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ON RECOGNITION OF STRONG GRAPH BUNDLES

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ABSTRACT. Graph bundles generalize the notion of covering graphs and graph products. Recently, an algorithm for recognition of graph bundles over triangle free bases with respect to the Cartesian product was found. Here we study relationship between strong and Cartesian graph bundles. An algorithm for recognition of graphs which have a representation as a graph bundle with connected fibre over a triangle free base with respect to the strong product of graphs is given.

1. Introduction

In topology, bundles are objects which generalize both covering spaces and Cartesian products ([2]). Analogously, graph bundles generalize the notion of covering graphs and graph products. Graph bundles can be defined with respect to arbitrary graph products ([19]). (For a classification of all possible associative graph products, see [11].) Various problems on graph bundles were studied recently, including edge coloring ([18]), maximum genus ([17]), isomorphism classes ([14]), characteristic polynomials ([15], [21]) and chromatic numbers ([12], [13]).

Finite connected graphs enjoy unique factorization with respect to the strong product of graphs ([3], [16]) and there is a polynomial algorithm for factoring strong product graphs ([7]). Contrarily, a graph may have more than one representation as a graph bundle. Natural questions therefore are to find all possible representations of a graph as a graph bundle or to decide whether a graph has at least one representation as a nontrivial graph bundle. This is much like the situation in the case of Cartesian product (see [20], [6], [22], [4], [8], [9] and the references there). A polynomial time algorithm for recognizing connected Cartesian graph bundles with connected fibres over a triangle free base was given in [10].

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In this paper we study the graph bundles with respect to the strong product of graphs. In particular we look at the relationship to Cartesian graph bundles.

The paper is organized as follows. Section 2 gives some basic definitions and recalls some well-known facts on graph bundles. In Section 3 the notion of weighted graph bundle is introduced. In Section 4 a graph transformation is defined yielding a correspondence between Cartesian and strong graph bundles. Observations are summarized in a form of an algorithm in Section 5.

2. Definitions

We will consider only finite connected simple graphs, i.e. graphs without loops and multiple edges. The vertex set of graph G will be denoted by $V(G)$ and the edge set will be denoted by $E(G)$. We write briefly uv for the edge $\{u,v\}$. Two edges are *adjacent* if they have a common vertex. $G \cong H$ denotes graph isomorphism, i.e. the existence of a bijection $b\colon V(G) \to V(H)$ such that the vertices g_1, g_2 are connected in G exactly when the vertices $b(g_1)$, $b(g_2)$ are connected in H. A maximal complete subgraph is called a *clique*. Vertices of a complete graph K_q will usually be denoted by $\{0,1,\dots,q-1\}$.

The *strong product* $G \boxtimes H$ of graphs G and H has as vertices the pairs (g,h) where $g \in V(G)$ and $h \in V(H)$. Vertices (g_1,h_1) and (g_2,h_2) are adjacent if either $\{g_1,g_2\}$ is an edge of G and $h_1 = h_2$ or if $g_1 = g_2$ and $\{h_1,h_2\}$ is an edge of H or if $\{g_1,g_2\}$ is an edge of G and $\{h_1,h_2\}$ is an edge of H.

EXAMPLE. $K_2 \boxtimes K_2 \cong K_4$.

Let G, B and F be graphs. A triple (G,p,B) is a (*strong*) *graph bundle* with *fiber* F over the *base graph* B if there is a mapping $p\colon G \to B$ which maps adjacent vertices of G to adjacent or identical vertices in B and the edges are mapped to edges or collapsed to a vertex, such that for each vertex $v \in V(B)$, $p^{-1}(v) \cong F$, and for each edge $e \in E(B)$, $p^{-1}(e) \cong K_2 \boxtimes F$. For a given graph G there may be several mappings $p_\iota\colon G \to B_\iota$ with the above properties. In such cases we write (G, p_ι, B_ι) to avoid confusion.

We say an edge e is *degenerate* if $p(e)$ is a vertex. Otherwise we call it *nondegenerate.*

The projection p induces a (*fundamental*) *factorization* of G into the graph consisting of isomorphic copies of the fibre F and the graph consisting of all nondegenerate edges.

(G,p,B) is *trivial* if either the base graph B or the fibre F has only one vertex. For brevity we will say a graph G is a *graph bundle* if there is a nontrivial representation of G as a graph bundle (G,p,B).

Assume (G,p,B) and $(G,\tilde{p},\tilde{B})$ are two nontrivial representations of G as graph bundles. If for each $\tilde{v} \in \tilde{B}$, $\tilde{p}^{-1}(\tilde{v}) \subseteq p^{-1}(v)$ for some $v \in B$, we define $B \leq \tilde{B}$. Clearly the relation $\leq$ induces a partial order in the set of all possible bases of G. B is a *maximal base* if it is maximal with respect to this relation. A fibre F is a *minimal fibre* if it is a fibre of a nontrivial representation of G with a maximal base.

The following relation, defined on the vertex set of G which was first defined in [3], proved to be useful in studies of the strong product. The equivalence relation S is defined as follows: $x\,S\,x$ and for $x \neq y$, $x\,S\,y$ if

- $xy \in E(G)$

and

- $N(x) = N(y)$, where $N(x) = \{x\} \cup \{y \mid xy \in E(G)\}$ is the *closed neighborhood* of a vertex x.

In [7], vertices x and y with $x\,S\,y$ are called *interchangeable.*

EXAMPLE. In any product $K_2 \boxtimes G$ vertices $(0,v)$ and $(1,v)$ are in relation S for any $v \in G$.

We define a graph G/S on equivalence classes of S as follows. Vertices are equivalence classes of S, $V(G/S) = \{[x] \mid x \in V(G)\}$, where $[x]$ denotes the S-equivalence class of the vertex x. By definition, two vertices $[x]$, $[y] \in V(G/S)$ are adjacent if there are vertices $v \in [x]$ and $u \in [y]$, which are connected in G. It is easy to see that then all pairs $v \in [x]$, $u \in [y]$ must be connected in G. (Hint: Assume $uv \in E(G)$ for $u \in [x]$ and $v \in [y]$. Then: $u \in [x] \implies u S x \implies N(u) = N(x) \implies xv \in E(G)$. Similarly, $xv \in E(G)$ implies $xy \in E(G)$.)

It is useful to remember sizes of equivalence classes of S. We therefore define the *weight function* $c_S\colon V(G/S) \to \mathbb{N}$ with $c_S([v]) = |[v]|$, where $|[v]|$ is the cardinality of the equivalence class of v. In the sequel we will most of the time consider the *weighted graph* $(G/S, c_S)$ rather than G/S.

Two weighted graphs $(G/S, c_S)$ and $(H/S, c_S)$ are, by definition, isomorphic if

- there is an isomorphism $\pi\colon G/S \to H/S$

such that

- $c_S(v) = c_S(\pi(v))$ for all $v \in V(G/S)$

It can be shown (see [3]) that:

LEMMA 1. *Two graphs G and H are isomorphic if and only if the corresponding weighted graphs $(G/S, c_S)$ and $(H/S, c_S)$ are isomorphic.*

For any edge $e \in E(B)$, $p^{-1}(e) \cong K_2 \boxtimes F$, by definition of strong graph bundle. We call the corresponding isomorphism between the two fibres defined by $(0,v) \mapsto (1,v)$ a *lifted isomorphism* or a lifted matching. It can be shown that, provided F is triangle free and $F \not\cong K_2$, the lifted matching is unique. (Hint: any vertex in $p^{-1}(e) \cong K_2 \boxtimes F$ has exactly one interchangeable pair.)

3. Weighted graph bundles

We define a *weighted graph bundle* as follows. $((G,c_G),p,(B,c_B))$ is a weighted graph bundle if for some weighted graph B with weight function c_B and some mapping p, (G,p,B) is a graph bundle and the weight function c_G on G satisfies

(a) for a nondegenerate edge $e = xy$: $c_G(x)/c_B(p(x)) = c_G(y)/c_B(p(y))$,
(b) for a degenerate edge $e = xy$, $p(x) = p(y) = z$, and for any edge $zu \in B$ and the corresponding lifted isomorphism $m: p^{-1}(z) \to p^{-1}(u)$, $c_G(x)/c_G(m(x)) = c_G(y)/c_G(m(y))$.

(See Fig. 1.)

Only (positive) integer valued weight functions will be used in this paper.

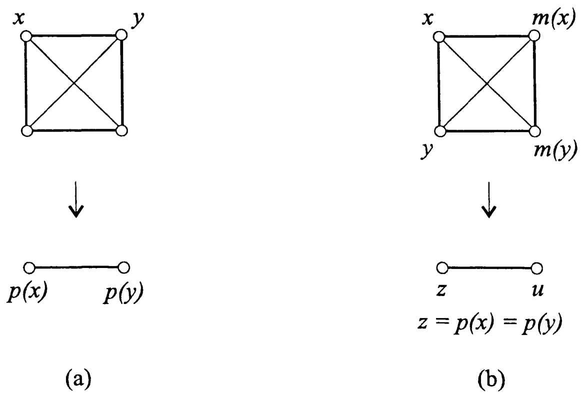

FIGURE 1. Definition of weighted graph bundles.

We wish to remark that the weighted graph bundles studied in [21] are not defined in the same way as here.

For any degenerate edge $e = xy$, $c_G(x)/c_G(y)$ can be understood as the weight of the corresponding directed edge in F. Similarly, for $y = m(x)$, $c_G(y)/c_G(x)$ can be seen as a weight of a directed nondegenerate edge of G or, alternatively, the weight of a directed edge $p(x)p(y)$ of B or the weight of the lifted isomorphism m.

EXAMPLE. Fig. 2 shows a strong graph bundle and the corresponding weighted graph bundle $\cdot/S$.

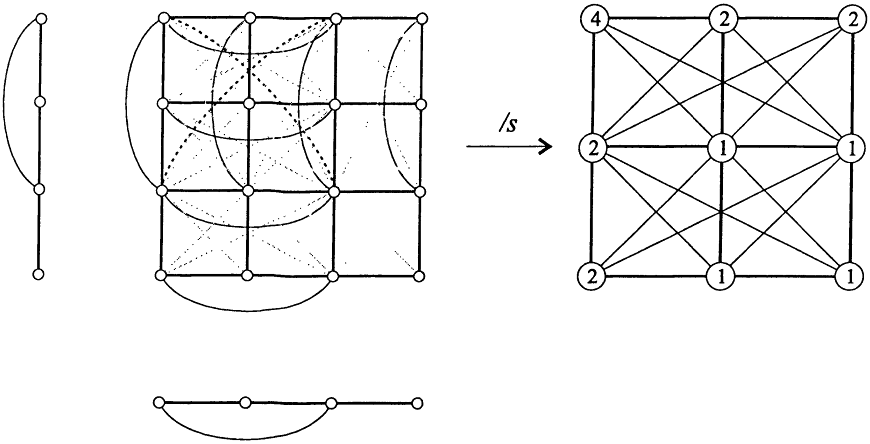

FIGURE 2. A strong product and the corresponding weighted graph.

A K_4 subgraph of G is *weighted bundle consistent*, if the weights $c_1 \leq c_2 \leq c_3 \leq c_4$ of the four vertices satisfy: $c_2 = c_1 k_{12}$, $c_3 = c_1 k_{13}$, $c_4 = c_2 k_{24} = c_2 k_{13}$ and $c_4 = c_3 k_{34} = c_3 k_{12}$ for some positive numbers k_{12} and k_{13}.

Clearly, if all weights are 1, then every K_4 is weighted bundle consistent. It is easy to see that in a weighted graph bundle, every K_4 in which there are both degenerate and nondegenerate edges must be weighted bundle consistent. (Hint: vertices of such a K_4 must belong to two fibres and can be labeled by, say x, y and $m(x)$, $m(y)$ for some lifted isomorphism m. From definition of weighted graph bundle we have $c(m(x))/c(x) = c(m(y))/c(y)$ and hence $k_{12} = k_{34}$. Similarly we obtain $k_{13} = k_{24}$.)

The above observations can be more formally written as:

PROPOSITION 1. *$((G, c_G), p, (B, c_B))$ is a weighted graph bundle if and only if (G, p, B) is a graph bundle in which every K_4 with both degenerate and non-degenerate edges is weighted bundle consistent.*

This gives us a straightforward idea for an algorithm for recognizing of weighted Cartesian graph bundles over a triangle free base: First, (ignoring the weights) find all representations of a graph as a Cartesian graph bundle over a triangle free base and second, check every K_4 with both degenerate and nondegenerate edges for weighted bundle consistency. A more elegant direct algorithm can be designed (for details, see [23]).

Recall from [10] that Cartesian graph bundles can be recognized in polynomial time. Here we write a slightly stronger statement.

THEOREM 1. *Let G be arbitrary graph. All representations of G as a Cartesian graph bundle with connected fibres over a triangle free base can be found in polynomial time.*

P r o o f. (Sketch) In [10] a polynomial algorithm is given which finds all representations of G as a graph bundle over a triangle free base with minimal fibres. These are obtained by computing a certain closure of equivalence classes of a certain equivalence relation, called δ^*. Following [10], all representations can be obtained by taking all unions of equivalence relations of δ^* and computing their closures. If G is a graph bundle, then it is easy to see that the number of δ^* equivalence classes is at most $\log_2 n$ (for Cartesian products, this was already observed in [6]).

Consequently, we can first compute the equivalence relation δ^*. If the number of δ^* classes is larger than $\log_2 n$, then G cannot be a graph bundle (and the algorithm stops). If the number of δ^* classes is less than $\log_2 n$, then the closures of all unions of δ^* classes can be computed in polynomial time. □

We now show that it is enough to study weighted graph bundles with trivial relation S.

PROPOSITION 2. *G/S is a strong weighted graph bundle if and only if G is a strong graph bundle with $B \neq K_p$ and $F \neq K_q$.*

We prove the proposition by proving two lemmas.

LEMMA 2. *Let G be a graph bundle with $B \neq K_p$ and $F \neq K_q$. Then $(G/S, c_S)$ has a representation as a nontrivial weighted graph bundle.*

P r o o f. Define a mapping $p_S \colon G/S \to \tilde{B}$ with $p_S([x]) := \{p(y) \mid y \in [x]\}$, where $[\cdot]$ denotes S equivalence classes in G. The graph $\tilde{B}$ has as vertices the sets $p_S([x])$ and $p_S([x])$ is adjacent to $p_S([y])$ if $xy \in E(G)$. Recall that then $uv \in E(G)$ for any $u \in [x]$ and $v \in [y]$.

We now sketch a proof that $(G/S, p_S, \tilde{B})$ is a graph bundle if (G, p, B) is.

We claim that:

For any edge xy, such that $y = m(x)$ for a lifted isomorphism m:
if $x\,S\,y$ then for any $z \in p^{-1}(p(x))$, $z\,S\,m(z)$.

Consider the closed neighborhood sets of x and y, $N(x) = N(y)$. The vertices of $N(x)$ are either in the same fibre as x or y or in some other fibre. For any vertex z not in the fibre of x or y, it can be shown that there are only two possibilities: either there are lifted isomorphisms m_{xw} and m_{yw} such that

$w = m_{xw}(x) = m_{yw}(y)$ or there is no lifted isomorphism with $m(x) = w$ and there is no lifted isomorphism with $m(y) = w$. All neighbors of x are determined by the neighbors given by lifted isomorphisms and by the neighbors in F. (Hint: The other neighbors are fourth vertices closing the product K_4's.)

Let $z \in p^{-1}(p(x))$. The lifted isomorphisms m_{xw} and m_{yw} ensure that the sets of neighbors which are given by lifted isomorphism edges of z and $m(z)$ are equal. z and $m(z)$ have the same sets of neighbors in the fibres of x and y because m is an isomorphism and because they are in the neighboring fibres of the strong product graph bundle. Since the other neighbors are already determined these, z and $m(z)$ have the same closed neighborhoods and hence $z\,S\,m(z)$. This concludes the proof of the claim.

Similarly, one can show that for a degenerate edge xy, if $x\,S\,y$ in G then the copies of these two vertices are in relation S in any copy of F. (Hint: Since x and y have the same neighbors, they have also the same neighbors when restricted to the fibre.)

Remark: In fact, it can be shown that $x S y$ implies $p(x) S p(y)$ and therefore $p([x]) = [p(x)]$. However, $p(x)\,S\,p(y)$ does not always imply $x\,S\,y$.

The relation S thus obeys the fibre structure of the graph bundle. This can be used to show that $p_S^{-1}(p_S(x))$ is isomorphic to F/S, and that p_S^{-1} of any edge of $\tilde{B}$ is isomorphic to $K_2 \boxtimes F/S$, i.e. that $(G/S, p_S, \tilde{B})$ is a graph bundle.

The weights c_B can naturally be defined to be the cardinalities of the sets $p_S([x])$, i.e. $c_B(p_S([x])) = |p_S([x])|$. With the weight functions c_S and c_B as defined above, $((G/S, c_S), p_S, (\tilde{B}, c_B))$ is a weighted graph bundle. We omit the details. □

LEMMA 3. *Let (H, c_H) be a weighted graph bundle with positive integer weights. Then there is (up to isomorphism) a unique graph bundle G such that $G/S \cong H$.*

Proof. Consider the following construction. Expand a vertex v of the weighted graph to complete subgraph of size $c_H(v)$. Connect each new vertex to all neighbours of v. Repeat this for all vertices (of weight > 1). It is easy to see that the resulting graph is a graph bundle, if the original graph is a weighted graph bundle. Lemma 1 implies uniqueness. □

Combining the above lemmas proves Proposition 2.

EXAMPLES. For graph bundles with $B = K_p$ and graph bundles with $F = K_q$ the corresponding weighted graphs G/S, H/S may not be (weighted) bundles. (See Fig. 3 and Fig. 4.) The reason is that the relation S collapses the base or the fibre of such representation. Another example is the product $K_p \boxtimes K_q = K_{pq}$ which has trivial relation S.

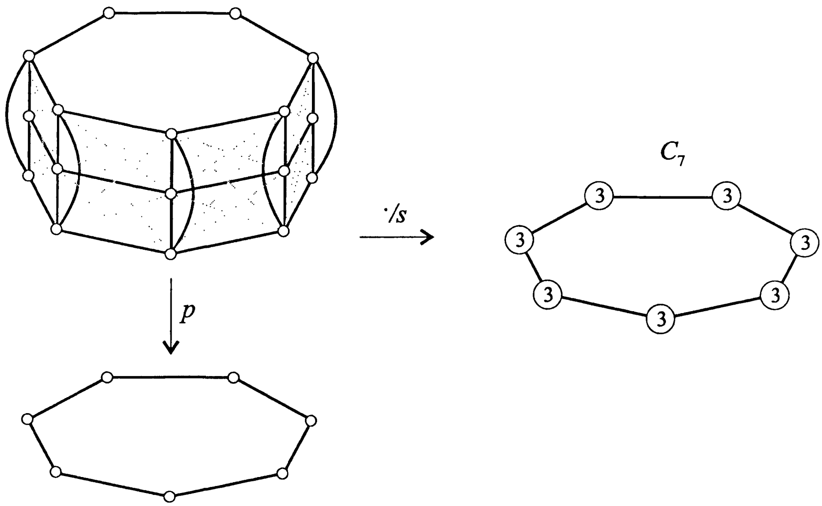

FIGURE 3. Example in which fibre is a complete graph.

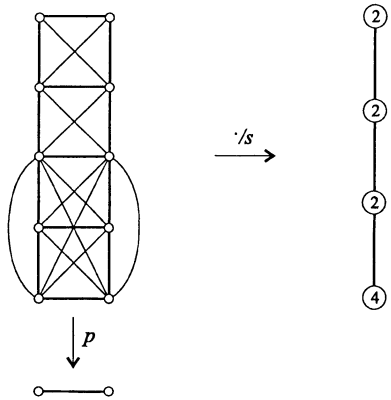

FIGURE 4. Example with base graph B complete graph.

Note that graph bundles with either base or fibre a complete graph are in fact products. It is well known that the strong product of an arbitrary graph with a complete graph is isomorphic to the lexicographic product with a complete graph. For recognition of such graphs the algorithm for recognition of lexicographic product graphs can be used ([5]). This algorithm is also used as a prephase in the algorithm for recognition strong product graphs in [7]. We may therefore restrict our attention to recognition of strong graph bundles with $B \neq K_p$ and $F \neq K_q$.

Alternatively, it can be shown that $G \cong K_d \boxtimes G'$ if and only if in the $(G/S, c_S)$

every vertex weight is divisible by d. Hence, for extracting complete factors we can simply divide all weights in $(G/S, c_S)$ by their greatest common divisor.

We may therefore turn attention to the problem of recognition of weighted graph bundles. We may also assume that the graph has trivial relation S because $(G/S)/S \cong G/S$, which can be seen easily.
(Graphs with $G/S \cong G$ are called *thin.*)

4. The transformation $\mathcal{C}$

We now define a transformation to an auxiliary edge-weighted graph $\mathcal{C}(G)$ as follows. Vertices of $\mathcal{C}(G)$ are cliques of G. Two vertices are connected, if the corresponding cliques share at least one edge.

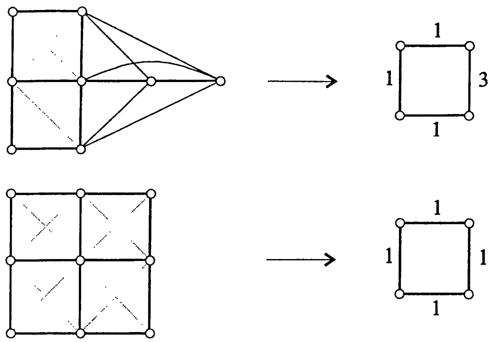

FIGURE 5. Example showing $\mathcal{C}(G)$ is not injective.

The transformation $\mathcal{C}$ is not injective as the example in Fig. 5 shows. Therefore we define weights of edges of $\mathcal{C}(G)$ to be the number of edges in the intersection of the corresponding cliques in G. An interesting question regarding $\mathcal{C}$ is whether G not being a strong bundle implies $\mathcal{C}(G)$ not being a Cartesian graph bundle. However, we do not address this question here.

We now restrict attention to graph bundles over a triangle free base. We will show that if G is a strong graph bundle, then $\mathcal{C}(G)$ must be a Cartesian graph bundle. First note:

LEMMA 4. *Let (G, p, B) be a strong graph bundle over a triangle free base B. Then cliques of G are projected in edges of B by the bundle projection p. Moreover, cliques in G are of the form $K_2 \boxtimes K_F$ where K_F is some clique of F.*

P r o o f . Clearly, cliques are projected into cliques by p.

Since B is triangle free, any clique of G is a subgraph of two neighboring fibres. In $K_2 \boxtimes F$, any complete subgraph is a subgraph of a $K_2 \boxtimes K_F$, where K_F is a clique in F. □

THEOREM 2. *Let G be a thin graph, i.e. $G/S \cong G$. If (G,p,B) is a strong graph bundle with triangle free base B (and $B \not\cong K_p$ and $F \not\cong K_q$), then $\mathcal{C}(G)$ has a representation as a Cartesian graph bundle over a triangle free base.*

P r o o f . Let $e = xy$ and $f = yz$ be any pair of adjacent edges of B. Since B is triangle free, $xz \notin E(B)$ and there is no edge between fibres $p^{-1}(x)$ and $p^{-1}(z)$ in G. By the lemma above, no clique in G can be projected to x, y and z.

The projection p of the strong graph bundle (G,p,B) has a natural meaning in the graph $\mathcal{C}(G)$. Every clique is projected to an edge of B since cliques of G are of the form $K_2 \boxtimes K_F$ for some clique K_F in F (by the lemma above).

Two cliques, which share a degenerate edge (or more degenerate edges) are projected to adjacent edges of B.

For any clique K, which is projected to e, there has to be a unique clique H of the same size, which shares edges with K and is projected to f. This is because G is a strong graph bundle.

It follows that $\mathcal{C}(G)$ is a Cartesian graph bundle over $l(B)$, the line graph of B. If we formally denote the projection p on $\mathcal{C}(G)$ by p', we may say $(\mathcal{C}(G), p', l(B))$ is a Cartesian graph bundle. □

In a special case, where both base and fibre are triangle free, we can give a statement in terms of line graphs.

COROLLARY 1. *If a graph G is K_5-free and G is a strong graph bundle over base $B \not\cong K_2$ with fibre $F \not\cong K_2$ then $\mathcal{C}(G)$ is a Cartesian graph bundle over $\ell(B)$ with fibre $\ell(F)$.*

If we wish to recognize strong graph bundles, we may proceed as follows.

First compute the G/S. Eliminate clique factors. Compute $\mathcal{C}(G/S)$. If $\mathcal{C}(G/S)$ is not a Cartesian bundle over a triangle free base, then we know G cannot be a strong bundle. But if $\mathcal{C}(G/S)$ is a Cartesian graph bundle, then we must be able to check whether G is. Note that although the transformation $\mathcal{C}$ may not be injective, we have started from G, so we know exactly which vertices and edges have been mapped to a particular clique, vertex of $\mathcal{C}(G/S)$. The check of G being a bundle can be based on the following lemma.

Lemma 5. *Let G be a strong graph bundle with trivial relation S. Let K_p and K_q be two adjacent vertices of $\mathcal{C}(G)$. If K_p and K_q are not in the same fibre of $\mathcal{C}(G)$, then every edge in the intersection of K_p and K_q is a degenerate edge in G.*

Proof. Recall that $\mathcal{C}(G)$ is a Cartesian graph bundle, if G is a strong graph bundle. We also know that the cliques K_p and K_q are of the form $K_2 \boxtimes$ a clique. It follows that $p = q$, $K_p \cap K_q \cong K_{p/2}$ and all edges of the intersection are degenerate in (G, p, B). We omit the details. □

We have to check that there are enough edges determined by the rule of the above lemma. This can be argued as follows. Assumptions that there are no triangles in B and $B \not\cong K_2$ imply there is a vertex v of degree more than 1 in B. It can be shown that all edges of the copy of fibre $p^{-1}(v)$ are intersections of cliques of G and are therefore all determined to be degenerate by the above lemma. Starting with a fibre, say $p^{-1}(v)$, it is clear how lifted matchings (uniquely) determine neighboring fibres. It is also clear that such fibres can be found efficiently — they are connected components of the subgraph of G induced on the degenerate edges determined by intersections of cliques. Hence we have:

Proposition 3. *Let G be a graph bundle with respect to the strong product over a triangle free base and with $B \not\cong K_2$ and $F \not\cong K_2$. Let $\big(\mathcal{C}(G), p', B'\big)$ be a representation of $\mathcal{C}(G)$ as a Cartesian graph bundle. Then the set of degenerate edges determined from this representation by application of Lemma* 5 *uniquely defines a fundamental factorization of G.*

5. Algorithm

The results obtained so far can be summarized in a form of an algorithm for recognition of strong graph bundles using the algorithm for recognition of Cartesian graph bundles over triangle free bases ([10]).

We conclude with some remarks.

In its present form, Step 3 of the algorithm involves computing all cliques of a graph, and hence this step of the algorithm may not be polynomial.

For graphs with cliques of maximal order at most $O(\log n)$, the transformation can be completed in polynomial time and for such graphs, this algorithm decides whether a graph is a strong graph bundle in polynomial time.

There are two main drawbacks of this algorithm: first it can fail to recognize some graph bundles with triangles in the base graph (this is inherited from the algorithm for recognition of Cartesian graph bundles) and second, it is not polynomial for graphs with large cliques. In both cases it seems that it should be possible to find an efficient solution, details are, however, not trivial at the first sight, so we leave improvement of this two drawbacks as open problems.

ALGORITHM

1. compute S
2. construct $G' = G/S$
3. compute $G'' = \mathcal{C}(G')$
4. **if** G'' is a not Cartesian graph bundle
   then return(G IS NOT a graph bundle with respect to strong product)
5. **for each** representation of G'' as Cartesian graph bundle **do begin**
   check if G' has the corresponding graph bundle representation **and**
   check if weights of G' are weighted bundle consistent
   if both checks true
   then return(G IS a graph bundle with respect to strong product)
   end
6. **if** no representation of G found in loop 5.
   then return(G IS NOT a graph bundle with respect to strong product)

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