CFP operations.

The next result shows the fundamental role of $\mathcal{M}$-compatible families of functions on the background of the CFP operations S, J and W and their connections to $\mathcal{M}$-admissible $(\mu, \mathfrak{m})$-covers.

THEOREM 3.1. *Let $\mathcal{M}$ be a divisible set and $\varphi_m(a)$ a family of arithmetical functions defined for $0 \le a < m$, $m \in \mathcal{M}$. Then $\{\varphi_m\}_{m\in\mathcal{M}}$ is $\mathcal{M}$-compatible if and only if*

$$\sum_{j=0}^{w} \mu_i \varphi_{b_i}(a_i) = 0 \tag{3.1}$$

for all $\mathcal{M}$-admissible $(\mu, \mathfrak{o})$-covers with identically vanishing covering function.

P r o o f. The given condition is sufficient, for the residue classes $a + jd$ (mod nd), $j = 0, 1, \dots, n-1$, each with weight 1 and the class a (mod d) with weight -1 form together a system with identically vanishing covering function, and (3.1) implies (1.5).

Conversely, let (1.1) be an $\mathcal{M}$-admissible $(\mu, \mathfrak{o})$-cover with identically vanishing covering function. Let N be a common multiple of moduli $b_1, \dots, b_w$. Then using the operation S we can lift every class a_i (mod b_i) to N/b_i classes modulo N. If we assign to each of these new classes $a_i + jb_i$ (mod N) the weight μ_i of the original class a_i (mod b_i), $1 \le i \le w$, we obtain a weighted collection of residue classes j (mod N), $j = 0, 1, \dots, N-1$, with the original covering function $\mathfrak{o}$.

Consider the sum

$$\sum_{t=0}^{N-1} \mathfrak{m}(t)\varphi_N(t)\,.$$

The sum is vanishing since the covering function $\mathfrak{m}$ is such. However, it can be modified along the ideas of the proof of Theorem 1.1 as follows

$$\begin{aligned}\sum_{t=0}^{N-1} \mathfrak{m}(t)\varphi_N(t) &= \sum_{s=0}^{N-1}\sum_{i=1}^{w} \mu_i [s \in a_i \pmod{b_i}]\varphi_N(s) \\ &= \sum_{i=1}^{w} \mu_i \sum_{j=0}^{N/b_i - 1} \varphi_{b_i \cdot \frac{N}{b_i}}(a_i + jb_i) \\ &= \sum_{i=1}^{w} \mu_i \varphi_{b_i}(a_i),\end{aligned}$$

and this finishes the proof. □

Corollary 3.2. ([Sun1989a]) *Let P be a set of primes and f a mapping into an $\mathbb{R}$-modul such that $\langle (x+r)/p, py\rangle \in \mathrm{Dom}(f)$ for all $r = 0,1,\dots,p-1$ if $p \in P$ and $\langle x, y\rangle \in \mathrm{Dom}(f)$. Then the following statements are equivalent*

(a) $\sum\limits_{r=0}^{p-1} f\big((x+r)/p, py\big) = f(x,y)$ *for all* $p \in P$, $\langle x,y\rangle \in \mathrm{Dom}(f)$.

(b) $\sum\limits_{s=1}^{w} \mu_s f\big((x+a_s)/b_s, b_s y\big) = \sum\limits_{t=1}^{k} \nu_t f\big((x+c_t)/d_t, d_t y\big)$ *for all* $\langle x,y\rangle \in \mathrm{Dom}(f)$ *whenever* $\sum\limits_{s=1}^{w} \mu_s[x \in a_s \pmod{b_s}] = \sum\limits_{t=1}^{k} \nu_s[x \in c_t \pmod{d_t}]$ *for each* $x \in \mathbb{Z}$ $(\mu_s, \nu_t \in \mathbb{R},\ 0 \le a_s < b_s,\ 0 \le c_t < d_t)$ *and any prime dividing* $\prod\limits_{s,t} b_s d_t$ *is contained in* P.

In the next result the relation (3.1) is given explicitly in terms of "parameters" characterizing a $(\mu, \mathfrak{m})$-cover. This form of rewriting of (3.1) is mainly motivated by applications in which various identities involving offsets $0,1,\dots,w-1$ and the modulus w of (1.2) appear (cf. notes in Section 6).

Corollary 3.3. *Let $\mathcal{M}$ be a divisible set and $\varphi_m(a)$ a family of arithmetical functions defined for $0 \le a < m$, $m \in \mathcal{M}$. Then $\{\varphi_m\}_{m\in\mathcal{M}}$ is $\mathcal{M}$-compatible if and only if*

$$\sum_{a=0}^{b_0-1} \mathfrak{m}(a)\varphi_{b_0}(a) = \sum_{i=1}^{w} \mu_i \varphi_{b_i}(a_i) \tag{3.2}$$

for all the $\mathcal{M}$-admissible $(\mu, \mathfrak{m})$-covers.

One of the most important class of families of compatible functions are generalized Kubert functions. We shall see in Section 4 that there are compatible functions which are not generalized Kubert functions. Walum [Walu1991] noticed that if F is a arithmetic function such that $F(dn, dr) = F(n.r)$ and $F(n + r, r) = F(n, r)$, then the identity

$$\sum_{j \pmod d} F(n + jd, rd) = \theta(d)F(n.r)$$

is sufficient for f defined by $f(p/q) = F(p, q)$ to fulfill (GKI).

The next result extends generalized Kubert identities to general systems of residue classes. Its proof for every admissible value of the variable x parallels the ideas of the proof of Theorem 1.2 starting with the identity

$$\phi(N) \sum_{t=0}^{N-1} \mathfrak{m}(t) f\left(\frac{x+t}{N}\right) = \phi(N) \sum_{t=0}^{N-1} \mathfrak{m}(t) f\left(\frac{x+t}{N}\right).$$

COROLLARY 3.4. *Let f be a (non-zero) function satisfying* (GKI*) *for every $m \in \mathbb{N}$. Then for every admissible value of x and a $(\mu, \mathfrak{m})$-cover* (1.1) *we have*

$$\phi(b_0) \sum_{a=0}^{b_0-1} \mathfrak{m}(a) f\left(\frac{x+a}{b_0}\right) = \sum_{i=1}^{w} \mu_i \phi(b_i) f\left(\frac{x+a_i}{b_i}\right).$$

COROLLARY 3.5. *Let ϕ be a non-vanishing arithmetical function. A (non-zero) function f satisfies* (GKI*) *if and only if ϕ is completely multiplicative and for each exact m-cover $\{a_i \pmod{b_i} : 1 \le i \le w\}$ with standardized offsets we have*

$$nf(x) = \sum_{i=1}^{w} \phi(b_i) f\left(\frac{x+a_i}{b_i}\right). \tag{3.3}$$

P r o o f. The sufficiency follows taking n copies of (1.2). □

From results of M i l n o r paper [Miln1983], there follows the necessity to consider with a generalized Kubert function $f(x)$ also the function f at the so called complementary argument $f(1-x)$. B e e b e e [Beeb1995] used analytic tools to prove the corresponding identities for Hurwitz zeta ζ_s and Lerch ell function ℓ_s in the case of exact covers. The next result presents an elementary proof of a generalization of Beebee's results based on properties of m-covers:

COROLLARY 3.6. *Let ϕ be a non-vanishing arithmetical function. A (non-zero) function f satisfies* (GKI*) *if and only if ϕ is completely multiplicative and for each exact m-cover $\{a_i \pmod{b_i} : 1 \le i \le w\}$ with standardized offsets we have*

$$nf(1-x) = \sum_{i=1}^{w} \phi(b_i) f\left(1 - \frac{x+a_i}{b_i}\right). \tag{3.4}$$

Proof. To the proof of (3.4) note that $\{(b_i - a_i - 1) \pmod{b_i} : 1 \le i \le w\}$ is also an exact m-cover with standardized offsets if such is (1.1). (This can be proved by an adaptation of the proof for exact covers given in [Znám1975]). Then substitute $1 - x$ for x and $b_i - a_i - 1$ for a_i in (3.3) to get (3.4). □

Note that using [Miln1983; Theorem 1] and comments following its Corollary it is possible to give a completely independent proof of Corollary 3.4 for the continuous $f\colon (0,1) \to C$ satisfying (GKI*) with $\phi(m) = m^{s-1}$ (or in other words (KI_s)).

There is another aspect of the involved identities. In [Beeb1978], [Beeb1991], [Beeb1992], [Beeb1994], [Beeb1995], [NoZn1974], [Frae1973], [Frae1975], [Poru1975], [Poru1976], [Poru1982], [Por1994a], [Por1994b] it was also shown that some identities of type (3.2) also characterize the exact covers or $(\mu, \mathfrak{m})$-covers. This means that the identities can be reversed in the sense that from their validity, it follows that the involved system of arithmetical sequences forms a $(\mu, \mathfrak{m})$-cover. For a general result in this direction we refer the reader to results announced in [Sun1989b]. Note that results of this type can also be proved using Lemma 2.4 once such a characterization is proved for a suitable particular function.

4. The descendants from Lerch functions

Lemma 2.4 shows a suggestive way how to construct functions satisfying Kubert functions. Another idea to construct (generalized) Kubert functions is a generalization of the idea of the so called Fourier-Bernoulli and holomorphic Bernoulli distribution (cf. [Lang1978; p. 65]). In this section we are to show that many of these and related functions can be derived from Lerch functions.

Lerch function is defined by

$$\mathfrak{K}(w, x, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi \mathrm{i}\, x}}{(w+n)^s}\,. \tag{4.1}$$

Properties of this function were investigated in some special cases already by Euler, then by Malmstén [Malm1849], Hurwitz [Hurw1882], and in general, but in another form, by Lipschitz [Lips1857]. It seems that it was not systematically studied before Lerch [Lerc1887], [Lerc1892], [Lerc1893], [Lerc1894]. Lerch [Lerc1894] called this function after *Lipschitz*. Suppose, for the sake of simplicity, that $w > 0$ is real. If $0 < \mathrm{Re}(s) < 1$ the series on the right hand side of (4.1) is convergent provided $x \notin \mathbb{Z}$, but if $\mathrm{Re}(s) > 1$ we have an absolutely convergent series. The series (4.1) converges also for complex x provided $\Im(x) > 0$. Here the series represents an entire transcendental function in s. It is also possible to extend the range of w to complex values. The function $\mathfrak{K}$ satisfies the following Lipschitz-Lerch functional equation ([Apos1952], [Lerc1887], [Lerc1892], [Lerc1894], [Lips1857]):

$$\frac{(2\pi)^s}{\Gamma(s)}\mathfrak{K}(w,x,1-s) = e^{\pi i(\frac{1}{2}s-2wx)}\,\mathfrak{K}(x,1-w,s)+e^{\pi i(-\frac{1}{2}s+2w(1-x))}\,\mathfrak{K}(1-x,w,s) \tag{4.2}$$

for $0 < w < 1$. Note that it is possible to write $\mathfrak{K}(1,-w,s)$ in the first term of the right hand side. Lerch [Lerc1887] or [Lerc1892; p. 21] proved that

$$\mathfrak{K}(w,x,s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{(1-w)t-2\pi i x}}{e^{t-2\pi i x}-1}t^{s-1}\,dt \tag{4.3}$$

provided $x,w \in (0,1)$ and $\mathrm{Re}(s) > 0$. The expression remains true also for $\Im(x) > 0$ or for complex w with $\mathrm{Re}(w) > 0$. The integral shows that the function $\mathfrak{K}(w,s,x)$ is analytic for x in the complex plane from which the lines $k - iy$ with $k \in \mathbb{Z}$ and $y \geq 0$ are cut off. Using G. N. Watson notation of the integral indicating that the path of integration is a loop beginning at ∞ on the upper border of the cut, going around the point 0 in the positive sense to the lower border and ending there at $-\infty$, we have ([Lerc1887])

$$\mathfrak{K}(w,x,s) = -\frac{\Gamma(1-s)}{2\pi i}\int_\infty^{(0+)} \frac{e^{-wz}(-z)^{s-1}}{1-e^{2\pi i x-z}}\,dz = \frac{\Gamma(1-s)}{2\pi i}\int_{-\infty}^{(0+)} \frac{e^{wz}\,z^{s-1}}{1-e^{2\pi i x+z}}\,dz\,, \tag{4.4}$$

where the contour (in the first integral) does not contain neither in the interior nor on it points of the form $2\pi i(x+k)$ for $k \in \mathbb{Z}$.

Lerch function $\mathfrak{K}$ is in a close connection with the function[2] (often called Lerch function, too)

$$\mathfrak{k}(w,z,s) = \sum_{n=0}^\infty \frac{z^n}{(w+n)^s} \tag{4.5}$$

defined through this series for $w \neq 0,-1,-2,\dots$, $|z| \leq 1$, $z \neq 1$ and $\mathrm{Re}(s) > 0$ or for $\mathrm{Re}(s) > 1$ if $z = 1$. Plainly,

$$\mathfrak{K}(w,x,s) = \mathfrak{k}\left(w, e^{2\pi i x}, s\right)\,. \tag{4.6}$$

LEMMA 4.1. *For every $m \in \mathbb{N}$ and all admissible values of s, z, w, x we have*

$$\sum_{k=0}^{m-1} z^{k/m}\mathfrak{k}\left(\frac{w+k}{m}, z, s\right) = m^s\mathfrak{k}(w, z^{1/m}, s)\,, \tag{4.7}$$

$$\sum_{k=0}^{m-1} e^{2\pi i xk/m}\,\mathfrak{K}\left(\frac{w+k}{m}, x, s\right) = m^s\mathfrak{K}\left(w, \frac{x}{m}, s\right)\,. \tag{4.8}$$

[2] denoted by Φ in [BaEr1953; §1.11]

Proof. Taking (4.6) into consideration it is sufficient to prove only (4.7). On the domain of absolute convergence of $\mathfrak{k}(w,x,s)$ we have

$$\begin{aligned}\sum_{k=0}^{m-1} z^{k/m}\mathfrak{k}\left(\frac{w+k}{m},z,s\right) &= \sum_{k=0}^{m-1} z^{k/m}\sum_{n=0}^{\infty}\frac{z^n}{\left(\frac{w+k}{m}+n\right)^s}\\ &= m^s\sum_{n=0}^{\infty}\sum_{k=0}^{m-1}\frac{z^{(k+mn)/m}}{(w+k+mn)^s}\\ &= m^s\mathfrak{k}(w,z^{1/m},s)\,.\end{aligned}$$

For other z,s use analytic continuation. □

The substitution $z^{1/m}\mapsto z$ leads to the following identity which can also be proved along the lines of the previous proof

$$\sum_{k=0}^{m-1} z^k\mathfrak{k}\left(\frac{w+k}{m},z^m,s\right) = m^s\mathfrak{k}(w,z,s)\,,$$

and after multiplication by z^w we get

$$\sum_{k=0}^{m-1} z^{w+k}\mathfrak{k}\left(\frac{w+k}{m},z^m,s\right) = m^s z^w\mathfrak{k}(w,z,s)\,. \tag{4.9}$$

Now define formally

$$\mathfrak{K}_\theta(w,x,s) = \sum_{n\in\mathcal{N}}\frac{\theta(n)\,\mathrm{e}^{2n\pi\mathrm{i}\,x}}{(w+n)^s}\,, \tag{4.10}$$

where $\mathcal{N}=\mathbb{Z}$, $\mathcal{N}=\mathbb{N}$, or $\mathcal{N}=\mathbb{N}\cup\{0\}$, and θ is an arithmetic function defined on $\mathbb{N}$ with the convention that the domain of θ is extended so that $\theta(-n)=-\theta(n)$, if necessary.

Lemma 4.2. *Let θ be a completely multiplicative function on $\mathbb{N}$, where the domain of θ is extended so that $\theta(n)=-\theta(n)$, if necessary. Then for every $m\in\mathbb{N}$ and all admissible values of s,z,w,x we have*

$$\sum_{k=0}^{m-1}\mathfrak{K}_\theta\left(mw,\pm\frac{x+k}{m},s\right) = \frac{\theta(m)}{m^{s-1}}\mathfrak{K}_\theta\left(\frac{w}{m},\pm x,s\right). \tag{4.11}$$

where on both sides the same sign should be taken.

P r o o f . On the domain of absolute convergence of $\mathfrak{K}_\theta(w, x. s)$ a standard algebraic manipulation gives

$$
\begin{aligned}
\sum_{k=0}^{m-1} \mathfrak{K}_\theta\left(mw, \pm\frac{x+k}{m}, s\right) &= \sum_{k=0}^{m-1}\sum_{n\in\mathcal{N}} \frac{\theta(n)}{(mw+n)^s}\, \mathrm{e}^{\pm 2\pi\,\mathrm{i}\, n(x+k)/m} \\
&= \sum_{n\in\mathcal{N}}\sum_{k=0}^{m-1} \frac{\theta(n)}{(mw+n)^s}\, \mathrm{e}^{\pm 2\pi\,\mathrm{i}\, xn/m}\, \mathrm{e}^{\pm 2\pi\,\mathrm{i}\, nk/m} \\
&= \sum_{n\in\mathcal{N}} \frac{\theta(n)}{(mw+n)^s}\, \mathrm{e}^{\pm 2\pi\,\mathrm{i}\, xn/m} \sum_{k=0}^{m-1} \mathrm{e}^{\pm 2\pi\,\mathrm{i}\ \imath k/m} \\
&= \sum_{d\in\mathcal{N}} \frac{\theta(dm)}{(mw+dm)^s}\, m\, \mathrm{e}^{\pm 2\pi dx} \\
&= \frac{\theta(m)}{m^{s-1}} \sum_{d\in\mathcal{N}} \frac{\theta(d)}{(w+d)^s}\, \mathrm{e}^{\pm 2\pi dx} \\
&= \frac{\theta(m)}{m^{s-1}} \mathfrak{K}_\theta(w, x, s)\,.
\end{aligned}
$$

For other s the result again follows by analytic continuation. □

4.1. Euler-Frobenius polynomials.

Define functions $\beta_n(x, \alpha)$ by the generating function (cf. [Apos1952])

$$
\frac{z\,\mathrm{e}^{xz}}{\alpha\,\mathrm{e}^{z} - 1} = \sum_{n=0}^{\infty} \beta_n(x, \alpha)\frac{z^n}{n!}\,. \tag{4.12}
$$

Functions $\beta_n(x, \alpha)$ are polynomials and represent in a certain sense a common generalization of Bernoulli and Euler polynomials. Namely

$$
\begin{aligned}
\beta_n(x, 1) &= B_n(x) && \text{for } n = 0, 1, 2 \dots, \\
\beta_n(x, -1) &= -nE_{n-1}(x)/2 && \text{for } n = 1, 2, 3 \dots.
\end{aligned} \tag{4.13}
$$

More generally, if $\alpha \in \mathbb{C}$ is an fth root of unity, then we can associate with α a series of f functions

$$
\beta_n(x, 1),\ \beta_n(x, \alpha),\ \beta_n(x, \alpha^2),\ \dots,\ \beta_n(x, \alpha^{f-1})\,. \tag{4.14}
$$

Taking (4.4) with $s = -n$, $n \geq 0$, and using Cauchy residue theorem we get

$$
\mathfrak{K}(w, \pm x, -n) = -\frac{\beta_{n+1}(w, \mathrm{e}^{\pm 2\pi\,\mathrm{i}\,x})}{n+1}\,. \tag{4.15}
$$

If, in (4.11), $\theta(n) = 1$ identically, then together with (4.15) we get

$$
\sum_{k=0}^{m-1} \beta_n\left(w, \mathrm{e}^{\pm 2\pi\,\mathrm{i}(x+k)/m}\right) = m^n \beta_n\left(\frac{w}{m}, \mathrm{e}^{\pm 2\pi\,\mathrm{i}\,x}\right) \tag{4.16}
$$

for $n = 1, 2, \ldots$. In both cases it is understood that the same sign is taken on both sides. On the other hand, (4.8) gives that for every $m \in \mathbb{N}$ and $n = 0, 1, 2 \ldots$ we have

$$\sum_{k=0}^{m-1} \mathrm{e}^{2\pi\,\mathrm{i}\,xk/m} \beta_n\left(\frac{w+k}{m}, \mathrm{e}^{2\pi\,\mathrm{i}\,x}\right) = m^{-n+1} \beta_n\left(w, \mathrm{e}^{2\pi\,\mathrm{i}\,x/m}\right). \tag{4.17}$$

Arguing with analytic continuation with respect to $z = \mathrm{e}^{2\pi\,\mathrm{i}\,x/m}$ we get the following identity holding for all complex z

$$\sum_{k=0}^{m-1} z^k \beta_n\left(\frac{w+k}{m}, z^m\right) = m^{-n+1} \beta_n(w, z). \tag{4.18}$$

Relation (4.17) implies that each β_n satisfies a compatibility relation. To see this note that $\mathrm{e}^{2\pi\,\mathrm{i}\,x} = \mathrm{e}^{2\pi\,\mathrm{i}\,x/m}$ if and only if

$$x = \frac{m}{m-1} t, \qquad t \in \mathbb{Z}.$$

Thus, if $m = 1 + Kf$ for some $K \in \mathbb{Z}$, and $t = Kh$ with $h \in \mathbb{Z}$, the number $\mathrm{e}^{2\pi\,\mathrm{i}\,x/m} = \mathrm{e}^{2\pi\,\mathrm{i}\,h/f}$ runs over fth roots of unity and moreover $\mathrm{e}^{2\pi\,\mathrm{i}\,x/m} = \mathrm{e}^{2\pi\,\mathrm{i}\,x}$. Therefore, if α is an fth root of unity and $m \equiv 1 \pmod{f}$, (4.17) gives the following compatibility relation each for $\beta_n(x, \alpha)$

$$\sum_{k=0}^{m-1} \alpha^k \beta_n\left(\frac{x+k}{m}, \alpha\right) = m^{-n+1} \beta_n(x, \alpha). \tag{4.19}$$

Polynomials $\beta_n(x, \alpha)$ form so called Appell set of polynomials with leading coefficients $\neq 1$. Their normalization leads to polynomials

$$\tau_n(x, \alpha) = \frac{\alpha - 1}{n+1} \beta_{n+1}(x, \alpha), \qquad \alpha \neq 1, \quad n = 0, 1, 2, \ldots, \tag{4.20}$$

introduced by E u l e r [Eule1755; pp. 487–491] and also studied by F r o b e n i u s [Frob1910]). Their generating function is

$$\frac{\alpha - 1}{\alpha\,\mathrm{e}^z - 1} \mathrm{e}^{xz} = \sum_{n=0}^{\infty} \tau_n(x, \alpha) \frac{z^n}{n!}, \qquad \alpha \neq 1.$$

A modified multiplication formula proved by C a r l i t z [Carl1953] for τ_n can be readily derived from (4.19), and is therefore left to the reader. Equivalent reformulation of the compatibility relations in terms of certain numbers associated with these polynomials can be found in [Por1999].

Note that from definition of $\beta_n(x, \alpha)$ we get the difference equation

$$\alpha \beta_n(x+1, \alpha) - \beta_n(x, \alpha) = n x^{n-1}. \tag{4.21}$$

4.2. Hurwitz zeta.

One of the important classical prototypes of functions satisfying Kubert identities is the Hurwitz zeta function

$$\zeta_s(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

defined in this way for $\mathrm{Re}(s) > 1$ and real $x > 0$. Since Hurwitz zeta is a special case of the function $\mathfrak{K}$, namely $\mathfrak{K}(x, 0, s)$, the relation (4.8) shows that $\zeta_s(x)$ satisfies Kubert identities in the form

$$\sum_{a=0}^{m-1} \zeta_s\left(\frac{x+a}{m}\right) = m^s \zeta_s(x), \tag{4.22}$$

which is true for every integer $m \geq 1$. This together gives the well-known fact that Hurwitz zeta function $\zeta_z(x)$ is a Kubert function for every complex constant $z \neq 0$. More precisely, $\zeta_z(x)$ satisfies (KI_s) with $s = 1 - z$. Clearly, this result can also be reformulated in the sense that the Hurwitz zeta function $\zeta_s(x)$ is a generalized m^s-Kubert function for every complex number $s \neq 0$.

Since

$$\zeta_{-s}(x) = -\frac{B_{s+1}(x)}{s+1}, \qquad s = 0, 1, \dots, \tag{4.23}$$

the compatibility relation for Hurwitz zeta implies Raabe multiplication formula for Bernoulli polynomials, and vice versa Raabe formula implies that Hurwitz zeta function satisfies Kubert identities.

Note that the Hurwitz zeta satisfies the difference equation

$$\zeta_s(x+1) - \zeta_s(x) = -\frac{1}{x^s}. \tag{4.24}$$

J. Milnor [Miln1983; p. 309] proved that the functions

$$\gamma_{1-s}(x) = \frac{\partial}{\partial s} \zeta_s(x)$$

satisfy Kubert identity (KI_s) with a correction term involving sth Bernoulli polynomial

$$\gamma_s(x) = \frac{B_s(x)}{s} \cdot \log m + m^{s-1} \sum_{k=0}^{m-1} \gamma_s\left(\frac{x+k}{m}\right), \qquad s = 1, 2, \dots. \tag{4.25}$$

4.3. Polygamma functions.

Substituting the Lerch identity ([Lerc1894; p. 18])

$$\gamma_1(x) = \log \Gamma(x) - \frac{1}{2} \log(2\pi)$$

into (4.25) with $s = 1$ we get the multiplication formula of Gauß and Legendre

$$\prod_{k=0}^{m-1} \Gamma\left(\frac{x+k}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-x} \Gamma(x). \tag{4.26}$$

Consequently, the digamma function (also called the logarithmic derivatives of the Gamma function) investigated by Legendre, Poisson and Gauß

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x)$$

also satisfies a variation of Kubert identity (KI_0), however, with a correction term. Namely

$$\psi(x) = \frac{1}{m} \sum_{k=0}^{m-1} \psi\left(\frac{x+k}{m}\right) + \log m\,. \tag{4.27}$$

Since the trigamma functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2}\,, \qquad z \neq 0, -1, 2, \dots\,,$$

coincides with Hurwitz zeta $\zeta_2(x)$, trigamma satisfies (KI_{-1}). Using Lemma 2.4 we get that, more generally, the classical polygamma function defined through

$$\psi^{(n)}(x) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} \log \Gamma(x)\,, \qquad n = 0, 1, 2, \dots\,,$$

also satisfies the Kubert identity in the form

$$\psi^{(n)}(x) = \delta_{n,0} \log m + m^{-(n+1)} \sum_{k=0}^{m-1} \psi^{(n)}\left(\frac{x+k}{m}\right).$$

where $\delta_{n,0}$ is the Kronecker delta.

For the polygamma function we have

$$\psi^{(n)}(x+1) - \psi^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}\,. \tag{4.28}$$

If we define for $n = 1, 2, \dots$

$$G^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} G^{(n-1)}(x) = \frac{1}{2^n}\left[\psi^{(n)}\left(\frac{x}{2}+\frac{1}{2}\right) - \psi^{(n)}\left(\frac{x}{2}\right)\right], \tag{4.29}$$

then

$$G^{(n)}(1+x) + G^{(n)}(x) = \frac{2(-1)^n n!}{x^{n+1}} \tag{4.30}$$

and for odd positive m we have

$$G^{(n)}(x) = \frac{1}{m^{n+1}} \sum_{k=0}^{m-1} (-1)^k G^{(n)}\left(\frac{x+k}{m}\right). \tag{4.31}$$

Milnor identity (4.25) can also be used for higher derivatives

$$\gamma_{1-s}^{(n)}(x) = \frac{\partial^n}{\partial x^n}\gamma_s(x), \qquad n = 0, 1, 2, \dots .$$

In corresponding identities for $\gamma_s^{(n)}(x)$, $s \in \mathbb{N}$, a correction term appears only if $s \leq n$.

4.4. Lerch ell function.

Another important function satisfying Kubert relations is the Lerch function $\ell_s(x)$, also called the periodic zeta function, which is defined as

$$\ell_s(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi \mathrm{i}\, nx}}{n^s}$$

for real x, $\operatorname{Re} s > 1$. It can be continued analytically by the formula of Jonquiére (cf. related relation (4.3)) for $\operatorname{Re} s > 0$ and to other values of s by

$$\ell_{s-1}(x) = \frac{1}{2\pi \mathrm{i}} \cdot \frac{\partial}{\partial x}\ell_s(x). \tag{4.32}$$

Lerch function $\ell_s(x)$ is a special case of $\mathfrak{K}(w, x, s)$ and (4.11) shows that Lerch function satisfies (KI_s).

The special role of the Hurwitz zeta and Lerch function is given by an interesting result of Milnor [Miln1983; p. 287] asserting that they span the two-dimensional complex vector space $\mathcal{K}_s$, $s \in \mathbb{C}$, of all continuous maps f: $(0,1) \to \mathbb{C}$ which satisfy the Kubert identity (KI_s) for every positive integer m, and every $x \in (0,1)$.

4.5. Polylogarithms.

Trivial decomposition

$$1 - z^m = (1-z)(1-\omega z)(1-\omega^2 z)\dots(1-\omega^{m-1} z), \tag{4.33}$$

where $\omega = e^{2\pi \mathrm{i}/m}$, provides another compatible type identity in the multip'icative form. It can be generalized to $(\mu, \mathfrak{m})$-cover (1.1) as follows:

$$\prod_{t=1}^{w}\left(1 - e^{2\pi \mathrm{i}(a_t + x)/b_t} \cdot z^{M/b_t}\right) = \prod_{t=1}^{b_0}\left(1 - e^{2\pi \mathrm{i}(t+x)/b_0} \cdot z^{M\ b_0}\right)^{\mathfrak{m}\ j)} .$$

where M is a common multiple of the moduli of (1.1). Taking logarithm of both sides of (4.33), dividing by x and integrating we get the so called *factorization theorem*

$$\mathrm{Li}_2(z^m) = m\big(\mathrm{Li}_2(z) + \mathrm{Li}_2(\omega z) + \cdots + \mathrm{Li}_2(\omega^{m-1} z)\big) \tag{4.34}$$

for the dilogarithm, which is the special case $n = 2$ of the polylogarithm function

$$\mathrm{Li}_n(z) = \frac{z}{1^n} + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots, \qquad |z| \le 1,$$

(using an integral representation can be analytically extended to a wider range of values of z). The factorization formula

$$\mathrm{Li}_n(z^m) = m^{n-1}\big(\mathrm{Li}_n(z) + \mathrm{Li}_n(\omega z) + \cdots + \mathrm{Li}_n(\omega^{m-1} z)\big) \tag{4.35}$$

for the polylogarithm function Li_n can be derived from[3] (4.11) firstly for $z = \mathrm{e}^{2\pi x/m}$, and for general z by analytic continuation. Putting $z = r\,\mathrm{e}^{\mathrm{i}\,\vartheta}$ in (4.35) and taking the real part we get

$$\mathrm{Li}_n(r^m, \vartheta) = m^{n-1} \sum_{k=0}^{m-1} \mathrm{Li}_n\left(r, \frac{\vartheta + 2\pi k}{m}\right).$$

4.6. Trigonometric functions.

Euler identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} = \pi \csc \pi x$$

together with Gauß–Legendre formula (4.26) multiplied with itself but with argument $1 - x$ (and the reversed ordering of the factors) gives the following formula known to Euler [Eule1748; §240]

$$\sin \pi x = 2^{m-1} \prod_{k=0}^{m-1} \sin \frac{(x+k)\pi}{m}. \tag{4.36}$$

Since $\int \cot x \,\mathrm{d}x = \ln(\sin x)$ (cf. Lemma 2.4), or from

$$\psi^{(n)}(x) - (-1)^n \psi^{(n)}(1-z) = -\pi \frac{\mathrm{d}^n}{\mathrm{d}x^n} \cot(\pi x)$$

and (4.27) for function ψ we get

$$\cot \pi x = \frac{1}{m} \sum_{k=0}^{m-1} \cot \frac{(x+k)\pi}{m}, \tag{4.37}$$

[3]or from (4.34) by the successive integration with respect to $\mathrm{d}(\log x) = \mathrm{d}x/x$

which again can be found in [Eule1748; §250]. Taking derivative of (4.37) gives the multiplication formula for the square of the cosecant function

$$\csc^2 \pi x = \frac{1}{m^2} \sum_{k=0}^{m-1} \csc^2 \frac{(x+k)\pi}{m} .$$

For the cosecant alone we have only an odd version. Using

$$G^{(n)}(x) + (-1)^n G^{(n)}(1-z) = 2\pi \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{1}{\sin(\pi x)}$$

for $n = 0$ we get from (4.31) for odd m

$$\csc \pi x = \frac{1}{m} \sum_{k=0}^{m-1} (-1)^k \csc \frac{(x+k)\pi}{m} , \qquad 2 \nmid m , \tag{4.38}$$

a formula again listed in [Eule1748; §237, §248]. Taking derivative of (4.38) we get a multiplication formula for $\csc x \cot x$, etc.

A further chain of identities results after the following algebraic manipulation of (4.36). Substitution $-x$ for x yields an odd analog of (4.36)

$$\sin \pi x = 2^{m-1} \prod_{k=0}^{m-1} \sin \frac{(x-k)\pi}{m} , \qquad 2 \nmid m . \tag{4.39}$$

On the other hand, the substitution $x \mapsto x - (2m+1)/2$ gives

$$(-1)^m \cos \pi x = 2^{2m} \prod_{k=0}^{2m} \cos \frac{(x-k)\pi}{2m+1} . \tag{4.40}$$

This leads for even m to a multiplication (or perhaps better factorization) formula for the cosine function

$$\cos \pi x = 2^{4m} \prod_{k=0}^{4m} \cos \frac{(x-k)\pi}{4m+1} . \tag{4.41}$$

Dividing (4.39) by (4.41), or using $\int \tan x \; \mathrm{d}x = -\ln \cos x$. we arrive at

$$\tan \pi x = \prod_{k=0}^{4m} \tan \frac{(x-k)\pi}{4m+1} . \tag{4.42}$$

5. Summation of functions

We saw in Section 4 that the examples of functions satisfying Kubert's type identities have certain relations to difference equations. The simplest difference equation is of the type

$$\underset{\omega}{\triangle} F(x) \stackrel{\text{def}}{=} \frac{F(x+\omega)-F(x)}{\omega} = \varphi(x)\,, \tag{5.1}$$

$$\underset{\omega}{\nabla} G(x) \stackrel{\text{def}}{=} \frac{G(x+\omega)+G(x)}{2} = \varphi(x)\,, \tag{5.2}$$

where the span (increment) ω is a nonzero constant. The operator $\underset{\omega}{\triangle}$ is the so called first (Nörlund) difference quotient. If $\omega = 1$, we shall simply write $\triangle$ and ∇. It is interesting to note that

$$\lim_{\omega\to 0} \underset{\omega}{\triangle} \varphi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\varphi\,.$$

The solutions of (5.1) will be called *sums* and those of (5.2) *alternating sums* of $\phi(x)$, resp. The general solution of (5.1) or of (5.2) can be represented as a sum of a particular solution and an arbitrary solution of $\underset{\omega}{\triangle} \pi(x) = 0$, or of $\underset{\omega}{\nabla} \mathfrak{p}(x) = 0$, respectively. The problem is to determine particular solutions having prescribed properties. These solutions are called *principal sums*. Thus for instance, polynomial $\varphi(x)$ should lead to a polynomial solution, or the solution should reproduce the known functions, as Γ function, etc. An immediate verification shows that

$$A - \omega \sum_{j=0}^{\infty} \varphi(x + j\omega) \qquad \text{or} \qquad A + \omega \sum_{j=1}^{\infty} \varphi(x - j\omega) \tag{5.3}$$

is a *formal* solution of (5.1) with an arbitrary constant A, and

$$2\sum_{j=0}^{\infty} (-1)^j \varphi(x + j\omega) \qquad \text{or} \qquad -2\sum_{j=1}^{\infty} (-1)^j \varphi(x - j\omega) \tag{5.4}$$

formally solves the equation (5.2). If these series converge, they represent (with $A = 0$) the basic prototypes of principal sums. For instance, if $\varphi(x) = \mathrm{e}^{-x}$, then

$$\frac{\omega\,\mathrm{e}^{-x}}{\mathrm{e}^{-\omega} - 1}$$

is the principal sum of (5.1). Another important example is given by

$$\varphi(x) = -\frac{1}{x^s}\,, \qquad s > 1\,.$$

Then the principal sum of (5.1) is the Hurwitz zeta (cf. (4.24)).

Nörlund wrote the constant A in (5.3) in the form $\int_c^\infty \phi(t)\ \mathrm{d}t$ and thus (5.3) gets the form

$$\int_c^\infty \varphi(t)\ \mathrm{d}t - \omega \sum_{s=0}^\infty \phi(x+s\omega)\,,$$

with an arbitrary constant c.

Unfortunately, the series and the integral here do not converge in general. Therefore Nörlund, and other authors apply some ingenious summation tricks to guarantee a convenient "expressibility" of the principal sums also in these cases. The general scheme of Nörlund's approach can be described as follows.

Suppose that $\varphi(x)$ is positive and continuous (complex or real) function of real $x \geq b$. He replaced the function $\varphi(x)$ on the right hand sides by another function $\phi(x,\eta)$ of x and of a new parameter $\eta > 0$, which is chosen so that

a) $\lim\limits_{\eta\to 0} \phi(x,\eta) = \varphi(x)$,

b) both $\int_c^\infty \phi(x,\eta)\,\mathrm{d}x$ and $\sum_{s=0}^\infty \phi(x+s\omega,\eta)$ converge.

Then, the function

$$F(x \mid \omega,\eta) = \int_c^\infty \phi(t,\eta)\ \mathrm{d}t - \omega \sum_{s=0}^\infty \phi(x+s\omega,\eta)\,.$$

solves the difference equation

$$\mathop{\triangle}_{\omega} F(x) = \phi(x.\eta), \tag{5.5}$$

and if we let $\eta \to 0$, equation (5.5) reduces to (5.1) and

$$F(x \mid \omega) = \lim_{\eta\to 0} F(x \mid \omega, \eta)$$

is the solution of (5.1), which is called the *principal sum* provided this limit exists uniformly and is independent of the choice of $\phi(x,\eta)$ subject to conditions a) and b). An analogical scheme is applied to equation (5.2).

Nörlund also used a special notation which is not available in the contemporary TeX font sets, therefore we will use the following adapted integral sign instead, i.e.

$$F(x \mid \omega) = \int_c^x\!\!\!\!\!\triangle\ \phi(z) \mathop{\triangle}_{\omega} z$$

will denote the principal sum of equation (5.1), and

$$G(x \mid \omega) = \oint \phi(x) \underset{\omega}{\nabla} x$$

the principal alternating sum of (5.2).

Nörlund applies the above scheme with

$$\phi(x, \eta) = \varphi(x)\, \mathrm{e}^{-\eta\lambda(x)}\,,$$

where $\lambda(x) = x^p (\log x)^q$ for some $p \geq 1,\ q \geq 0$.

Convenient sufficient conditions for the fulfilment of the above requirements is given by the following assumptions ([Nörl1924; pp. 48–49]):

(1) $\varphi(x)$ has for $x \geq b$ a continuous derivative of order m for some m such that

$$\lim_{x\to\infty} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \varphi(x) = 0$$

and moreover

(2a) $\sum\limits_{n=0}^{\infty} (-1)^n \frac{\mathrm{d}^m}{\mathrm{d}x^m} \varphi(x + n\omega)$ is uniformly convergent in the interval $b \leq x \leq b+\omega$, and consequently in every interval $b \leq x \leq B$ for any arbitrarily large B,

(2b) $\sum\limits_{n=0}^{\infty} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \varphi(x + n\omega)$ is uniformly convergent in the interval $b \leq x \leq b+\omega$ (and consequently in every interval $b \leq x \leq B$ for any arbitrarily large B).

Conditions (1) and (2a) are, of course, applicable in the case when the principal alternating sum exists, and (1) and (2b) for the principal sum of $\varphi(x)$. Thus, for instance, we get ([Nörl1924; p. 53])

$$\oint_0^x r z^{r-1} \triangle z = B_r(x) \qquad \text{and} \qquad \oint x^r \nabla x = E_r(x)\,,$$

or more generally

$$\oint_0^x r z^{r-1} \underset{\omega}{\triangle} z = \omega^r B_r\left(\frac{x}{\omega}\right).$$

In connection with function listed in Section 4 it follows from the definition of the principal sum that it satisfies the compatibility property, called the multiplication theorem ([Nörl1924; p. 44])

$$\sum_{s=0}^{m-1} F\left(x+\frac{s\omega}{m}\,\Big|\,\omega\right)=mF\left(x\,\Big|\,\frac{\omega}{m}\right). \tag{5.6}$$

whereas for principal alternating sum of (5.2) we have

$$\sum_{s=0}^{m-1}(-1)^s F\left(x+\frac{s\omega}{m}\,\Big|\,\omega\right)=-\frac{\omega}{2}G\left(x\,\Big|\,\frac{\omega}{m}\right). \tag{5.7}$$

$$\sum_{s=0}^{m-1}(-1)^s G\left(x+\frac{s\omega}{m}\,\Big|\,\omega\right)=G\left(x\,\Big|\,\frac{\omega}{m}\right). \tag{5.8}$$

Moreover

$$G(x\mid\omega)=\frac{2}{\omega}\left[F(x\mid\omega)-F(x\mid 2\omega)\right]. \tag{5.9}$$

Application of these relations to the $\varphi(x)=x^r$, $r\in\{0,1,2,\dots\}$. leads to the mentioned multiplication (or compatibility) relations for Bernoulli and Euler polynomial, and the known relation between them. The same pattern is also followed by the psi function and function G from Section 4, etc. Here

$$\psi(x\mid\omega)=\mathop{\text{S}}_{1}^{x}\frac{1}{z}\mathop{\triangle}_{\omega} z=\psi\left(\frac{x}{\omega}\right)+\log\omega\,.$$

If we apply the above scheme to $\varphi(x)=\frac{1}{2}\log x$, then we arrive at the logarithm of the Γ-function. Γ-function satisfies the difference equation

$$u(x+1)=xu(x). \tag{5.10}$$

The alternating sum function γ related to the Γ-function and defined through (cf. [Nörl1924; p. 115])

$$\log\gamma(x\mid\omega)=\frac{1}{2}\,\mathop{\text{S}}\log x\mathop{\nabla}_{\omega} x$$

can be seldom found in book on special functions. It satisfies the difference equation

$$\gamma(x+1)\gamma(x)=x\,.$$

It follows from the above theory that it also satisfies a multiplication type formula

$$\frac{\gamma(x)\gamma\left(x+\frac{2}{m}\right)\cdots\gamma\left(x+\frac{m-1}{m}\right)}{\gamma\left(x+\frac{1}{m}\right)\gamma\left(x+\frac{3}{m}\right)\cdots\gamma\left(x+\frac{m-2}{m}\right)}=\frac{\gamma(mx)}{\sqrt{m}}$$

for odd m.

Note that the difference equation (5.10) is, besides the Γ-function, also satisfied be the complementary Γ-function defined by

$$\Gamma_c(x) = (1 - e^{2\pi i x})\Gamma(x) = \int_{\infty}^{(0+)} t^{x-1} e^{-t} \, dt .$$

Combining (4.26) and (4.33) we get the compability relation

$$\prod_{k=0}^{m-1} \Gamma_c\left(\frac{x+k}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-x} \Gamma_c(x) .$$

For complementary argument theorems, and the relations between the functions of the complementary arguments x, $1-x$, which we shortly touched e.g. in Corollary 3.6, we refer the reader again to [Nörl1924].

Another interesting result towards that mentioned in Lemma 2.4 is the following one ([Nörl1924; p. 55]):

LEMMA 5.1. *If the above conditions* (1), *and* (2a), *or* (1) *and* (2b) *are fulfilled, then*

$$\frac{d}{dx} \oint_a^x \varphi(z) \underset{\omega}{\Delta} z = \oint_a^x \varphi'(z) \underset{\omega}{\Delta} z + \varphi(a) ,$$

$$\frac{d}{dx} \oint \varphi(z) \underset{\omega}{\nabla} z = \oint_a^x \varphi'(z) \underset{\omega}{\Delta} z .$$

We have seen in Section 4 that there are compatible functions which are not Kubert functions. Thus the above lemma extends techniques of Lemma 2.4 to more general spaces of functions.

J. Beebee [Beeb1995; Definition 3] introduced the following problem:

Let φ be a function defined in the interval $[x_0, x_1]$ with values in an Abelian group. Let (1.1) be an exact cover, not necessarily with standardized offsets. Then the identity

$$\sum_k \varphi(x+k)[x_0 \le x+k \le x_1] = \sum_{i=1}^{w} \sum_{\alpha} \varphi(x + a_i + \alpha b_i)[x_0 \le x + a_i + \alpha b_i \le x_1]$$

is called an *exact sum*. He asked (p. 9) whether exact sums lead to functional identities like the Kubert ones.

The above considerations show that the answer is positive provided $x_1 = \infty$, the Abelian group is a subset of complex numbers and the series in (5.3) is convergent. Moreover, the question can be correspondingly extended to functions of the type $\phi(x, \eta)$. We leave the details to the reader.

We have considered up to now only the case of real argument x. For complex arguments x, ω the corresponding theory is developed in [Nörl1924; Chapter 4] or [Nörl1927; Chapter III]. Cauchy's residue theorem can be. of course, used here for the evaluation of the sums in (5.3) and (5.4). For the evaluation of (5.3) the function $\pi \cot(\pi z)$ with residue $+1$ for integral values z provides a good service. This idea was used by Beebee in his development of the theory exact sums. On the other hand, the evaluation of (5.4) can be often successfully finished using $\pi/\sin(\pi x)$ having residue $(-1)^s$ for integral values $x = s$. By the way, function $\big(\pi/\sin(\pi x)\big)^2$ can also be used in evaluation of (5.3). We refer the reader for more details about this standard technique to [Lind1905: Chapter III], [Nörl1924], [Nörl1927].

6. Selected applications

In this section we shall briefly outline some previously published instances of Theorem 1.2. They are of two types in general. Under the first category come the results used to characterize various forms of $(\mu, \mathfrak{m})$-covers, mostly exact covers. To the second category belong papers, for instance [Beeb1978], [Beeb1991]. [Beeb1992], [Beeb1994], [Por1994a], [Por1994b], [Por1999] in which this Theorem is used to reprove or extend some previously known identities proved originally by different methods, again mostly employing exact covers. To be more concrete. cases covered by Theorem 1.2 for *exact covers* were investigated and proved for the following functions:

[Beeb1991]:

- $\varphi_d(a) = 2\sin\big((x+a)\pi/d\big)$ in multiplicative form,
- $\varphi_d(a) = (1/d)\cot\big((x+a)\pi/d\big)$,
- $\varphi_d(a) = \big(1/d^2\big)\csc^2\big((x+a)\pi/d\big)$.

In fact the sin function already appears in [Stei1958].

[Frae1973], [Frae1975]: contains identities for exact covers involving Bernoulli and Euler polynomials, i.e.

- $\varphi_d(a) = d^{n-1}B_n\big((x+a)/d\big)$,
- $\varphi_{2d+1}(a) = (2d+1)^n(-1)^a E_n\big((x+a)/(2d+1)\big)$.

[Beeb1992]: a variation of the Fraenkel result extending a recurrence for Bernoulli numbers proved in [DeRo1991]. A further generalization can be found in [Por1999].

[Beeb1994]: The Pochhammer symbol

$$(z)_0 = 1\,, \qquad (z)_n = z(z+1)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}$$

for $n = 1, 2, \ldots$ satisfies trivially a multiplicative compability relation

$$(z)_m = m^m \prod_{a=0}^{m-1} \left(\frac{z+a}{m}\right)_1\,,$$

or extended for an exact cover (1.1) in the form

$$(z)_N = \prod_{i=1}^{w} b_i^{N/b_i} \left(\frac{z+a_1}{b_i}\right)_{N/b_i}\,, \tag{6.1}$$

where N is the least common multiple of the moduli $b_1, \ldots, b_w$. Using Gauß's identity

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!n^z}{(z)_n}$$

it is proved in [Beeb1994] that the function

$$\varphi_b(a) = \begin{cases} \dfrac{\Gamma\left(\frac{z}{b}\right)}{b^{1-\frac{z}{b}}} & \text{if } a = 0\,, \\[2ex] \dfrac{\Gamma\left(\frac{z+a}{b}\right)}{b^{-\frac{z}{b}}\Gamma\left(\frac{a}{b}\right)} & \text{if } a \neq 0 \end{cases}$$

is a $\mathbb{N}$-compatible function.

Corresponding extension involving Bernoulli and Euler polynomials (and numbers) for *m-covers or for general* $(\mu, \mathfrak{m})$*-covers* can be found in [Poru1975], [Poru1976], [Poru1982], [Por1994a], and [Por1994b]. As a special case of identities with the first Bernoulli polynomial Hermite formula involving the greatest integer function is mentioned in [Por1994a] and [Por1994b]. These Hermite formulas are actually a consequence of relations of the type (2.3). The Bernoulli polynomials and the greatest integer function are, among others, also mentioned by S u n [Sun1989a] already in 1989, where the author noticed that the function

$$\varphi_d(a) = d^{\frac{z+a}{d}-\frac{1}{2}}\Gamma\left(\frac{z+a}{d}\right)(2\pi)^{-1/2}$$

is an example of a compatible function (in the multiplicative form) which does not fulfill Kubert identities. This is so due to the appearance of factor $(2\pi)^{-1/2}$ (note that the first factor is of the type $d^{B_1(z)}$, and thus satisfying compability relation in the multiplicative form). This factor $(2\pi)^{-1/2}$ can be removed e.g. taking

$$\frac{d^{\frac{z+a}{d}-\frac{1}{2}}\Gamma\left(\frac{z+a}{d}\right)}{d^{\frac{a}{d}-\frac{1}{2}}\Gamma\left(\frac{a}{d}\right)}\,,$$

which gives the Beebee's function mentioned above.

The reader can found further examples of such functions in the part 4 of the present paper. Beebee [Beeb1995] used some of those which satisfy (generalized) Kubert identities. His proofs of involving Lerch ell and Hurwitz zeta functions are based on Jonquirè formula or on the loop integrals we mentioned in part 4. We showed thus that these result can be derived in a "more elementary" way based on the fact that these functions are $\mathbb{N}$-compatible. Beebee also introduces functions denoted by $V_n(x;a,b)$ and $U(x,z;a,b)$ which in our notation can be written as

$$V_n(x;a,b) = \beta_n\big(x, \mathrm{e}^{-2\pi\,\mathrm{i}\,a/b}\big)\,, \qquad U_n(x,z;a,b) = z^{x+a}\beta_n\left(\frac{x+a}{b}, z^b\right).$$

Since $V_k(0;a,b) = \beta_k\big(0, \mathrm{e}^{-2\pi\,\mathrm{i}\,a/b}\big)$ is an Appell polynomial, its expansion in powers of x (cf. e.g. [Por1999; (1.12)]) implies

$$V_n(x;a,b) = \sum_{r=0}^{n} \binom{n}{r} V_{n-r}(0;a,b)x^r$$

what is Beebee formula (21) in [Beeb1995]. The same relation also implies

$$U_n(x,z;a,b) = z^{x+a}\beta_n\left(\frac{x+a}{b}, z^b\right) = z^{x+a}\sum_{r=0}^{n}\binom{n}{r} V_{n-r}(0,z^b)\left(\frac{x+a}{b}\right)^r.$$

what is the second relation of part a) of Beebee [Beeb1995; Theorem 7]. Further, since $U_n(x,z;0,1) = z^x\beta_n(x,z)$, the expansion

$$\beta_n(x,\alpha) = f^{n-1}\sum_{r=0}^{f-1}\alpha^r B_n\left(\frac{x+r}{f}\right)$$

implies part b) of [Beeb1995; Theorem 7], provided z is an fth root of unity.

Relation (4.16) implies that $\varphi_d(a) = d^{-n}\beta_n\big(dw, \mathrm{e}^{\pm 2\pi\,\mathrm{i}(x+a)/d}\big)$ form a compatible family. Then Corollary 3.3 together with these facts gives the following extension of one of Beebee's results to general $(\mu,\mathfrak{m})$-covers.

COROLLARY 6.1. *Let $n = 0,1,2\dots$ and $w,x\in\mathbb{R}$. If (1.1) is a $(\mu,\mathfrak{m})$-cover, then*

$$\sum_{t=0}^{b_0-1}\mathfrak{m}(t)b_0^{-n}\beta_n\big(b_0w, \mathrm{e}^{\pm2\pi\,\mathrm{i}(x+t)/b_0}\big) = \sum_{t=0}^{w}\mu_t b_t^{-n}\beta_n\big(b_tw, \mathrm{e}^{\pm2\pi\,\mathrm{i}(x+a_t)/b_t}\big).$$

Similarly, (4.18) shows that $\varphi_d(a) = z^a d^{n-1}\beta_n\left(\frac{x+a}{d}, z^d\right)$ is a compatible family.

COROLLARY 6.2. *Let* $n = 0, 1, 2\dots$ *and* $x \in \mathbb{R}$ *and* $z \in \mathbb{C}$. *If* (1.1) *is a* $(\mu, \mathfrak{m})$*-cover, then*

$$\sum_{t=0}^{b_0-1} \mathfrak{m}(t) z^t b_0^{n-1} \beta_n\left(\frac{x+t}{b_0}, z^{b_0}\right) = \sum_{t=0}^{w} \mu_t z^{a_t} b_t^{n-1} \beta_n\left(\frac{x+a_t}{b_t}, z^{b_t}\right).$$

If $z = \alpha$ is an fth root of unity, then we get the following generalization of identities proved in [Poru1975; Theorem 4, D_2] and [Por1994b; Theorem 1] if instead of functions (4.13) those of (4.14) are taken:

COROLLARY 6.3. *Let* $n = 0, 1, 2\dots$ *and* $x \in \mathbb{R}$, *and let* α *be an* f*th root of unit. If* (1.1) *is a* $(\mu, \mathfrak{m})$*-cover, then*

$$\sum_{t=0}^{b_0-1} \mathfrak{m}(t) \alpha^t b_0^{n-1} \beta_n\left(\frac{x+t}{b_0}, \alpha^{b_0}\right) = \sum_{t=0}^{w} \mu_t \alpha^{a_t} b_t^{n-1} \beta_n\left(\frac{x+a_t}{b_t}, \alpha^{b_t}\right).$$

Note that the last identity yields a compact form of writing the Raabe and Nielsen multiplicative formulas for exact covers (1.2) with odd or even w.

We leave to the reader to write down further identities with functions we mentioned in Section 4. Thus, for instance, Beebee function $L_s(w, z)$ of [Beeb1995] equals the function $z^w \mathfrak{k}(w, z, s)$ in our notation. Then (4.15) implies due to (4.6) that

$$\mathfrak{k}(w, \mathrm{e}^{2\pi \mathrm{i} x}) = -\frac{b_{n+1}(w, \mathrm{e}^{2\pi \mathrm{i} x})}{n+1},$$

which using analytic continuation gives

$$L_s(w, z) = -\frac{z^w \beta_{n+1}(w, z)}{n+1}.$$

Relation (4.9) implies that functions of the type $f(a, d) = z^{w+a} d^{-s} \mathfrak{k}\left(\frac{w+a}{d}, z^d, s\right)$ are compatible, etc.

Also note that in many cases it is not necessary to suppose that the offsets are standardized, e.g. in Corollary 6.1. This is true, for functions which are constructed using (2.3), or for trigonometric function cot, or when the offsets appear in a suitable form in the argument of an exponential, etc.

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