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ASYMPTOTIC BEHAVIOUR OF A CLASS OF THIRD ORDER DELAY-DIFFERENTIAL EQUATIONS

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ABSTRACT. Sufficient conditions in terms of coefficient functions or a delay-differential inequality are obtained so that delay-differential equations of the form

$$y'''(t) + a(t)y''(t) + b(t)y'(t) + c(t)y\big(g(t)\big) = 0 \qquad (*)$$

have the property (B), that is, every nonoscillatory solution $y(t)$ of $(*)$ satisfies $y(t)y^{(i)}(t) > 0$, $0 \le i \le 3$, for large t, where $a, b, c, g \in C\big([\sigma,\infty),\mathbb{R}\big)$, $\sigma \in \mathbb{R}$, such that $a(t) \le 0$, $b(t) \le 0$, $c(t) < 0$, $g(t) \le t$ and $g(t) \to \infty$ as $t \to \infty$.

1.

In this paper, we study the asymptotic behaviour of solutions of a class of third order delay-differential equations of the form

$$y'''(t) + a(t)y''(t) + b(t)y'(t) + c(t)y\big(g(t)\big) = 0\,, \tag{1.1}$$

where $a \in C^2\big([\sigma,\infty),\mathbb{R}\big)$, $b \in C^1\big([\sigma,\infty),\mathbb{R}\big)$ and $c \in C\big([\sigma,\infty),\mathbb{R}\big)$ such that $a(t) \le 0$, $b(t) \le 0$, $c(t) < 0$, $\sigma \in \mathbb{R}$, and $g \in C\big([\sigma,\infty),\mathbb{R}\big)$ such that $g(t) \le t$, $g(t) \to \infty$ as $t \to \infty$. Equation (1.1) may be written as

$$\big(r(t)y''(t)\big)' + q(t)y'(t) + p(t)y\big(g(t)\big) = 0\,, \tag{1.2}$$

where $r(t) = \exp\Big(\int\limits_\sigma^t a(s)\,\mathrm{d}s\Big)$, $q(t) = b(t)r(t) \le 0$ and $p(t) = c(t)r(t) < 0$. If $g(t) \equiv t$, then (1.1) takes the form

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0 \tag{1.3}$$

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which has been studied by many authors in recent years (see [1], [4], [6], [9], [10]). In [1], Ahmad and Lazer obtained the following criteria for the asymptotic behaviour of solutions of (1.3):

LEMMA 1.1. *A necessary and sufficient condition for* (1.3) *to have an oscillatory solution is that for an arbitrary nonoscillatory solution* $u(t)$ *of* (1.3) *the following conditions hold:*

$$u(t)u'(t)u''(t)u'''(t) \neq 0 \qquad \textit{for} \quad t \geq t_0 \geq \sigma$$

and

$$\operatorname{sgn} u(t) = \operatorname{sgn} u'(t) = \operatorname{sgn} u''(t) = \operatorname{sgn} u'''(t)\,, \qquad t \geq t_0 \geq \sigma\,.$$

Further, $\lim\limits_{t\to\infty} |u(t)| = \lim\limits_{t\to\infty} |u'(t)| = \infty$ *and* $\lim\limits_{t\to\infty} |u''(t)| = \lim\limits_{t\to\infty} |u'''(t)| = \infty$ *if* $\lim\limits_{t\to\infty} c(t) \neq 0$.

In [10], Parhi and Das obtained the following result:

LEMMA 1.2. *Suppose that* $a'(t) \geq 0$, $c(t) - b'(t) + a''(t) < 0$ *and*

$$\int\limits_0^\infty \left[-\frac{2a^3(t)}{27} + \frac{a(t)b(t)}{3} - c(t) - \frac{2a(t)a'(t)}{3} + b'(t) \right.$$
$$\left. -a''(t) - \frac{2}{3\sqrt{3}} \left(\frac{a^2(t)}{3} - b(t) + 2a'(t) \right)^{3/2} \right] \mathrm{d}t = \infty\,.$$

Then (1.3) *has an oscillatory solution.*

From the proof of Lemmas 1.1 and 1.2 it is clear that such techniques cannot be applied to derive similar results for (1.1). This is due to the presence of delay in (1.1). However, the study of the asymptotic behaviour of solutions of (1.1) is possible because of the canonical transformation due to Trench [12] and some comparison results by Kusano and Naito [8].

For results concerning property (A)/(A′), the reader is referred to [3], [11].

By a solution of (1.1) we mean a thrice continuously differentiable function $y\colon [T_y, \infty) \to \mathbb{R}$, $T_y \geq \sigma$, which satisfies (1.1) for $t \geq T_y$. Such a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

DEFINITION. Following Kiguradze [7], we say that (1.1) or (1.2) *has property* (B) if every nonoscillatory solution $y(t)$ of the equation satisfies

$$y(t)y^{(i)}(t) > 0\,, \qquad 0 \leq i \leq 3\,, \tag{1.4}$$

for $t \geq t_0 \geq \sigma$.

In Section 2, we obtain sufficient conditions in terms of the coefficient functions of (1.1) so that the equation has property (B). We have used a delay-differential inequality to establish that (1.1) has property (B) in Section 3.

2.

Setting

$$Ly = \big(r(t)y''\big)' + q(t)y' , \tag{2.1}$$

one may write (1.2) as

$$Ly(t) + p(t)y\big(g(t)\big) = 0 . \tag{2.2}$$

It is easy to see that the operator (2.1) may be written in the form

$$Ly \equiv \frac{1}{v(t)}\left(\frac{r(t)}{v^{-2}(t)}\left(\frac{y'}{v(t)}\right)'\right)' , \tag{2.3}$$

where $v(t)$ is a positive solution of the second order linear differential equation

$$\big(r(t)v'\big)' + q(t)v = 0 , \qquad t \in [\sigma,\infty) . \tag{2.4}$$

Lemma 2.1. *Equation* (2.4) *admits a positive increasing solution* $v(t)$ *satisfying*

$$\int_\sigma^\infty v(t)\,\mathrm{d}t = \infty \qquad \textit{and} \qquad \int_\sigma^\infty \frac{\mathrm{d}t}{v^2(t)r(t)} < \infty . \tag{2.5}$$

Proof. Suppose that $v(t)$ is a solution of (2.4) with $v(\sigma) > 0$ and $v'(\sigma) > 0$. From the continuity of $v'(t)$ it follows that there exists a $\delta > 0$ such that $v'(t) > 0$ for $t \in [\sigma, \sigma + \delta)$. We claim that $v'(t) > 0$ for $t \ge \sigma$. If not, then there exists a $t_1 > \sigma$ such that $v'(t_1) = 0$ and $v'(t) > 0$ for $t \in [\sigma, t_1)$. Integrating (2.4) from σ to t_1, we obtain a contradiction. Thus $v(t) > 0$ and $v'(t) > 0$ for $t \ge \sigma$. Consequently, $\int_\sigma^\infty v(t)\,\mathrm{d}t = \infty$. Further, $\big(r(t)v'(t)\big)' \ge -q(t)v(t) \ge 0$ for $t \ge \sigma$ implies that $r(t)v'(t) \ge r(t_0)v'(t_0)$ for $t \ge t_0 \ge \sigma$. Hence

$$v^2(t) > \big(r(t_0)v'(t_0)\big)^2 \left(\int_{t_0}^t \frac{\mathrm{d}s}{r(s)}\right)^2 .$$

Thus, for $t > t_1 > t_0$,

$$\begin{aligned}\int_{t_1}^t \frac{\mathrm{d}s}{r(s)v^2(s)} &< \frac{1}{\big(r(t_0)v'(t_0)\big)^2}\int_{t_1}^t \frac{\mathrm{d}s}{r(s)\Big(\int_{t_0}^s \frac{\mathrm{d}\theta}{r(\theta)}\Big)^2}\\ &< \frac{1}{\big(r(t_0)v'(t_0)\big)^2}\,\frac{1}{\int_{t_0}^{t_1}\frac{\mathrm{d}s}{r(s)}} < \infty .\end{aligned}$$

Hence $\int_\sigma^\infty \frac{\mathrm{d}t}{r(t)v^2(t)} < \infty$. This completes the proof of the lemma. □

THEOREM 2.2. *Equation* (2.2) *can be represented essentially uniquely in the canonical form*

$$Ly(t) + p(t)y(g(t)) = 0, \tag{2.2c}$$

where

$$Ly = \frac{1}{r_3(t)}\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}\left(\frac{y}{r_0(t)}\right)'\right)'\right)', \tag{2.6}$$

$r_i \in C([\sigma,\infty),\mathbb{R})$ *such that* $r_i(t) > 0$, $0 \le i \le 3$, *and* $\int_\sigma^\infty r_i(t)\,\mathrm{d}t = \infty$, $i = 1,2$.

Proof. In view of Lemma 2.1, the operator Ly given by (2.1) may be written in the form (2.3). Since

$$\int_\sigma^\infty \frac{\mathrm{d}t}{r(t)v^2(t)} < \infty,$$

then proceeding as in the proof of Lemma 2 of Trench [12], one may write (2.3) in the form

$$Ly = \frac{1}{\tilde r_3(t)}\left(\frac{1}{\tilde r_2(t)}\left(\frac{1}{\tilde r_1(t)}\left(\frac{y}{\tilde r_0(t)}\right)'\right)'\right)', \tag{2.7}$$

where $\tilde r_0(t) = 1$, $\tilde r_1(t) = v(t)\int_t^\infty \frac{\mathrm{d}s}{r(s)v^2(s)}$, $\tilde r_2(t) = \frac{1}{r(t)v^2(t)}\left(\int_t^\infty \frac{\mathrm{d}s}{r(s)v^2(s)}\right)^{-2}$, $\tilde r_3(t) = v(t)\int_t^\infty \frac{\mathrm{d}s}{r(s)v^2(s)}$. Clearly, $\int_\sigma^\infty \tilde r_2(t)\,\mathrm{d}t = \infty$. If $\int_\sigma^\infty \tilde r_1(t)\,\mathrm{d}t = \infty$, then we set $r_i(t) = \tilde r_i(t)$, $0 \le i \le 3$. If $\int_\sigma^\infty \tilde r_1(t)\,\mathrm{d}t < \infty$, then (2.7) may be written in the form (2.6), where

$$r_0(t) = \tilde r_0(t)\int_t^\infty \tilde r_1(s)\,\mathrm{d}s, \qquad r_1(t) = \tilde r_1(t)\left(\int_t^\infty \tilde r_1(s)\,\mathrm{d}s\right)^{-2},$$

$$r_2(t) = \tilde r_2(t)\int_t^\infty \tilde r_1(s)\,\mathrm{d}s, \qquad r_3(t) = \tilde r_3(t).$$

Clearly, $\int_\sigma^\infty r_i(t)\,\mathrm{d}t = \infty$, $i = 1,2$. Thus the theorem is proved. □

Setting $L_0 y = y/r_0(t)$ and $L_i y = (L_{i-1}y)'/r_i(t)$, $1 \le i \le 3$, we see that (2.2c) may be written as

$$L_3 y(t) + p(t)y(g(t)) = 0. \tag{2.2c}$$

DEFINITION. Equation (2.2c) is said *to have property* (B′) if every nonoscillatory solution $y(t)$ of the equation satisfies

$$y(t)L_iy(t) > 0\,, \qquad 0 \le i \le 3\,, \tag{2.8}$$

for $t \ge t_0 \ge \sigma$.

LEMMA 2.3. *If $y(t)$ is a nonoscillatory solution of (2.2c), then either*

$$\operatorname{sgn} L_0y(t) = \operatorname{sgn} L_1y(t) = \operatorname{sgn} L_2y(t) = \operatorname{sgn} L_3y(t)$$

or

$$\operatorname{sgn} L_0y(t) = \operatorname{sgn} L_1y(t) = \operatorname{sgn} L_3y(t) \ne \operatorname{sgn} L_2y(t)$$

for $t \ge t_0 \ge \sigma$.

The proof is straightforward and is thus omitted.

Remark. Lemma 2.3 is true for $g(t) \equiv t$. It holds whether L_3y is given by (2.6) or (2.7).

Remark. Lemma 5 in [8] holds for $\tau(t) \equiv t$.

THEOREM 2.4. *Let $g \in C^1\big([\sigma,\infty),\mathbb{R}\big)$ with $g'(t) > 0$ for $t \ge \sigma$. If the canonical ordinary differential equation*

$$L_3y + \frac{p\big(g^{-1}(t)\big)r_3\big(g^{-1}(t)\big)}{g'\big(g^{-1}(t)\big)r_3(t)}y = 0 \tag{2.9}$$

has property (B′), *then* (2.2c) *has property* (B′).

P r o o f. Let $y(t)$ be a nonoscillatory solution of (2.2c). Without loss of generality, we may assume that $y(t) > 0$ and $y\big(g(t)\big) > 0$ for $t \ge t_0 > \sigma$. Hence $L_0y(t) > 0$, $L_1y(t) > 0$ and $L_3y(t) > 0$ for large t due to Lemma 2.3. To complete the proof of the theorem, it is enough to prove, in view of Lemma 2.3, that $L_2y(t) > 0$ for large t. If possible, suppose that $L_2y(t) < 0$ for $t \ge t_1 > t_0$. Integrating (2.2c) from t $(> t_1)$ to ∞, we obtain

$$-L_2y(t) > \int_t^\infty r_3(s_3)\,|p(s_3)|\,y\big(g(s_3)\big)\,\mathrm{d}s_3\,.$$

Further integrating from $t > t_1$ to ∞ yields

$$L_1y(t) > \int_t^\infty r_2(s_2)\bigg(\int_{s_2}^\infty r_3(s_3)\,|p(s_3)|\,y\big(g(s_3)\big)\,\mathrm{d}s_3\bigg)\,\mathrm{d}s_2\,.$$

Integrating the above inequality from t_1 to t we get

$$L_0y(t) >$$

$$> K + \int_{t_1}^{t} r_1(s_1)\left(\int_{s_1}^{\infty} r_2(s_2)\left(\int_{s_2}^{\infty} r_3(s_3)\,|p(s_3)|\,y\big(g(s_3)\big)\,\mathrm{d}s_3\right)\mathrm{d}s_2\right)\mathrm{d}s_1$$

$$> K + \int_{t_1}^{t} r_1(s_1)\left(\int_{s_1}^{\infty} r_2(s_2)\left(\int_{g(s_2)}^{\infty} \frac{r_3\big(g^{-1}(\theta)\big)\,\big|p\big(g^{-1}(\theta)\big)\big|\,y(\theta)}{g'\big(g^{-1}(\theta)\big)}\,\mathrm{d}\theta\right)\mathrm{d}s_2\right)\mathrm{d}s_1$$

$$> K + \int_{t_1}^{t} r_1(s_1)\left(\int_{s_1}^{\infty} r_2(s_2)\left(\int_{s_2}^{\infty} \frac{r_3\big(g^{-1}(\theta)\big)\,\big|p\big(g^{-1}(\theta)\big)\big|\,y(\theta)}{g'\big(g^{-1}(\theta)\big)}\,\mathrm{d}\theta\right)\mathrm{d}s_2\right)\mathrm{d}s_1$$

$$> K + \int_{t_1}^{t} r_1(s_1)\left(\int_{s_1}^{\infty} r_2(s_2)\left(\int_{s_2}^{\infty} \frac{r_3\big(g^{-1}(\theta)\big)\,\big|p\big(g^{-1}(\theta)\big)\big|\,r_0(\theta)L_0y(\theta)}{g'\big(g^{-1}(\theta)\big)}\,\mathrm{d}\theta\right)\mathrm{d}s_2\right)\mathrm{d}s_1$$

where $K = L_0y(t_1) > 0$. Thus from [8; Lemma 5] it follows that the integral equation

$$u(t) = K + \int_{t_1}^{t} r_1(s_1)\left(\int_{s_1}^{\infty} r_2(s_2)\left(\int_{s_2}^{\infty} \frac{r_3\big(g^{-1}(\theta)\big)\,\big|p\big(g^{-1}(\theta)\big)\big|\,r_0(\theta)u(\theta)}{g'\big(g^{-1}(\theta)\big)}\,\mathrm{d}\theta\right)\mathrm{d}s_2\right)\mathrm{d}s_1$$

admits a solution $u \in C\big([t_1,\infty),(0,\infty)\big)$ satisfying

$$K \le u(t) \le L_0y(t)\,, \qquad t \ge t_1\,.$$

Setting $z(t) = r_0(t)u(t) > 0$, $t \ge t_1$, we notice that $z(t) > 0$ for $t \ge t_1$ and it satisfies the equation

$$L_0z(t) = K + \int_{t_1}^{t} r_1(s_1)\left(\int_{s_1}^{\infty} r_2(s_2)\left(\int_{s_2}^{\infty} \frac{r_3\big(g^{-1}(\theta)\big)\,\big|p\big(g^{-1}(\theta)\big)\big|\,z(\theta)}{g'\big(g^{-1}(\theta)\big)}\,\mathrm{d}\theta\right)\mathrm{d}s_2\right)\mathrm{d}s_1$$

$$> 0\,.$$

Hence

$$L_1z(t) = \int_{t}^{\infty} r_2(s_2)\left(\int_{s_2}^{\infty} \frac{r_3\big(g^{-1}(\theta)\big)\,\big|p\big(g^{-1}(\theta)\big)\big|\,z(\theta)}{g'\big(g^{-1}(\theta)\big)}\,\mathrm{d}\theta\right)\mathrm{d}s_2 > 0\,,$$

$$L_2z(t) = -\int_{t}^{\infty} \frac{r_3\big(g^{-1}(\theta)\big)\,\big|p\big(g^{-1}(\theta)\big)\big|\,z(\theta)}{g'\big(g^{-1}(\theta)\big)}\,\mathrm{d}\theta < 0$$

and thus $z(t)$ is a solution of the equation

$$L_3 z + \frac{r_3\big(g^{-1}(t)\big)p\big(g^{-1}(t)\big)}{g'\big(g^{-1}(t)\big)r_3(t)} z = 0$$

since $L_2 z(t) < 0$, we get a contradiction to the assumption that (2.9) has property (B$'$). Hence the theorem is proved. □

Remark. We may recall that $L_3 y$ is given by (2.7) if

$$\int_\sigma^\infty \tilde{r}_1(t)\, \mathrm{d}t = \infty \tag{2.10}$$

and it is given by (2.6) if

$$\int_\sigma^\infty \tilde{r}_1(t)\, \mathrm{d}t < \infty\,. \tag{2.11}$$

THEOREM 2.5. *Suppose that* (2.10) *holds and* (2.4) *admits a solution* $v(t)$ *satisfying* (2.5) *and*

$$v'(t) \le \big(v(t)r(t)\big)^{-1} \left(\int_t^\infty \frac{\mathrm{d}s}{r(s)v^2(s)} \right)^{-1}. \tag{2.12}$$

Let $g \in C^1\big([\sigma,\infty),\mathbb{R}\big)$ *such that* $g'(t) > 0$ *for* $t \ge \sigma$. *If*

$$\big(r(t)y''\big)' + q(t)y' + \frac{p\big(g^{-1}(t)\big)\tilde{r}_3\big(g^{-1}(t)\big)}{g'\big(g^{-1}(t)\big)\tilde{r}_3(t)} y = 0 \tag{2.13}$$

has property (B), *then*

$$L_3 y + \frac{p\big(g^{-1}(t)\big)\tilde{r}_3\big(g^{-1}(t)\big)}{g'\big(g^{-1}(t)\big)\tilde{r}_3(t)} y = 0 \tag{2.14}$$

has property (B$'$), *where* $L_3 y$ *is given by* (2.7).

Proof. Let $y(t)$ be a nonoscillatory solution of (2.14). We may take $y(t) > 0$ for $t \ge t_0 \ge \sigma$. Since $y(t)$ is a solution of (2.13) which has property (B), then $y'(t) > 0$, $y''(t) > 0$ and $y'''(t) > 0$ for $t \ge t_1 > t_0$. Thus $L_0 y(t) > 0$ and $L_3 y(t) > 0$ for $t \ge t_1$. Further

$$L_1 y(t) = \frac{1}{\tilde{r}_1(t)}\big(L_0 y(t)\big)' = \frac{1}{\tilde{r}_1(t)} \left(\frac{y(t)}{\tilde{r}_0(t)} \right)' = \frac{y'(t)}{\tilde{r}_1(t)} > 0 \qquad \text{for} \quad t \ge t_1\,.$$

From the assumption (2.12) it follows that $\tilde{r}_1'(t) \le 0$ for $t \ge t_1$ and hence $L_2 y(t) > 0$ for $t \ge t_1$. Thus (2.14) has property (B$'$). This completes the proof of the theorem. □

THEOREM 2.6. *Suppose that* $g \in C^1([\sigma,\infty),\mathbb{R})$ *such that* $g'(t) > 0$ *for* $t \ge \sigma$, $a'(t) \ge 0$ *and* $c^*(t) - b'(t) + a''(t) < 0$, *where*

$$c^*(t) = \frac{p(g^{-1}(t))\tilde{r}_3(g^{-1}(t))}{g'(g^{-1}(t))\tilde{r}_3(t)r(t)}\,.$$

If

$$\int_\sigma^\infty \Big[-\frac{2a^3(t)}{27} + \frac{a(t)b(t)}{3} - c^*(t) - \frac{2a(t)a'(t)}{3} + b'(t) - a''(t) - \frac{2}{3\sqrt{3}}\Big(\frac{a^2(t)}{3} - b(t) + 2a'(t)\Big)^{3/2}\Big]\,\mathrm{d}t = \infty\,,$$

then (2.13) *has property* (B).

The proof follows from Lemmas 1.1 and 1.2.

THEOREM 2.7. *Let* (2.10) *hold and* $0 \le \lim\limits_{t\to\infty} \tilde{r}_1(t) < \infty$. *Let* $g \in C^1([\sigma.\infty),\mathbb{R})$ *be such that* $g'(t) > 0$ *for* $t \ge \sigma$. *If* (2.13) *has property* (B), *then* (2.14) *has property* (B′), *where* $L_3 y$ *is given by* (2.7).

Proof. Let $y(t)$ be a nonoscillatory solution of (2.14). Let $y(t) > 0$ for $t \ge t_0 \ge \sigma$. Since $y(t)$ is a solution of (2.13), from the given condition it follows that $y'(t) > 0$, $y''(t) > 0$ and $y'''(t) > 0$ for $t \ge t_1 > t_0$. Thus $\lim\limits_{t\to\infty} y'(t) = \infty$. On the other hand, it is clear that $L_0 y(t) > 0$ and $L_3 y(t) > 0$ for $t \ge t_1$. Since $\tilde{r}_0(t) = 1$, then $L_1 y(t) > 0$ for $t \ge t_1$. In view of Lemma 2.3 and the remark that follows, $L_2 y(t) > 0$ or < 0 for $t \ge t_2 > t_1$. If $L_2 y(t) < 0$ for $t \ge t_2$, then $0 \le \lim\limits_{t\to\infty} L_1 y(t) < \infty$. Hence $\lim\limits_{t\to\infty} y'(t) = \lim\limits_{t\to\infty} \tilde{r}_1(t) L_1 y(t) < \infty$, a contradiction. Thus the theorem is proved. □

THEOREM 2.8. *Suppose that the conditions of Theorem* 2.6 *are satisfied. Let* (2.10) *hold. If either* $0 \le \lim\limits_{t\to\infty} \tilde{r}_1(t) < \infty$ *or* (2.4) *admits a solution* $v(t)$ *satisfying* (2.5) *and* (2.12), *then* (2.2c) *has property* (B′), *where* $L_3 y$ *is given by* (2.7).

The proof follows from Theorems 2.4 – 2.7.

EXAMPLE 1. Consider

$$y'''(t) - \frac{1}{t}y''(t) - \frac{3}{t^2}y'(t) - \mathrm{e}\Big(1 - \frac{1}{t} - \frac{3}{t^2}\Big)y(t-1) = 0$$

for $t \ge 3$. The associated second order equation

$$\Big(\frac{3}{t}v'\Big)' - \frac{9}{t^3}v = 0$$

admits a solution $v(t) = t^3$ which satisfies (2.5) and (2.12). Clearly, $\tilde{r}_1(t) = \frac{1}{12t} = \tilde{r}_3(t)$, $\tilde{r}_2(t) = 48t^3$ and $\tilde{r}_0(t) = 1$. Thus (2.10) holds and the given equation may be written in the canonical form as

$$12t\Big(\frac{1}{48t^3}(12ty')'\Big)' - 3\,\mathrm{e}\Big(\frac{1}{t} - \frac{1}{t^2} - \frac{3}{t^3}\Big)y(t-1) = 0\,, \qquad t \ge 3\,.$$

As $g(t) = t - 1$, $g^{-1}(t) = t + 1$ and $p(t) = -3\,\mathrm{e}\big(\frac{1}{t} - \frac{1}{t^2} - \frac{3}{t^3}\big)$, one may easily verify that

$$c^*(t) = -\,\mathrm{e}\,t^2\left[\frac{1}{(t+1)^2} - \frac{1}{(t+1)^3} - \frac{3}{(t+1)^4}\right]$$

and all the conditions of Theorem 2.6 are satisfied. Thus from Theorem 2.8 it follows that the above canonical equation has property (B′). In particular, $y(t) = \mathrm{e}^t$ is a positive solution of the equation such that $L_0y(t) = y(t)/\tilde{r}_0(t) > 0$, $L_1y(t) = \frac{1}{\tilde{r}_1(t)}\big(L_0y(t)\big)' > 0$, $L_2y(t) = \frac{1}{\tilde{r}_2(t)}\big(L_1y(t)\big)' > 0$ and $L_3y(t) = \frac{1}{\tilde{r}_3(t)}\big(L_2y(t)\big)' > 0$.

THEOREM 2.9. *Let* (2.11) *hold and* $g \in C^1\big([\sigma,\infty),\mathbb{R}\big)$ *such that* $g'(t) > 0$ *for* $t \ge \sigma$. *If*

$$\big(r(t)y''\big)' + q(t)y' + \frac{p\big(g^{-1}(t)\big)r_3\big(g^{-1}(t)\big)}{g'\big(g^{-1}(t)\big)r_3(t)}y = 0 \tag{2.15}$$

has property (B), *then* (2.9) *has property* (B′), *where* L_3y *is given by* (2.6).

P r o o f. Let $y(t)$ be a nonoscillatory solution of (2.9). We may assume that $y(t) > 0$ for $t \ge t_0 \ge \sigma$. Since $y(t)$ is a solution of (2.15), then from the given hypothesis it follows that $y'(t) > 0$, $y''(t) > 0$ and $y'''(t) > 0$ for $t \ge t_1 > t_0$. Thus $\lim\limits_{t\to\infty} y(t) = \infty$. Clearly, $L_0y(t) > 0$ and $L_3y(t) > 0$ for $t \ge t_1$. From Lemma 2.3 it follows that $L_1y(t) > 0$ and $L_2y(t) > 0$ or < 0 for $t \ge t_1$. If possible, let $L_2y(t) < 0$ for $t \ge t_1$. Hence $0 \le \lim\limits_{t\to\infty} L_1y(t) < \infty$. Further, $\big(L_0y(t)\big)' = r_1(t)L_1y(t)$ implies that

$$\begin{aligned} r_1(t)L_1y(t) &= \frac{y'(t)}{r_0(t)} - \frac{y(t)r_0'(t)}{r_0^2(t)} = \frac{y'(t)}{r_0(t)} + \frac{\tilde{r}_1(t)y(t)}{r_0^2(t)} \\ &> r_1(t)y(t)\,. \end{aligned}$$

Taking the limit as $t \to \infty$ we get $\infty = \lim\limits_{t\to\infty} y(t) \le \lim\limits_{t\to\infty} L_1y(t) < \infty$, a contradiction. This completes the proof of the theorem. □

THEOREM 2.10. *Suppose that* $g \in C^1\big([\sigma,\infty),\mathbb{R}\big)$ *such that* $g'(t) > 0$ *for* $t \ge \sigma$, $a'(t) \ge 0$ *and* $c^{**}(t) - b'(t) + a''(t) < 0$, *where*

$$c^{**}(t) = \frac{p\big(g^{-1}(t)\big)r_3\big(g^{-1}(t)\big)}{g'\big(g^{-1}(t)\big)r_3(t)r(t)}\,.$$

If

$$\int_0^\infty \left[-\frac{2a^3(t)}{27} + \frac{a(t)b(t)}{3} - c^{**}(t) - \frac{2a(t)a'(t)}{3} + b'(t) \right.$$

$$\left. -a''(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + 2a'(t)\right)^{3/2} \right] \mathrm{d}t = \infty\,,$$

then (2.15) *has property* (B).

The proof follows from Lemma 1.1 and 1.2.

THEOREM 2.11. *Let the conditions of Theorem* 2.10 *hold. If* (2.11) *holds, then* (2.2c) *has property* (B′), *where* $L_3 y$ *is given by* (2.6).

The theorem follows from Theorems 2.4, 2.9 and 2.10.

THEOREM 2.12. *Suppose that* (2.11) *holds and* $\int_\sigma^\infty p(t)\,\mathrm{d}t = -\infty$, *where* $p(t) = c(t)\exp\left(\int_\sigma^t a(s)\,\mathrm{d}s\right)$. *If* (2.2c) *has property* (B′) *with* $L_3 y$ *as in* (2.6), *then* (1.1) *has property* (B).

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) and hence of (1.2). We may assume, without loss of generality, that $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_0 \ge \sigma$. Since $y(t)$ is a solution of (2.2c) which has property (B′), we have $L_0 y(t) > 0$, $L_1 y(t) > 0$, $L_2 y(t) > 0$ and $L_3 y(t) > 0$ for $t \ge t_1 > t_0$. Hence $\lim_{t\to\infty} L_1 y(t) = \infty$. If $\beta > \alpha > t_1$ be such that $y''(\alpha) \ge 0$, $y''(\beta) \le 0$ and $y'(t) > 0$ for $t \in (\alpha, \beta)$, then integrating (1.2) from α to β we obtain

$$0 \ge r(\beta)y''(\beta) - r(\alpha)y''(\alpha) = -\int_\alpha^\beta q(t)y'(t)\,\mathrm{d}t - \int_\alpha^\beta p(t)y(g(t))\,\mathrm{d}t > 0\,,$$

a contradiction. Hence $y'(t) > 0$ or ≤ 0 for large t. If $y'(t) \le 0$ for $t \ge t_2 > t_1$, then $\lim_{t\to\infty} y(t) < \infty$. On the other hand, $(L_0 y(t))' = r_1(t) L_1 y(t)$ implies that

$$\begin{aligned} r_1(t)L_1y(t) &= \frac{y'(t)}{r_0(t)} - \frac{y(t)r_0'(t)}{r_0^2(t)} = \frac{y'(t)}{r_0(t)} + \frac{y(t)\tilde{r}_1(t)}{r_0^2(t)} \\ &\le y(t)r_1(t)\,, \end{aligned}$$

since $L_3 y$ is given by (2.6) in this case. Thus $\lim_{t\to\infty} y(t) = \infty$. a contradiction. Hence $y'(t) > 0$ for $t \ge t_2 > t_1$. Consequently from (1.2) it follows that

$(r(t)y''(t))' > 0$ for $t \geq t_2$. If $y''(t) < 0$ for $t \geq t_3 > t_2$, then integrating (1.2) from t_3 to t $(t > t_3)$ we get

$$r(t)y''(t) = r(t_3)y''(t_3) - \int_{t_3}^{t} q(s)y'(s)\,\mathrm{d}s - \int_{t_3}^{t} p(s)y(g(s))\,\mathrm{d}s$$

$$> r(t_3)y''(t_3) - y(g(t_3)) \int_{t_3}^{t} p(s)\,\mathrm{d}s.$$

Thus $y''(t) > 0$ for large t, a contradiction. Hence $y''(t) > 0$ for $t \geq t_3$. From (1.1) we get $y'''(t) > 0$ for $t \geq t_3$. This proves that (1.1) has property (B), which completes the proof of the theorem. □

COROLLARY 2.13. *Let the conditions of Theorem* 2.10 *hold. If* (2.11) *holds and* $\int_{\sigma}^{\infty} p(t)\,\mathrm{d}t = -\infty$, *then* (1.1) *has property* (B), *where* $p(t) = c(t)\exp\Big(\int_{\sigma}^{t} a(s)\,\mathrm{d}s\Big)$. *Further,* $\lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |y'(t)| = \infty$ *and* $\lim_{t\to\infty} |y''(t)| = \lim_{t\to\infty} |y'''(t)| = \infty$ *if* $\lim_{t\to\infty} c(t) \neq 0$.

This follows from Theorems 2.11 and 2.12.

EXAMPLE 2. Consider

$$y'''(t) - \frac{1}{t}y''(t) - \frac{8}{t^2}y'(t) - \mathrm{e}\Big(1 - \frac{1}{t} - \frac{8}{t^2}\Big)y(t-1) = 0 \qquad (2.16)$$

for $t \geq 4$. The associated second order equation

$$\Big(\frac{4}{t}v'\Big)' - \frac{32}{t^3}v = 0$$

admits a solution $v(t) = t^4$ satisfying (2.5). Clearly, $\tilde{r}_0(t) = 1$, $\tilde{r}_1(t) = \frac{1}{24t^2} = \tilde{r}_3(t)$ and $\tilde{r}_2(t) = 144t^5$. Thus $\int_4^{\infty} \tilde{r}_1(t)\,\mathrm{d}t < \infty$ and hence (2.11) is satisfied. One may calculate $r_0(t) = \frac{1}{24t}$, $r_1(t) = 24$, $r_2(t) = 6t^4$ and $r_3(t) = \frac{1}{24t^2}$. Thus (2.16) may be written in the canonical form as

$$24t^2\bigg(\frac{1}{6t^4}\Big(\frac{1}{24}(24ty(t))'\Big)'\bigg)' - 4\,\mathrm{e}\Big(\frac{1}{t} - \frac{1}{t^2} - \frac{8}{t^3}\Big)y(t-1) = 0 \qquad (2.17)$$

for $t \geq 4$. Since

$$c^{**}(t) = -\,\mathrm{e}\,t^3\bigg(\frac{1}{(t+1)^3} - \frac{1}{(t+1)^4} - \frac{8}{(t+1)^5}\bigg),$$

then all the conditions of Theorem 2.10 hold. From Theorem 2.11 it follows that (2.17) has property (B′). In particular, $y(t) = \mathrm{e}^t$ is a positive solution of (2.17) with $L_0y(t) > 0$, $L_1y(t) > 0$, $L_2y(t) > 0$ and $L_3y(t) > 0$. Further, by Corollary 2.13, (2.16) has property (B). In particular, $y(t) = \mathrm{e}^t$ is a positive solution of (2.16) with $y'(t) > 0$, $y''(t) > 0$ and $y'''(t) > 0$.

THEOREM 2.14. *Suppose that* (2.10) *holds and* $0 < \lim_{t\to\infty} \tilde{r}_1(t)$. *If* (2.2c) *has property* (B$'$) *with* $L_3 y$ *as in* (2.7), *then* (1.1) *has property* (B).

P r o o f. Let $y(t)$ be a nonoscillatory solution of (1.1). Hence $y(t)$ is a solution of (1.2). We may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_0 \ge \sigma$. Since $y(t)$ is a solution of (2.2c), then $L_0 y(t) > 0$, $L_1 y(t) > 0$, $L_2 y(t) > 0$ and $L_3 y(t) > 0$ for $t \ge t_1 > t_0$, where $L_3 y$ is given by (2.7). Thus $\lim_{t\to\infty} L_1 y(t) = \infty$. Further, $L_1 y(t) > 0$ implies that $y'(t) > 0$. From (1.2) it follows that $y''(t) > 0$ or < 0 for large t. If $y''(t) < 0$ for large t, then $0 \le \lim_{t\to\infty} y'(t) < \infty$. However, $y'(t) = \tilde{r}_1(t) L_1 y(t)$ implies that $\lim_{t\to\infty} y'(t) = \infty$. Hence $y''(t) > 0$ for $t \ge t_2 > t_1$. Consequently, $y'''(t) > 0$ for $t \ge t_2$ due to (1.1). Thus (1.1) has property (B) and hence the theorem is proved. □

COROLLARY 2.15. *Let the conditions of Theorem* 2.6 *hold. If* (2,10) *holds and* $0 < \lim_{t\to\infty} \tilde{r}_1(t) < \infty$, *then* (1.1) *has property* (B). *Further,* $\lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |y'(t)| = \infty$ *and* $\lim_{t\to\infty} |y''(t)| = \lim_{t\to\infty} |y'''(t)| = \infty$ *if* $\lim_{t\to\infty} c(t) \ne 0$.

This follows from Theorems 2.8 and 2.14.

THEOREM 2.16. *If* g *is monotonic increasing,* (2.10) *holds and*

$$\int_\sigma^\infty c(t) \exp\left(\int_\sigma^t a(s)\, \mathrm{d}s \right) \mathrm{d}t = -\infty\,,$$

then (1.1) *has property* (B).

P r o o f. Let $y(t)$ be a nonoscillatory solution of (1.1). We may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_0 \ge \sigma$. Clearly, $y(t)$ satisfies (1.2) and (2.2c), where $L_3 y$ is given by (2.7). Since $L_0 y(t) = y(t)/\tilde{r}_0(t) = y(t) > 0$, $t \ge t_0$, it follows from Lemma 2.3 that $L_1 y(t) > 0$, that is, $y'(t) > 0$ for $t \ge t_1 > t_0$. Integrating (1.2) from t_1 to t ($t > t_1$) we obtain

$$\begin{aligned} r(t)y''(t) &= r(t_1)y''(t_1) - \int_{t_1}^t q(s)y'(s)\, \mathrm{d}s - \int_{t_1}^t p(s)y(g(s))\, \mathrm{d}s \\ &> r(t_1)y''(t_1) - y(g(t_1)) \int_{t_1}^t p(s)\, \mathrm{d}s\,. \end{aligned}$$

Thus $y''(t) > 0$ for large t, say, for $t \ge t_2 > t_1$. From (1.1) it follows that $y'''(t) > 0$ for $t \ge t_2$. Hence (1.1) has property (B). This completes the proof of the theorem. □

EXAMPLE 3. Consider

$$y'''(t) - e^{t/2} y(t/2) = 0, \qquad t \ge 1. \tag{2.18}$$

The associated second order equation $v'' = 0$ has a solution $v(t) = t$ satisfying (2.5). In this case, $\tilde r_0(t) = \tilde r_1(t) = \tilde r_2(t) = \tilde r_3(t) = 1$. Thus (2.10) holds. Since $c^*(t) = -2\,e^t$, then all the conditions of Corollary 2.15 are satisfied and hence (2.18) has property (B). In particular, $y(t) = e^t$ is a positive solution of (2.18) with $y'(t) > 0$, $y''(t) > 0$ and $y'''(t) > 0$.

This example also illustrates Theorem 2.16 as $p(t) = -\,e^{t/2}$.

Remark. Example 1 illustrates Theorem 2.16. We may note that $y(t) = e^t$ is a positive solution of the equation with $y'(t) > 0$, $y''(t) > 0$ and $y'''(t) > 0$. However, Corollary 2.15 cannot be applied to this example as $\lim\limits_{t\to\infty} \tilde r_1(t) = 0$.

3.

In this section, we show, using a delay-differential inequality, that (1.1) has property (B).

THEOREM 3.1. *Suppose that* (2.11) *holds and* $g \in C^1\big([\sigma,\infty),\mathbb{R}\big)$ *such that* $g'(t) > 0$ *for* $t \ge \sigma$. *If*

$$z'(t) + F(t)z\big(g(t)\big) \ge 0 \tag{3.1}$$

has no eventually negative solutions, then (2.2c) *has property* (B′), *where* L_3y *is given by* (2.6) *and*

$$F(t) = -r_3(t)p(t)r_0\big(g(t)\big)\Big[R_1\big(g(g(t))\big) - R_1\big(g(g(g(t)))\big)\Big]\times$$
$$\times\Big[R_2\big(g(t)\big) - R_2\big(g(g(t))\big)\Big],$$

and

$$R_i(t) = \int_\sigma^t r_i(s)\,ds, \qquad i = 1,2.$$

P r o o f. Let $y(t)$ be a nonoscillatory solution of (2.2c), where L_3y is given by (2.6). We may assume that $y(t) > 0$ and $y\big(g(t)\big) > 0$ for $t \ge t_0 > \sigma$. Hence $L_iy(t) > 0$ for $t \ge t_1 > t_0$, $i = 0,1,3$, in view of Lemma 2.3. We claim that $L_2y(t) > 0$, $t \ge t_1$. If not, $L_2y(t) < 0$, $t \ge t_1$, by Lemma 2.3. From (2.2c) we get, for $t \ge t_1$,

$$\big(L_2y(t)\big)' = r_3(t)\,|p(t)|\,r_0\big(g(t)\big)L_0y\big(g(t)\big).$$

Further, $(L_0y(t))' = r_1(t)L_1y(t)$ implies that

$$\int_{g(g(t))}^{g(t)} r_1(s)L_1y(s)\,\mathrm{d}s < L_0y(g(t)) < L_0y(t)\,,$$

for $t \geq t_2 > t_1$, that is,

$$\begin{aligned} L_0y(t) &> L_1y(g(t)) \int_{g(g(t))}^{g(t)} r_1(s)\,\mathrm{d}s \\ &> L_1y(g(t))\Big[R_1(g(t)) - R_1(g(g(t)))\Big] \end{aligned}$$

for $t \geq t_2$. Thus, for $t \geq t_3 > t_2$, we have

$$(L_2y(t))' > r_3(t)\,|p(t)|\,r_0(g(t))L_1y(g(g(t)))\Big[R_1(g(g(t))) - R_1(g(g(g(t))))\Big]\,. \tag{3.2}$$

On the other hand, $(L_1y(t))' = r_2(t)L_2y(t)$ implies that

$$\begin{aligned} -L_1y(t) < L_1y(g^{-1}(t)) - L_1y(t) &= \int_t^{g^{-1}(t)} r_2(s)L_2y(s)\,\mathrm{d}s \\ &< L_2y(g^{-1}(t)) \int_t^{g^{-1}(t)} r_2(s)\,\mathrm{d}s \\ &< L_2y(g^{-1}(t))\,[R_2(g^{-1}(t)) - R_2(t)] \end{aligned}$$

for $t \geq t_1$, that is,

$$L_1y(g(t)) > -L_2y(t)[R_2(t) - R_2(g(t))]$$

for $t \geq t_4 > t_3$. Thus, for $t \geq t_5 > t_4$, we have

$$L_1y(g(g(t))) > -L_2y(g(t))\Big[R_2(g(t)) - R_2(g(g(t)))\Big]\,. \tag{3.3}$$

Hence (3.2) and (3.3) yield

$$\begin{aligned} (L_2y(t))' > -r_3(t)\,|p(t)|\,r_0(g(t))L_2y(g(t))\Big[R_1(g(g(t))) - R_1(g(g(g(t))))\Big] \times \\ \times \Big[R_2(g(t)) - R_2(g(g(t)))\Big]\,. \end{aligned}$$

Consequently, $L_2y(t) < 0$ is a solution of (3.1), a contradiction which completes the proof of the theorem. □

COROLLARY 3.2. *Let the conditions of Theorem* 3.1 *hold. If* $\int\limits_{\sigma}^{\infty} p(t)\ \mathrm{d}t = -\infty$, *where* $p(t) = c(t)\exp\Big(\int\limits_{\sigma}^{\infty} a(s)\ \mathrm{d}s\Big)$, *then* (1.1) *has property* (B).

This follows from Theorems 2.12 and 3.1.

Remark. It is well known (see [5; p. 46]) that if $g(t) < t$ and

$$\liminf_{t\to\infty} \int\limits_{g(t)}^{t} F(s)\ \mathrm{d}s > \frac{1}{\mathrm{e}},$$

then (3.1) has no eventually negative solutions.

THEOREM 3.3. *Suppose that* (2.10) *holds and* $g \in C^1\big([\sigma,\infty),\mathbb{R}\big)$ *such that* $g'(t) > 0$ *for* $t \ge \sigma$. *If*

$$z'(t) + \tilde{F}(t)z\big(g(t)\big) \ge 0$$

does not admit eventually negative solutions, that (2.2c) *has property* (B′), *where* $L_3 y$ *is given by* (2.7) *and*

$$\begin{aligned}\tilde{F}(t) = -\tilde{r}_3(t)p(t)\tilde{r}_0\big(g(t)\big)\Big[\tilde{R}_1\big(g(g(t))\big) - \tilde{R}_1\big(g(g(g(t)))\big)\Big]\times \\ \times\Big[\tilde{R}_2\big(g(t)\big) - \tilde{R}_2\big(g(g(t))\big)\Big],\end{aligned} \tag{3.4}$$

and

$$\tilde{R}_i(t) = \int\limits_{\sigma}^{t} \tilde{r}_i(s)\ \mathrm{d}s, \qquad i = 1, 2.$$

The proof is similar to that of Theorem 3.1 and hence is omitted.

COROLLARY 3.4. *Let* (2.10) *hold,* $0 < \lim\limits_{t\to\infty} \tilde{r}_1(t)$ *and* $g \in C^1\big([\sigma,\infty),\mathbb{R}\big)$ *such that* $g'(t) > 0$ *and* $g(t) < t$. *If*

$$\liminf_{t\to\infty} \int\limits_{g(t)}^{t} \tilde{F}(s)\ \mathrm{d}s > 1/\mathrm{e},$$

where $\tilde{F}$ *is given by* (3.4), *then* (1.1) *has property* (B).

This follows from Theorems 2.14 and 3.3.

In [2], D ž u r i n a obtained the following results.

THEOREM 3.5. (see [2; Corollary 1]) *Let* $\tau \in C([\sigma,\infty),\mathbb{R})$ *such that* $\tau(t) > t$ *and* $w(t) = g(\tau(t)) < t$. *If either*

$$\liminf_{t\to\infty} \int_{w(t)}^{t} Q(s)\, ds > 1/e,$$

or

$$\limsup_{t\to\infty} \int_{w(t)}^{t} Q(s)\, ds > 1,$$

then (2.2c) *has property* (B'), *where*

$$Q(t) = -r_2(t) \int_{t}^{\tau(t)} r_3(s)p(s)r_0(g(s))\big(R_1(g(s)) - R_1(t_1)\big)\, ds$$

for sufficiently large t *with* $g(t) > t_1$.

THEOREM 3.6. (see [2; Corollary 3]) *Let* $\tau \in C([\sigma,\infty),\mathbb{R})$ *such that* $\tau(t) > t$ *and* $w(t) = g(\tau(t)) < t$. *If either*

$$\liminf_{t\to\infty} \int_{w(t)}^{t} \tilde{Q}(s)\, ds > 1/e,$$

or

$$\limsup_{t\to\infty} \int_{w(t)}^{t} \tilde{Q}(s)\, ds > 1,$$

then (2.2c) *has property* (B'), *where*

$$\tilde{Q}(t) = -r_2(t) \int_{t}^{\tau(t)} r_3(s)p(s)r_0(g(s))R_1(g(s))\, ds.$$

Remark. We note that in above theorems $g(t) < w(t) < t$. Further, D ž u r i n a 's theorems cannot be applied to the following example whereas Corollary 3.4 can be applied.

EXAMPLE 4. Consider

$$y'''(t) - 128\frac{1}{t^3}y(t/2) = 0, \qquad t > 1. \tag{3.5}$$

Here the associated second order equation is given by $v'' = 0$ and $\tilde r_0(t) = \tilde r_1(t) = \tilde r_2(t) = \tilde r_3(t) = 1$. Further, $g(t) = t/2$ implies that $g\big(g(t)\big) = t/4$ and $g\big(g\big(g(t)\big)\big) = t/8$. Thus $\tilde R_1(t) = \tilde R_2(t) = t - 1$, $\tilde F(t) = 4/t$ and hence $\liminf\limits_{t\to\infty} \int\limits_{t/2}^{t} \tilde F(s)\,\mathrm{d}s = 4\log 2 > 1/\mathrm{e}$. From Corollary 3.4 it follows that (3.5) has property (B). However, neither Theorem 3.5 nor Theorem 3.6 can be applied to (3.5) because choosing $\tau(t) = t + 1$, we notice that $w(t) = \frac{t+1}{2} < t$,

$$Q(t) = 128 \int\limits_t^{t+1} \frac{1}{s^3}\Big(\frac{s}{2} - 1\Big)\,\mathrm{d}s - 128\tilde R_1(t_1) \int\limits_t^{t+1} \frac{\mathrm{d}s}{s^3}$$

and

$$\tilde Q(t) = 128 \int\limits_t^{t+1} \frac{1}{s^3}\Big(\frac{s}{2} - 1\Big)\,\mathrm{d}s\,.$$

Hence

$$\lim_{t\to\infty} \int\limits_{(t+1)/2}^{t} Q(s)\,\mathrm{d}s = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \int\limits_{(t+1)/2}^{t} \tilde Q(s)\,\mathrm{d}s = 0\,.$$

We may note that the canonical form of (3.5) is (3.5) itself.

Džurina's theorems and our Corollary 3.2 apply to the following example:

EXAMPLE 5. Consider

$$y'''(t) - y'(t) - 6\,\mathrm{e}^t\, y(t/2) = 0\,, \qquad t \ge 0\,. \tag{3.6}$$

The equation $v'' - v = 0$ admits a solution $v(t) = \mathrm{e}^t$ satisfying (2.5). In this case $\tilde r_1(t) = \frac12\,\mathrm{e}^{-t}$ and hence

$$\int\limits_0^{\infty} \tilde r_1(t)\,\mathrm{d}t = \frac12 < \infty\,.$$

Further, $r_0(t) = \frac12\,\mathrm{e}^{-t} = r_3(t)$ and $r_1(t) = 2\,\mathrm{e}^t = r_2(t)$. Clearly, $g(t) = t/2$ implies that $g\big(g(t)\big) = t/4$ and $g\big(g\big(g(t)\big)\big) = t/8$. Hence $R_1\big(g\big(g(t)\big)\big) - R_1\big(g\big(g\big(g(t)\big)\big)\big) = 2\big(\mathrm{e}^{t/4} - \mathrm{e}^{t/8}\big)$, $R_2\big(g(t)\big) - R_2\big(g\big(g(t)\big)\big) = 2\big(\mathrm{e}^{t/2} - \mathrm{e}^{t/4}\big)$ and $F(t) = 6\big(\mathrm{e}^{t/4} + \mathrm{e}^{-t/8} - 1 - \mathrm{e}^{t/8}\big)$. Thus

$$\int\limits_{t/2}^{t} F(s)\,\mathrm{d}s = 6\,\mathrm{e}^{t/8}\Big(4\,\mathrm{e}^{t/8} - 12 - \frac12 t\,\mathrm{e}^{-t/8}\Big) - 6\big(8\,\mathrm{e}^{-t/8} - 8\,\mathrm{e}^{-t/16}\big)\,.$$

Consequently,

$$\liminf_{t\to\infty} \int_{t/2}^{t} F(s)\, \mathrm{d}s > 1/\mathrm{e}\,.$$

Since all the conditions of Corollary 3.2 are satisfied, then (3.6) has property (B). It is easy to see that all the conditions of Theorems 3.5 and 3.6 are satisfied. Clearly, $y(t) = \mathrm{e}^{2t}$ is a solution of (3.6) with $y'(t) > 0$, $y''(t) > 0$ and $y'''(t) > 0$.

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