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SPECTRAL PROPERTIES OF GENERAL SELF-ADJOINT, EVEN ORDER DIFFERENTIAL OPERATORS

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ABSTRACT. Necessary and sufficient condition for discreteness and boundedness below of the spectrum of the full-term singular differential operator $\ell(y) \equiv \frac{1}{w(t)} \sum\limits_{k=0}^{n} (-1)^k \big(p_k(t)y^{(k)}\big)^{(k)}$, $t \in [a,\infty)$, is established. This condition is based on a recently proved generalized reciprocity principle for ℓ and on the relationship between spectral properties of ℓ and oscillation of a certain associated $(2n-2)$-order differential equation. An application to "Euler-type" fourth order operator is given.

1. Introduction

In this paper we deal with spectral properties of the full-term self-adjoint differential operator

$$\ell(y) \equiv \frac{1}{w(t)} \sum_{k=0}^{n} (-1)^k \big(p_k(t)y^{(k)}\big)^{(k)} \tag{1.1}$$

where $t \in I := [a,\infty)$, $p_n^{-1}, p_{n-1}, \dots, p_0, w \in L_{\mathrm{loc}}(I)$, $p_n, w > 0$ on I are real valued functions, for some $a \in \mathbb{R}$. In particular, we look for necessary and sufficient conditions for ℓ to possess the so-called property **BD**:

> The spectrum of any self-adjoint extension of the minimal differential operator generated by ℓ in the weighted Hilbert space $L^2_w(I)$ (with the inner product $(y,z)_w = \int_I w(t)y(t)\overline{z(t)}\,\mathrm{d}t$) is discrete and bounded below.

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The paper is organized as follows. In this introductory section we recall some basic facts of spectral theory of singular differential operators and oscillation of self-adjoint equations. In Section 2 we list auxiliary results. Main Theorem of Section 3 gives necessary and sufficient condition for property **BD** of ℓ. In Section 4 we discuss a special case of Main Theorem — the case of fourth order operators. In this case the oscillation of the unpleasant equation (3.1) can be readily determined in view of deeply developed oscillation theory of the second order equations. We illustrate the application on a special fourth order operator of the form $\ell_N(y) \equiv \frac{1}{w(t)} N^*\big(r(t)N(y)\big)$, where $N(y) \equiv y'' + \frac{1}{4t^2} y$ is the second order "Euler-type" operator. Appendices are devoted to technical computations for the reciprocal operator ℓ_R and for the proof of Theorem F.

First recall briefly the basic facts of spectral theory of singular differential operators (a comprehensive treatment of this topic may be found e.g. in [8], [11], [14]). Denote the quasi-derivatives

$$
\begin{aligned}
y^{[j]} &= y^{(j)}\,, \qquad j = 0, 1, \dots, n-1\,,\\
y^{[n]} &= p_n y^{(n)}\,,\\
y^{[n+j]} &= p_{n-j} y^{(n-j)} - \big(y^{[n+j-1]}\big)'\,, \qquad j = 1, \dots, n\,,
\end{aligned}
$$

and let

$$
\mathcal{D}(M) = \Big\{ y \in L^2_w(I) :\ y^{[k]} \in AC(I)\,,\ k = 0, \dots, 2n-1\,,\ y^{[2n]} \in L^2_w(I) \Big\}\,.
$$

The differential operator $M\colon \mathcal{D}(M) \subseteq L^2_w(I) \to L^2_w(I)$ given by $M(y) = \ell(y)$, $y \in \mathcal{D}(M)$, and its adjoint $M_0 := M^*$ are called respectively the *maximal* and the *minimal operator* defined by ℓ. Any self-adjoint extension K of M_0 (which exists, since the functions p_k are real, i.e., M_0 has the same deficiency indices γ_+, γ_- for which $n \le \gamma_+ = \gamma_- \le 2n$ holds) satisfies $M_0 \subseteq K \subseteq M$ and all self-adjoint extensions of M_0 have the same essential spectrum.

One of the most important problems in the theory of singular differential operators is to find conditions which guarantee that the essential spectrum of any self-adjoint extension K of M_0 is empty, i.e. K has a spectrum which is discrete and bounded below — the so-called property **BD** ([10]). Property **BD** means, roughly speaking, that the singular operator behaves like a regular one since it is known that the spectrum of regular operators consists only of eigenvalues of finite multiplicities with the only possible cluster point at ∞.

Spectral properties of singular differential operators of the form (1.1) are closely related to the oscillation theory of self-adjoint, even order differential equations. Two points $t_1, t_2 \in I$ are said to be *conjugate* relative to the equation

$$
\ell_1(y) \equiv \sum_{k=0}^{n} (-1)^k \big(p_k(t) y^{(k)}\big)^{(k)} = 0 \tag{1.2}
$$

if there exists a nontrivial solution y of this equation for which $y^{(i)}(t_1) = 0 = y^{(i)}(t_2)$, $i = 0, \dots, n-1$. Equation (1.2) is said to be *nonoscillatory* if there exists $c \in I$ such that the interval (c, ∞) contains no pair of points conjugate relative to (1.2), in the opposite case (1.2) is said to be *oscillatory at* ∞.

Besides the definition of (non)oscillation for linear differential equations defined by means of (non)existence of an arbitrarily large pair of conjugate points, we will need another (stronger) definition of oscillation properties, introduced by Nehari and Levin. To distinguish these concepts from the above given ones, we shall speak about *N-disconjugacy*, *N-nonoscillation*, etc.

DEFINITION 1.1. Consider a linear differential equation of nth order

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_0(t)y = 0\,, \tag{1.3}$$

where $p_k \in C(I)$. This equation is said to be *N-disconjugate* on an interval $I_0 \subseteq I$ whenever every nontrivial solution of (1.3) has at most $n-1$ zeros on I_0, every zero counted according to its multiplicity. Equation (1.3) is said to be *N-nonoscillatory* if there exists $c \in I$ such that this equation is N-disconjugate on $(c, +\infty)$.

Self-adjoint equations (1.2) are closely related to linear Hamiltonian systems (LHS)

$$x' = A(t)x + B(t)u\,, \qquad u' = -C(t)x - A^T(t)u\,, \tag{1.4}$$

where A, B, C are $n \times n$ matrices, continuous on I, B and C symmetric. Equation (1.2) can be rewritten into LHS (1.4) by the following way: let y be a solution of (1.2) and set $x = \left(y^{[0]}, \dots, y^{[n-1]}\right)^T$, $u = \left(y^{[2n-1]}, \dots, y^{[n]}\right)^T$. Then (x, u) is a solution of LHS (1.4) with

$$\begin{aligned} B(t) &= \operatorname{diag}\{0, \dots, 0, p_n^{-1}(t)\}\,, \\ C(t) &= -\operatorname{diag}\{p_0(t), \dots, p_{n-1}(t)\}\,, \\ A(t) &= A_{i,j} = \begin{cases} 1 & \text{for } j = i+1\,,\ i = 1, \dots, n-1\,, \\ 0 & \text{elsewhere.} \end{cases} \end{aligned} \tag{1.5}$$

In this case we say that the solution (x, u) is *generated* by y. Simultaneously with (1.4) we will consider its matrix analogy

$$X' = A(t)X + B(t)U\,, \qquad U' = -C(t)X - A^T(t)U\,, \tag{1.6}$$

where X, U are $n \times n$ matrices. A solution (X, U) of (1.6) is said to be *isotropic* if $X^T(t)U(t) - U^T(t)X(t) \equiv 0$. An alternative terminology is prepared [9], self-conjoined [12], self-conjugate [13]; our terminology is due to [3]. An isotropic solution (X, U) of (1.6) is said to be *principal at* ∞ if X is nonsingular on an interval $[c, \infty) \subseteq I$ and

$$\lim_{t \to \infty} \left(\int_c^t X^{-1}(s)B(s)X^{T-1}(s)\, \mathrm{d}s \right)^{-1} = 0\,.$$

Let $(\tilde{X},\tilde{U})$ be a solution of (1.6) which is linearly independent of the principal solution (X,U) (i.e. (X,U), $(\tilde{X},\tilde{U})$ form the base of the solution space of (1.6)), then $(\tilde{X},\tilde{U})$ is said to be *nonprincipal at* ∞. A system $y_1,\dots,y_n$ of solutions of (1.2) is said to form the *principal* (*nonprincipal*) *system at* ∞ if the solution (X,U) of the corresponding LHS (1.6) whose columns are generated by $y_1,\dots,y_n$ is principal (nonprincipal) at ∞. For example, concerning the equation $y^{(2n)}=0$, the functions $y_i=t^{i-1}$, $i=1,\dots,n$, form the principal system of solutions while $\tilde{y}_i=t^{n+i-1}$, $i=1,\dots,n$, form the nonprincipal one. The principal (nonprincipal) system of solutions at ∞ exists whenever (1.2) is nonoscillatory.

Let g, h be solutions of (1.2). Then the expression

$$\{g,h\}:=\sum_{j=1}^{n}\left(g^{[j-1]}h^{[2n-j]}-g^{[2n-j]}h^{[j-1]}\right) \tag{1.7}$$

is called *Lagrange's bracket* of g and h.

2. Auxiliary results

The following fundamental result relates oscillation and spectral theories of singular differential operators.

THEOREM A. ([8]) *The following are equivalent.*

(i) *The operator ℓ possesses property* **BD**.

(ii) *The equation $\ell(y)=\lambda y$ is nonoscillatory for every $\lambda\in\mathbb{R}$.*

(iii) *For every $\lambda\in\mathbb{R}$ there exists $N\in\mathbb{R}$ such that*

$$\int_N^\infty\left[\sum_{k=0}^{n}p_k(t)\left(y^{(k)}\right)^2\right]\,\mathrm{d}t\geq\lambda\int_N^\infty w(t)y^2(t)\,\mathrm{d}t$$

for any $y\in W^{n,2}(N,\infty)$ with compact support in $[N,+\infty)$.

One of the most important tools in the oscillation theory are transformations of LHS's and self-adjoint differential expressions.

THEOREM B. ([2], [4]) *Let $H,K\in C^1(I)$ be $n\times n$ matrices such that H is nonsingular and $H^TK\equiv K^TH$ on I. Then the transformation*

$$x=H(t)\tilde{x}\,,\qquad u=K(t)\tilde{x}+H^{T^{-1}}(t)\tilde{u} \tag{2.1}$$

transforms (1.4) *into a linear Hamiltonian system*

$$\tilde{x}'=\tilde{A}(t)\tilde{x}+\tilde{B}(t)\tilde{u}\,,\qquad \tilde{u}'=-\tilde{C}(t)\tilde{x}-\tilde{A}^T(t)\tilde{u}\,, \tag{2.2}$$

where

$$\tilde{A} = H^{-1}(-H' + AH + BK),$$
$$\tilde{B} = H^{-1}BH^{T-1},$$
$$\tilde{C} = H^T(K' + CH + A^T K) + K^T(-H' + AH + BK).$$

Suppose that (1.2) is disconjugate on an interval $I_0 \subseteq I$. Then there exists a symmetric solution $Q = (Q_{ij})_{i,j=1}^n$ of the Riccati matrix differential equation

$$Q' + A^T(t)Q + QA(t) + QB(t)Q + C(t) = 0, \tag{2.3}$$

with A, B, C given by (1.5). Denote $q_i(t) := -Q_{i+1,n}(t)$, $i = 0, \dots, n-1$, and consider the nth order differential operator

$$L(y) \equiv y^{(n)} + \frac{q_{n-1}(t)}{p_n(t)} y^{(n-1)} + \dots + \frac{q_0(t)}{p_n(t)} y. \tag{2.4}$$

The adjoint operator L^* is of the form

$$L^*(z) \equiv (-1)^n z^{(n)} + (-1)^{n-1} \left(\frac{q_{n-1}(t)}{p_n(t)} z\right)^{(n-1)} + \dots - \left(\frac{q_1(t)}{p_n(t)} z\right)' + \frac{q_0(t)}{p_n(t)} z. \tag{2.5}$$

Using these operators we have the following statement concerning factorization of ℓ_1.

THEOREM C. ([3]) *Let* (1.2) *be disconjugate on* $I_0 \subseteq I$. *Then for any* $y \in C^{2n}(I_0)$ *we have*

$$\ell_1(y) = L^*(p_n(t)L(y)), \tag{2.6}$$

where L, L^* *are given by* (2.4), (2.5).

We remark that the factorization of ℓ_1 is not unique — it depends on the choice of a symmetric solution Q of the Riccati equation (2.3).

In [5] the following generalized reciprocity principle for ℓ is proved.

THEOREM D. *Suppose that* (1.2) *is nonoscillatory at* $+\infty$ *and let* (2.6) *be any factorization of* ℓ_1 *near* $+\infty$. *Then the equation*

$$\ell_1(y) = w(t)y, \qquad \text{i.e.} \quad L^*(p_n(t)L(y)) = w(t)y, \tag{2.7}$$

is nonoscillatory at $+\infty$ *if and only if the "reciprocal equation"*

$$(\ell_R(z) \equiv)\ L\left(\frac{1}{w(t)} L^*(z)\right) = \frac{1}{p_n(t)} z \tag{2.8}$$

is nonoscillatory at $+\infty$.

This is a generalization of the well known reciprocity principle for two-term differential equation of the form

$$(-1)^n \big(p_n(t) y^{(n)}\big)^{(n)} = w(t) y\,, \tag{2.9}$$

which states that (2.9) is nonoscillatory at $+\infty$ if and only if

$$(-1)^n \left(\frac{1}{w(t)} y^{(n)}\right)^{(n)} = \frac{1}{p_n(t)} y \tag{2.10}$$

is nonoscillatory at $+\infty$ ([1]).

Remark 2.1. The operator ℓ_R given by the left hand side of (2.8) is self-adjoint, even order differential operator and hence it is of the form

$$\ell_R(z) = \sum_{k=0}^{n} (-1)^k \big(r_k(t) z^{(k)}\big)^{(k)} \tag{2.11}$$

with the leading coefficient $r_n = \frac{1}{w}$, see [11; Chapter I]. Computing the coefficients r_k, $k = 0, \dots, n$, means essentially to differentiate the products of functions in $\ell_R(z)$, see Appendix A. However, these coefficients depend both on the original coefficients p_k, $k = 0, \dots, n$, and on the weight function w. For one-term operators

$$\tilde{\ell}(y) \equiv \frac{(-1)^n}{w(t)} \big(p_n(t) y^{(n)}\big)^{(n)} \tag{2.12}$$

this is not the case since the reciprocal operator is then again a one-term operator with the only leading coefficient $r_n = \frac{1}{w}$.

Recently O. Došlý proved a general necessary condition for ℓ having property **BD**.

THEOREM E. ([5]) *Suppose that* (1.2) *is nonoscillatory at* $+\infty$ *and let* (2.6) *be any factorization of* ℓ_1 *near* $+\infty$. *Let* $z_1, \dots, z_n$ *be a principal system of solutions at* $+\infty$ *of the equation*

$$\ell_R(z) = 0\,, \qquad \textit{i.e. of the equation} \quad L\left(\frac{1}{w(t)} L^*(z)\right) = 0\,. \tag{2.13}$$

If the operator ℓ *has property* **BD** *in the weighted Hilbert space* $L^2_w(I)$, *then for any* $c = (c_1, \dots, c_n)^T \in \mathbb{R}^n$ *we have*

$$\lim_{t\to\infty} \frac{\int_t^\infty p_n^{-1} \left[c_1 z_1(s) + \cdots + c_n z_n(s)\right]^2 \,\mathrm{d}s}{c^T \left[\int^t X^{-1}(s) \bar{B}(s) X^{T-1}(s)\,\mathrm{d}s\right]^{-1} c} = 0\,, \tag{2.14}$$

where X is the Wronski matrix of $z_1,\dots,z_n$ and $\bar{B} = \operatorname{diag}\{0,\dots,0,w\}$.

The aim of this paper is to prove the converse — requiring an additional nonoscillation of a certain associated $(2n-2)$-order equation.

THEOREM F. *Let $h \neq 0$ on I be such that the quasi-derivatives (with respect to the operator ℓ_R) $h^{[j]} \in AC(I)$, $j = 0,\dots,2n-1$, $h^{[2n]} = \ell_R(h) \in L^2(I)$. Then the transformation $z = h(t)u$ yields*

$$h(t)\ell_R(z) = \sum_{k=0}^{n}(-1)^k\big(R_k(t)u^{(k)}\big)^{(k)}.$$

For the coefficients R_i we have: $R_n = r_n h^2$ and for $i = 1,\dots,n$

$$\begin{aligned}
R_{i-1} = -r_{i-1}h^2 + (-1)^{n-i+1}h\sum_{k=0}^{n-i}\tbinom{n-i}{k}\Big[&\tbinom{n}{i-k-1}\big(r_n h^{(n-i+k+1)}\big)^{(n-i-k+1)}\\
&+\tbinom{n}{i-k-2}\big(r_n h^{(n-i+k+2)}\big)^{(n-i-k)}\Big]\\
+h\sum_{k=0}^{n-i-1}(-1)^{k+1}\sum_{l=0}^{k}\tbinom{k}{l}\Big[&\tbinom{i+k}{i-k+l-1}\big(r_{i+k}h^{(2k-l+1)}\big)^{(l+1)}\\
&+\tbinom{i+k}{i-k+l-2}\big(r_{i+k}h^{(2k-l+2)}\big)^{(l)}\Big]\\
+\sum_{k=i}^{n}\Big[\tbinom{k-1}{i-2}h^{(k-i+1)}K_{k,i} - \tbinom{k-1}{i-1}&h^{(k-i)}K_{k,i-1}\Big],
\end{aligned}$$

where $K_{i,j}$ are entries of the matrix K, see the proof – formulas (B.2) – (B.4), (B.6).

Particularly, $R_0 = h\ell_R(h)$ and

$$R_{n-1} = n(n-1)\,r_n\big({h'}^2 - hh''\big) - nh\big(r_n h'\big)' - r_{n-1}h^2.$$

Moreover, z is a solution of (2.13) *if and only if $u = \frac{z}{h}$ is a solution of equation*

$$\sum_{k=0}^{n}(-1)^k\big(R_k(t)u^{(k)}\big)^{(k)} = 0. \tag{2.16}$$

Proof. See Appendix B. □

Remark 2.2.

(i) Observe that by taking $h = z_i$ (a solution of (2.13)) we get $R_0 \equiv 0$.

(ii) Observe also that if g and h are solutions of (1.2), then $v = u' = \left(\frac{g}{h}\right)'$ is a solution of $(2n-2)$-order equation

$$\sum_{k=1}^{n}(-1)^{k-1}\big(R_k v^{(k-1)}\big)^{(k-1)} = \kappa, \tag{2.17}$$

where κ is a real constant. In [7; Lemma 2.3], it is shown that $\kappa = \{g,h\}$. Moreover, if $\{g,h\} = 0$, then the transformation $v = \left(\frac{g}{h}\right)' \tilde{v}$ transforms (2.17) (with $\kappa = 0$) into

$$\sum_{k=1}^{n} (-1)^{k-1} \left(\tilde{R}_k \tilde{v}^{(k-1)}\right)^{(k-1)} = 0\,, \tag{2.18}$$

where $\tilde{R}_1, \dots, \tilde{R}_n$ are obtained by an application of Theorem F, with $\tilde{R}_1 \equiv 0$.

(iii) In [2], it is shown that quadratic functionals transform upon the transformation $z = hu$ essentially in the same way as their Euler-Lagrange equations, i.e.

$$\int_N^{+\infty} \left[\sum_{k=0}^{n} r_k \left(z^{(k)}\right)^2\right] \mathrm{d}t = \int_N^{+\infty} \left[\sum_{k=0}^{n} R_k \left(u^{(k)}\right)^2\right] \mathrm{d}t$$

for any $z \in W_0^{n,2}(N,+\infty)$, where $u = \frac{z}{h}$.

To prove the main statement we need also the following Wirtinger-type inequality ([10]).

Lemma 2.1. *Let $P\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $P'(t) \neq 0$ on $[a,b]$. Then*

$$\int_a^b |P'| z^2 \,\mathrm{d}t \le 4 \int_a^b \frac{P^2}{|P'|} z'^2 \,\mathrm{d}t$$

for any $z \in W_0^{1,2}(a,b)$.

For the application in Section 4 we give here the following Leighton-Wintner-type nonoscillation criterion for the second order equation of the form

$$\left(p(t)y'\right)' + \lambda q(t) y = 0\,, \tag{2.19}$$

where $p^{-1}, q \in L_{\mathrm{loc}}(I)$, $p > 0$, $q \ge 0$ on I, $\lambda > 0$ constant.

Lemma 2.2. *Suppose that one of the following hypotheses holds:*

$$\int^{\infty} \frac{1}{p} = +\infty \qquad \textit{and} \qquad \lim_{t\to\infty} p(t)q(t) \left(\int^{t} \frac{\mathrm{d}s}{p(s)}\right)^2 =: \Omega^2 < +\infty\,, \tag{2.20}$$

or

$$\int^{\infty} \frac{1}{p} < +\infty \qquad \textit{and} \qquad \lim_{t\to\infty} p(t)q(t) \left(\int_t^{\infty} \frac{\mathrm{d}s}{p(s)}\right)^2 =: \Omega^2 < +\infty\,. \tag{2.21}$$

Then (2.19) *is nonoscillatory for all* $\lambda < \frac{1}{4\Omega^2}$.

Proof. The well-known Leighton-Wintner criterion states that (2.19) is nonoscillatory provided

$$\int^{\infty} \frac{1}{p} = +\infty \qquad \text{and} \qquad \lim_{t\to\infty} \lambda \int^{t} \frac{ds}{p(s)} \int_{t}^{\infty} q(s)\, ds < \tfrac{1}{4}.$$

Using l'Hospital's rule the last limit takes the form

$$\lim_{t\to\infty} \frac{\lambda \int_t^{\infty} q(s)\, ds}{\left(\int^t \frac{ds}{p(s)}\right)^{-1}} = \lim_{t\to\infty} \lambda p(t) q(t) \left(\int^t \frac{ds}{p(s)}\right)^2 = \lambda\Omega^2 < \tfrac{1}{4}.$$

If $\int^{\infty} \frac{1}{p} < +\infty$, the transformation $y = h(t)z$ with $h(t) = \int_t^{\infty} \frac{ds}{p(s)}$ transforms (2.19) into the equation

$$(\tilde{p}(t)z')' + \lambda\, \tilde{q}(t)z = 0\,, \tag{2.22}$$

where $\tilde{p}(t) = p(t)\Bigl(\int_t^{\infty} \frac{ds}{p(s)}\Bigr)^2$ and $\tilde{q}(t) = q(t)\Bigl(\int_t^{\infty} \frac{ds}{p(s)}\Bigr)^2$. Now $\int^{\infty} \frac{1}{\tilde{p}} = +\infty$ and hence, by the first part, if

$$\lim_{t\to\infty} \tilde{p}(t)\tilde{q}(t) \left(\int_t^{\infty} \frac{ds}{\tilde{p}(s)}\right)^2 =: \Omega^2 < +\infty\,,$$

then (2.22) (and hence (2.19)) is nonoscillatory for $\lambda < \frac{1}{4\Omega^2}$. However, the last limit equals (2.21) since $\int^t \frac{ds}{\tilde{p}(s)} = \Bigl(\int_t^{\infty} \frac{ds}{p(s)}\Bigr)^{-1}$. □

3. BD criterion for full-term differential operators

We say that a system $z_1, \dots, z_{2n}$ of solutions of (2.13) forms an *ordered* system at $+\infty$ if $z_j > 0$, $j = 1, \dots, 2n$, for large t and

$$\frac{z_j}{z_{j+1}} \to 0\,, \qquad j = 1, \dots, 2n-1\,,$$

for $t \to +\infty$. Such system of solutions exists whenever (2.13) is N-nonoscillatory.

Let $z_1, \dots, z_n$, $\tilde{z}_1, \dots, \tilde{z}_n$ be an ordered system of solutions of (2.13). Then the functions $z_1, \dots, z_n$ form a principal system of solutions of (2.13) and the functions $\tilde{z}_1, \dots, \tilde{z}_n$ form a nonprincipal one (see [3; Chapter III]). Denote by (X, U) and $(\tilde{X}, \tilde{U})$ the solutions of the associated linear Hamiltonian system generated respectively by $z_1, \dots, z_n$ and $\tilde{z}_1, \dots, \tilde{z}_n$ and let $S = S_{i,j} = X^T\tilde{U} - U^T\tilde{X}$ be the (constant) Wronskian-type matrix of (X, U) and $(\tilde{X}, \tilde{U})$.

Main Theorem. *Suppose* (1.2) *is N-nonoscillatory at* $+\infty$ *and let* $z_1, \dots, z_n$, $\tilde{z}_1, \dots, \tilde{z}_n$, (X, U), $(\tilde{X}, \tilde{U})$, S *be as above. Let* (2.6) *be any factorization of* ℓ_1 *near* $+\infty$ *with* L, L^* *given by* (2.4), (2.5). *Suppose that there exists an index* $i \in \{1, \dots, n\}$ *and* $\lambda_0 > 0$ *such that the equation*

$$\sum_{k=0}^{n-1} (-1)^k \big(R_{k+1}(t) v^{(k)}\big)^{(k)} = \lambda \frac{S_{i,l} W_{l,i}^2(t)}{W'(t) W_{l,i}(t) - W(t) W'_{l,i}(t)}\, v \tag{3.1}$$

is nonoscillatory for all $\lambda < \lambda_0$, *where*

$$l := \min\{j \in \{1, \dots, n\} \mid S_{i,j} \neq 0\},$$

the functions $R_k(t)$, $k = 1, \dots, n$, *are given in Theorem F with* $h = z_i$ *and* $W := W(\tilde{z}_1, \dots, \tilde{z}_n)$, $W_{l,i} := W(\tilde{z}_1, \dots, \tilde{z}_{l-1}, z_i, \tilde{z}_{l+1}, \dots, \tilde{z}_n)$ *are the Wronskians of the functions in brackets. Then the operator* ℓ *given by* (1.1) *possesses property* **BD** *if and only if*

$$\lim_{t\to\infty} \frac{W(t)}{S_{i,l} W_{l,i}(t)} \int_t^\infty \frac{z_i^2(s)}{p_n(s)}\, \mathrm{d}s = 0. \tag{3.2}$$

Remark 3.1. N-nonoscillation of (2.13) easily follows from the requirement of N-nonoscillation of (1.2).

Proof.

$\Longrightarrow$: By Theorem E we have that for $c = e_i = (0, \dots, 1, \dots, 0)^T \in \mathbb{R}^n$ (with 1 at the ith entry)

$$\lim_{t\to\infty} \frac{\int_t^\infty p_n^{-1} z_i^2(s)\, \mathrm{d}s}{e_i^T \left[\int^t X^{-1}(s) \bar{B}(s) X^{T-1}(s)\, \mathrm{d}s\right]^{-1} e_i} = 0$$

with $\bar{B} = \operatorname{diag}\{0, \dots, 0, w\}$. Next we will show that

$$\lim_{t\to\infty} \frac{e_i^T \left[\int^t X^{-1}(s) \bar{B}(s) X^{T-1}(s)\, \mathrm{d}s\right]^{-1} e_i}{S_{i,l} \frac{W_{l,i}(t)}{W(t)}} = 1. \tag{3.3}$$

By a direct computation one may verify that $(X^{-1}\tilde{X})' = X^{-1}\bar{B}X^{T-1}S$ and so

$$\left[\int^{t} X^{-1}(s)\bar{B}(s)X^{T-1}(s)\,\mathrm{d}s\right]^{-1} = S\tilde{X}^{-1}X\,.$$

In [3; Chapter III], it is proved that

$$\left(\tilde{X}^{-1}X\right)_{j,i} = \frac{W(\tilde{z}_1,\dots,\tilde{z}_{j-1},z_i,\tilde{z}_{j+1},\dots,\tilde{z}_n)}{W(\tilde{z}_1,\dots,\tilde{z}_n)} \tag{3.4}$$

and according to the definition of $z_1,\dots,z_n$, $\tilde{z}_1,\dots,\tilde{z}_n$ we have for $j > k$

$$\lim_{t\to\infty}\left(\tilde{X}^{-1}X\right)_{j,i}/\left(\tilde{X}^{-1}X\right)_{k,i} = 0\,. \tag{3.5}$$

Thus

$$\begin{aligned}
&e_i^T\left[\int^{t} X^{-1}(s)\bar{B}(s)X^{T-1}(s)\,\mathrm{d}s\right]^{-1} e_i\\
&= \left(e_i^T S\right)\left(\tilde{X}^{-1}Xe_i\right)\\
&= \left(S_{i,1},\dots,S_{i,n}\right)\cdot\left((\tilde{X}^{-1}X)_{1,i},\dots,(\tilde{X}^{-1}X)_{n,i}\right)^T\\
&= S_{i,1}\left(\tilde{X}^{-1}X\right)_{1,i} + \dots + S_{i,n}\left(\tilde{X}^{-1}X\right)_{n,i}\\
&= S_{i,l}\left(\tilde{X}^{-1}X\right)_{l,i}\left[1 + \frac{S_{i,l+1}(\tilde{X}^{-1}X)_{l+1,i}}{S_{i,l}(\tilde{X}^{-1}X)_{l,i}} + \dots + \frac{S_{i,n}(\tilde{X}^{-1}X)_{n,i}}{S_{i,l}(\tilde{X}^{-1}X)_{l,i}}\right].
\end{aligned}$$

According to (3.4), (3.5), the validity of (3.3) is now easily seen.

$\Longleftarrow$: By Theorem A it suffices to show that the equation

$$\left(\ell_1(y) =\right)\ L^*\left(p_n(t)L(y)\right) = \lambda w(t)y \tag{3.7}$$

is nonoscillatory for every $\lambda \in \mathbb{R}$, which holds if and only if the equation

$$\left(\ell_R(z) =\right)\ L\left(\frac{1}{w(t)}L^*(z)\right) = \lambda\frac{1}{p_n(t)}z \tag{3.8}$$

is nonoscillatory for every $\lambda \in \mathbb{R}$ (Theorem D). The final step is to show that the quadratic functional

$$\mathcal{F}_R(z) \equiv \int_N^{+\infty}\left[zL\left(\frac{1}{w}L^*(z)\right) - \lambda\frac{1}{p_n}z^2\right]\mathrm{d}t$$

is positive for all $z \in W_0^{n,2}(N,+\infty)$, $z \not\equiv 0$, for some $N \in \mathbb{R}$.

Since (1.2) is nonoscillatory it follows that (3.7) is nonoscillatory for every $\lambda \le 0$.

Put $M(t) := \frac{S_{i,l} W_{l,i}(t)}{W(t)}$ and let $\lambda > 0$ be arbitrary. Suppose (3.2) holds. Then for $\varepsilon := \frac{\lambda_0}{8\lambda}$ there exists $N_1 \in \mathbb{R}$ such that

$$\frac{1}{M(t)} \int_t^{\infty} \frac{z_i^2(s)}{p_n(s)}\, ds < \varepsilon \qquad \text{for} \quad t \in [N_1, +\infty)\,.$$

For any $N_2 \ge N_1$ and for any $z \in W_0^{n,2}(N_2, +\infty)$ it follows (C-S stands for Cauchy-Schwartz inequality)

$$\begin{aligned}
0 &\le \int_{N_2}^{+\infty} \frac{1}{p_n} z^2\, dt = \int_{N_2}^{+\infty} \frac{z_i^2}{p_n}\Big(\frac{z}{z_i}\Big)^2\, dt \\
&= 2 \int_{N_2}^{+\infty} \frac{z_i^2}{p_n}\left[\int_{N_2}^{t} \frac{z}{z_i}\Big(\frac{z}{z_i}\Big)'\, ds\right] dt \\
&= 2 \int_{N_2}^{+\infty} \frac{z}{z_i}\Big(\frac{z}{z_i}\Big)' M \frac{1}{M}\left[\int_t^{\infty} \frac{z_i^2}{p_n}\, ds\right] dt \\
&\le 2\varepsilon \int_{N_2}^{+\infty} \sqrt{|M'|}\,\Big|\frac{z}{z_i}\Big| \frac{M}{\sqrt{|M'|}}\Big|\Big(\frac{z}{z_i}\Big)'\Big|\, dt \\
&\overset{\text{C-S}}{\le} 2\varepsilon \left(\int_{N_2}^{+\infty} |M'|\Big(\frac{z}{z_i}\Big)^2\, dt\right)^{1/2} \left(\int_{N_2}^{+\infty} \frac{M^2}{|M'|}\Big(\frac{z}{z_i}\Big)'^2\, dt\right)^{1/2} \\
&\overset{\text{Lemma 2.1}}{\le} 4\varepsilon \int_{N_2}^{+\infty} \frac{M^2}{|M'|}\Big(\frac{z}{z_i}\Big)'^2\, dt\,.
\end{aligned}$$

Thus for $N \ge N_2$ large enough we have

$$\mathcal{F}_R(z) = \int_N^{+\infty}\left[z\ell_R(z) - \frac{\lambda}{p_n} z^2\right] dt \ge \int_N^{+\infty}\left[\sum_{k=0}^{n} r_k\big(z^{(k)}\big)^2 - 4\lambda\varepsilon \frac{M^2}{|M'|}\Big(\frac{z}{z_i}\Big)'^2\right] dt$$

$$\left|\begin{array}{l} z=h(t)u \\ h=z_i \\ v=u'=\big(\frac{z}{z_i}\big)' \\ R_0\equiv 0 \end{array}\right| \overset{\substack{\text{Theorem F} \\ \text{Remark 2.2(iii)}}}{=} \int_N^{+\infty}\left[\sum_{k=0}^{n-1} R_{k+1}\left(v^{(k)}\right)^2 - \frac{\lambda_0}{2}\frac{M^2}{|M'|} v^2\right] dt \ge 0\,,$$

since the last integral is the quadratic functional of (3.1) with $\lambda = \frac{\lambda_0}{2} < \lambda_0$ and

$$\frac{M^2}{|M'|} = \frac{M^2}{-M'} = \frac{S_{i,l} W_{l,i}^2}{W' W_{l,i} - W W_{l,i}'} .$$

The proof is complete. □

Remark 3.2.

(i) The result of the above theorem is a natural generalization of Theorem 4.1 of [6] where only one-term operators (2.12) are taken into consideration. This field has been extensively explored in [7] for the fourth order differential operators where explicit (3.2)-like criteria for property **BD** of $\bar{\ell}(y) \equiv \frac{1}{w(t)}\big(r(t)y''\big)''$ are formulated.

(ii) In the Main Theorem we need the concept of N-disconjugacy (N-non-oscillation) for (1.2). One-term operators of the form (2.12) are disconjugate (N-disconjugate) in their nature.

(iii) It is possible to formulate Main Theorem also for the so-called *normalized* system of solution of (2.13), i.e. for such system for which the corresponding Wronskian-type matrix S is the identity matrix. In that case it is not necessary to require the existence of $2n$ positive solutions $z_1, \dots, \tilde{z}_n$ of (2.13) but it suffices to have the only one — for the transformation of the quadratic functional $\mathcal{F}_R$.

(iv) We are making use of the solution Q of the Riccati equation (2.3). However, solving (2.3) is as complicated as solving the original equation, so the result yields a practical use "only" for the operators ℓ for which either the solution of (2.3) is known or its factorization is available from somewhere else. Therefore, to apply Main Theorem, one may proceed from the other way around. Let be given a disconjugate differential operator

$$N(y) \equiv y^{(n)} + q_{n-1}(t) y^{(n-1)} + \cdots + q_0(t) y \tag{3.9}$$

of order n and let be given a (fixed) weight function w. Let us consider a disconjugate self-adjoint differential operator

$$\ell_N(y) \equiv \frac{1}{w(t)} N^*\big(r(t) N(y)\big) \tag{3.10}$$

of order $2n$ with the leading coefficient r. We want to find (3.2)-like condition which guarantees the property **BD** of ℓ_N in $L_w^2\big([a, +\infty)\big)$. In this way one may study operators at least of this particular form (in fact every disconjugate operator of order $2n$ can be factorized in this way — Theorem C). We illustrate the above procedure on the example of the next section.

4. A special fourth order operator

Let $N(y) = N^*(y) = y'' + \frac{1}{4t^2}y$, $r^{-1} \in L_{\mathrm{loc}}(I)$, $r > 0$ on I. Then we have

$$\ell_N(y) = \frac{1}{w(t)}\left\{\left(r(t)y''\right)'' + \left(\frac{r(t)}{2t^2}y'\right)' + \left[\left(\frac{r(t)}{4t^2}\right)'' + \frac{r(t)}{16t^4}\right]y\right\}. \tag{4.1}$$

The reciprocal operator ℓ_{NR} takes the form

$$\ell_{NR}(z) = N\left(\frac{1}{w(t)}N(z)\right) = \left(r_2(t)z''\right)'' - \left(r_1(t)z'\right)' + r_0(t)z, \tag{4.2}$$

i.e. $r_2(t) = \frac{1}{w(t)}$, $r_1(t) = -\frac{1}{2t^2w(t)}$, $r_0(t) = \left(\frac{1}{4t^2w(t)}\right)'' + \frac{1}{16t^4w(t)}$.

Notation. To simplify expressions and formulas we use an abbreviated notation for integrals — without the integration variable t, s, $\mathrm{d}t$, $\mathrm{d}s$ and without the lower integration limit. For example, under the symbol $\int^t sw\ln\frac{t}{s}$ we understand the function $t \mapsto \int_c^t sw(s)\ln\frac{t}{s}\,\mathrm{d}s$, or the symbol $\int^\infty tw$ means the constant $\int_c^\infty tw(t)\,\mathrm{d}t$, for some $c \in \mathbb{R}$.

The fundamental system of solutions of the reciprocal equation

$$\ell_{NR}(z) = 0 \tag{4.3}$$

is formed by the functions

$$\sqrt{t},\quad \sqrt{t}\ln t,\quad \sqrt{t}\int^t sw\ln\tfrac{t}{s},\quad \sqrt{t}\int^t sw\ln s\ln\tfrac{t}{s}.$$

Next we give the explicit description of principal and nonprincipal systems of solutions of (4.3). We distinguish the following cases:

(A) $\int^\infty tw = +\infty$. Then

$$z_1 = \sqrt{t},\quad z_2 = \sqrt{t}\ln t,\quad \tilde{z}_1 = \sqrt{t}\int^t sw\ln\tfrac{t}{s},\quad \tilde{z}_2 = \sqrt{t}\int^t sw\ln s\ln\tfrac{t}{s}.$$

(B) $\int^\infty tw < +\infty$, $\int^\infty tw\ln t = +\infty$. Then

$$z_1 = \sqrt{t},\qquad z_2 = \sqrt{t}\ln t\int_t^\infty sw + \sqrt{t}\int^t sw\ln s,$$

$$\tilde{z}_1 = \sqrt{t}\ln t,\qquad \tilde{z}_2 = \sqrt{t}\int^t sw\ln s\ln\tfrac{t}{s}.$$

(**C**) $\int^{\infty} tw\ln t < +\infty$, $\int^{\infty} tw\ln^2 t = +\infty$. Then

$$z_1 = \sqrt{t}\int_t^{\infty} sw\ln\tfrac{s}{t}\,, \qquad\qquad z_2 = \sqrt{t}\,,$$

$$\tilde{z}_1 = \sqrt{t}\ln t\int_t^{\infty} sw\ln s + \sqrt{t}\int^{t} sw\ln^2 s\,, \qquad \tilde{z}_2 = \sqrt{t}\ln t\,.$$

(**D**) $\int^{\infty} tw\ln^2 t < +\infty$. Then

$$z_1 = \sqrt{t}\int_t^{\infty} sw\ln\tfrac{s}{t}\,, \qquad z_2 = \sqrt{t}\int_t^{\infty} sw\ln s\ln\tfrac{s}{t}\,,$$

$$\tilde{z}_1 = \sqrt{t}\,, \qquad \tilde{z}_2 = \sqrt{t}\ln t\,.$$

One may easily verify that in all the above cases the solutions z_1, z_2, $\tilde{z}_1$, $\tilde{z}_2$ form an ordered system of solutions of (4.3). The matrix $S = S_{ij} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in all the cases (A) – (D).

Equation (3.1) takes the form

$$-\left(\frac{h^2}{w(t)}v'\right)' + R_1(t)v = \lambda G(t)\,v\,, \tag{4.4}$$

where $G(t) := S_{i,l}W_{l,i}^2(t)/\big(W'(t)W_{l,i}(t) - W(t)W_{l,i}'(t)\big)$. By Remark 2.2(ii), if g and h are solutions of (4.3) such that $\{g,h\} = 0$, then (4.4) is transformed into the equation

$$\left(\frac{W^2(g,h)}{wh^2}\tilde{v}'\right)' + \lambda G\,\frac{W^2(g,h)}{h^4}\,\tilde{v} = 0\,, \tag{4.5}$$

where $W(g,h)$ is the Wronskian of g and h. Applying Lemma 2.2 we get the final criterion for property **BD** of ℓ_N in every single case (A) – (D).

For illustration, we give it here only for the case (A) but one may proceed quite similarly in the other cases (c.f. [7; Criterions (A) – (D)]).

Criterion A. *Let* $\int^{\infty} tw = +\infty$.

(i) Case (A1)[1]. *If*

$$\lim_{t\to\infty} \frac{\left(\int^{t} sw\right)^2}{t^2 w \int^{t} sw \ln \frac{t}{s}} =: \Psi < +\infty\,, \tag{4.6}$$

then the operator ℓ_N *given by* (4.1) *has property* **BD** *if and only if*

$$\lim_{t\to\infty} \frac{\int^{t} sw \int^{t} sw \ln^2 s - \left(\int^{t} sw \ln s\right)^2}{\int^{t} sw} \int_t^{\infty} \frac{s}{r} = 0\,. \tag{4.7}$$

(ii) Case (A2)[2]. *If*

$$\lim_{t\to\infty} \frac{\left(\int^{t} sw \ln^2 s\right)^2}{t^2 w \ln^3 t \int^{t} sw \ln s \ln \frac{t}{s}} =: \Omega < +\infty\,, \tag{4.8}$$

then the operator ℓ_N *given by* (4.1) *has property* **BD** *if and only if*

$$\lim_{t\to\infty} \frac{\int^{t} sw \int^{t} sw \ln^2 s - \left(\int^{t} sw \ln s\right)^2}{\int^{t} sw \ln^2 s} \int_t^{\infty} \frac{s \ln^2 s}{r} = 0\,. \tag{4.9}$$

Example 4.1.

(i) Let $w(t) = \frac{1}{t^2}$. Then $\Psi = 2$ and Criterion A(i) states that ℓ_N has property **BD** if and only if

$$\lim_{t\to\infty} \ln^3 t \int_t^{\infty} \frac{s}{r(s)}\, \mathrm{d}s = 0\,.$$

[1]We have $h = z_1 = \sqrt{t}$, i.e. $i = 1$, $l = 2$, $S_{i,l} = -1$ and take $g = z_2 = \sqrt{t}\ln t$ (we could have taken also $g = \tilde{z}_1$ but the choice of the function g has no effect on the finiteness of the limit (4.6)). Then $W = W(\tilde{z}_1, \tilde{z}_2) = \int^{t} sw \int^{t} sw \ln^2 s - \left(\int^{t} sw \ln s\right)^2$ and $W_{l,i} = W(\tilde{z}_1, z_1) = -\int^{t} sw$.

[2]We have $h = z_2 = \sqrt{t}\ln t$, i.e. $i = 2$, $l = 1$, $S_{i,l} = 1$ and take $g = z_1 = \sqrt{t}$ (we could have taken also $g = \tilde{z}_2$). Then $W = W(\tilde{z}_1, \tilde{z}_2)$ and $W_{l,i} = W(z_2, \tilde{z}_2) = \int^{t} sw \ln^2 s$.

(ii) Let $w(t) = \frac{1}{t \ln t}$. Then $\Psi = +\infty$ and $\Omega = 1$. Thus part (i) of Criterion A does not apply to this case. However, part (ii) states that ℓ_N has property **BD** if and only if

$$\lim_{t\to\infty} \frac{t}{\ln^2 t} \int_t^\infty \frac{s \ln^2 s}{r(s)}\, \mathrm{d}s = 0 .$$

(iii) By a direct computation one may verify applicability of Criterion A (resp. (B) – (D)) to weight functions of the form $w(t) = t^\alpha \ln^\beta t$ or $w(t) = t^\alpha \mathrm{e}^{\gamma t}$, $\alpha, \beta, \gamma \in \mathbb{R}$, or to any "sufficiently smooth" weight functions.

Appendix A. The reciprocal operator ℓ_R

Differentiating the products in L and L^* we get

$$\begin{aligned}
\ell_R(z) &= L\Big(\frac{1}{w} L^*(z)\Big) \\
&= \sum_{i=0}^{n} \frac{q_i}{p_n} \left[\frac{1}{w} \sum_{k=0}^{n} (-1)^k \Big(\frac{q_k}{p_n}\, z\Big)^{(k)}\right]^{(i)} \\
&= \sum_{i=0}^{n} \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{m=0}^{i} (-1)^k \tbinom{k}{j} \tbinom{i}{m} \frac{q_i}{p_n} \left[\frac{1}{w} \Big(\frac{q_k}{p_n}\Big)^{(k-j)}\right]^{(i-m)} z^{(j+m)} ,
\end{aligned} \tag{A.1}$$

defining $q_n := p_n$.

Lemma A.1. *Let f be an integrable function. Then*

$$\int^t \bigg(\int^{s_1} \cdots \bigg(\int^{s_k} f(\tau)\, \mathrm{d}\tau \bigg)\, \mathrm{d}s_k \cdots \bigg)\, \mathrm{d}s_1 = \frac{1}{k!} \int^t (t-s)^k f(s)\, \mathrm{d}s .$$

P r o o f . By induction, integrating by parts, we get the result. □

To find the explicit form of the coefficients r_k, $k = 0, \dots, n-1$, ($r_n = \frac{1}{w}$) we may proceed as follows — formally insert the polynomial functions $1, t, \dots, t^{n-1}$ into $\ell_R(z)$.

Lemma A.2. *For the coefficients r_k of ℓ_R we have*

$$r_0 = \ell_R(1) ,$$

$$r_k = \frac{(-1)^k}{k!(k-1)!} \int^t (t-s)^{k-1} \left[\ell_R(s^k) - \sum_{j=0}^{k-1} (-1)^j \frac{k!}{(k-j)!} \big(r_j s^{k-j}\big)^{(j)}\right] \mathrm{d}s$$

$$\textit{for} \quad k = 1, \dots, n-1 .$$

P r o o f. Clearly, for $z \equiv 1$ we have $\ell_R(1) = r_0$. The equality

$$\ell_R(z) - r_0 z = \sum_{i=2}^{n} (-1)^i \big(r_i z^{(i)}\big)^{(i)} - (r_1 z')'$$

implies that

$$\int^{t} \big[\ell_R(z) - r_0 z\big] \, \mathrm{d}s = \sum_{i=2}^{n} (-1)^i \big(r_i z^{(i)}\big)^{(i-1)} - r_1 z' .$$

Taking $z = t$ we get

$$r_1 = -\int^{t} \big[\ell_R(s) - r_0 s\big] \, \mathrm{d}s .$$

In the analogous way we proceed in the case of r_k, $k = 1, \ldots, n-1$.

$$\ell_R(z) - \sum_{j=0}^{k-1} (-1)^j \big(r_j z^{(j)}\big)^{(j)} = \sum_{i=k+1}^{n} (-1)^i \big(r_i z^{(i)}\big)^{(i)} + (-1)^k \big(r_k z^{(k)}\big)^{(k)} .$$

Integrating k-times (Lemma A.1) and inserting $z = t^k$ yields

$$(-1)^k k! \, r_k = \frac{1}{(k-1)!} \int^{t} (t-s)^{k-1} \Big[\ell_R(s^k) - \sum_{j=0}^{k-1} (-1)^j \frac{k!}{(k-j)!} \big(r_j s^{k-j}\big)^{(j)}\Big] \, \mathrm{d}s .$$

$\square$

In Lemma A.2 we use the operator ℓ_R itself but the explicit form of ℓ_R is already known, c.f. (A.1).

COROLLARY A.1. *For the fourth order operator ℓ_R $(n = 2)$ we have*

$$\ell_R(z) = (r_2 z'')'' - (r_1 z')' + r_0 z ,$$

where

$$r_2 = \frac{1}{w} ,$$

$$r_1 = \left(\frac{q_1}{w p_2}\right)' + 2\frac{1}{w}\left(\frac{q_1}{p_2}\right)' - 2\frac{q_0}{w p_2} + \frac{q_1^2}{w p_2^2} ,$$

$$r_0 = \left(\frac{q_0}{w p_2}\right)'' - \left[\frac{1}{w}\left(\frac{q_1}{p_2}\right)'\right]'' - \frac{q_1}{p_2}\left[\frac{1}{w}\left(\frac{q_1}{p_2}\right)'\right]'$$
$$+ \frac{q_0^2}{w p_2^2} + \frac{q_1}{p_2}\left(\frac{q_0}{w p_2}\right)' - \frac{q_0}{w p_2}\left(\frac{q_1}{p_2}\right)' ,$$

q_0, q_1 *are defined in the same way as in* (2.4).

Another method for finding $r_0, \dots, r_n$ is the following: put together the terms of (A.1) belonging correspondingly to $r_0 z$, $(r_1 z')'$, $\dots$, $\left(r_n z^{(n)}\right)^{(n)}$. However, this might cause technical difficulties for the higher order operators (computed for $n = 2$).

Appendix B. Proof of Theorem F

Proof. Equation (2.13) may be rewritten into LHS (1.4) with A given by (1.5), $B = \operatorname{diag}\{0, \dots, 0, r_n^{-1}\}$ and $C = -\operatorname{diag}\{r_0, \dots, r_{n-1}\}$. The result is then obtained via the transformation of Theorem B.

In [2] it was shown that there exist $n \times n$ matrices $H(t)$, $K(t)$ satisfying the assumptions of Theorem B such that (2.1) transforms LHS corresponding to equation (2.13) into a LHS corresponding to (2.16), i.e. $\tilde{A} = A$, $\tilde{B} = \operatorname{diag}\{0, \dots, R_n^{-1}\}$ and $\tilde{C}$ diagonal. Then $R_i = -\tilde{C}_{i+1,i+1}$, $i = 0, \dots, n-1$.

Particularly,

$$H_{i,j} = \tbinom{i-1}{j-1} h^{(i-j)}, \qquad i \ge j.$$

Then both H and H^{-1} are lower triangular and $(H^{-1})_{i,j} = \binom{i-1}{j-1} \left(\frac{1}{h}\right)^{(i-j)}$, $i \ge j$. The identity $\tilde{A} = A$ reads as

$$HA = -H' + AH + BK, \tag{B.1}$$

which necessarily implies

$$K_{n,j} = \tbinom{n}{j-1} r_n h^{(n-j+1)}, \qquad j = 1, \dots, n. \tag{B.2}$$

The entries $K_{i,j}$, $i \ge j$, (i.e. the entries below and on the diagonal) we determine in such a way that the matrix $D := K' + CH + A^T K + KA$ is upper triangular, i.e. $D_{i,j} = 0$ for $i \ge j$. This equality implies

$$K_{i-1,j} = -K'_{i,j} - K_{i,j-1} - C_{i,i} H_{i,j}, \qquad i \ge j,$$

where we define $K_{i,j} := 0$ for $i = 0$ or $j = 0$. Hence, by induction, for $i \ge j$

$$\begin{aligned} K_{i,j} &= (-1)^{n-i} \sum_{k-0}^{n-i} \tbinom{n-i}{k} K_{n,j-k}^{(n-i-k)} \\ &\quad + \sum_{k=0}^{n-i-1} (-1)^k \sum_{l=0}^{k} \tbinom{k}{l} \left(C_{i+k+1,i+k+1} H_{i+k+1,j-k+l}\right)^{(l)} \end{aligned} \tag{B.3}$$

and substituting from (B.2) we get

$$K_{i,j} = (-1)^{n-i} \sum_{k=0}^{n-i} \binom{n-i}{k} \binom{n}{j-k-1} \left(r_n h^{(n-j+k+1)}\right)^{(n-i-k)} + \sum_{k=0}^{n-i-1} (-1)^k \sum_{l=0}^{k} \binom{k}{l} \binom{i+k}{j-k+l-1} \left(r_{i+k} h^{(i-j+2k-l+1)}\right)^{(l)}, \qquad i \geq j. \tag{B.4}$$

We use the usual agreement that $\binom{n}{k} = 0$ if $k > n$ or $k < 0$.

The remaining elements of K (i.e. the entries above the diagonal) we define so as to satisfy $H^T K = K^T H$. Writing K in the form $K = \bar{K} + \tilde{K}$ where $\bar{K}$ is lower triangular (its entries are given by (B.4)) and $\tilde{K}$ is upper triangular with zeros on the diagonal, we have

$$0 = H^T K - K^T H = H^T(\bar{K} + \tilde{K}) - (\bar{K}^T + \tilde{K}^T)H. \tag{B.5}$$

The matrix $H^T \bar{K} - \bar{K}^T H$ is antisymmetric, hence it can be written in the form $H^T \bar{K} - \bar{K}^T H = U - U^T$, where U is lower triangular with zeros on the diagonal, i.e.

$$U_{i,j} = \begin{cases} (H^T \bar{K} - \bar{K}^T H)_{i,j} = \sum_{k=i}^{n} \left(H_{k,i} K_{k,j} - H_{k,j} K_{k,i}\right) & \text{if } i > j, \\ 0 & \text{if } i \leq j. \end{cases}$$

Consequently, (B.5) is satisfied if $\tilde{K} = H^{T-1} U^T$. So for $i < j$

$$K_{i,j} = \tilde{K}_{i,j} = \sum_{k=i}^{j-1} \binom{k-1}{i-1} \left(\tfrac{1}{h}\right)^{(k-i)} \sum_{l=j}^{n} \left[\binom{l-1}{j-1} h^{(l-j)} K_{l,k} - \binom{l-1}{k-1} h^{(l-k)} K_{l,j}\right]. \tag{B.6}$$

This completes the definition of K.

Finally, since (B.1) holds, we have

$$\tilde{C} = H^T(K' + CH + A^T K) + K^T HA = H^T(K' + CH + A^T K + KA) = H^T D.$$

Thus $\tilde{C}$ is upper triangular (it's a product of two such matrices). Since $\tilde{C}$ is also symmetric, it is diagonal. For the diagonal entries we have

$$\tilde{C}_{i,i} = H_{i,i}(K'_{i,i} + C_{i,i} H_{i,i} + K_{i-1,i} + K_{i,i-1}),$$

where $H_{i,i} = h$, $C_{i,i} = r_{i-1}$ and

$$K'_{i,i} \overset{(B.4)}{=} (-1)^{n-i} \sum_{k=0}^{n-i} \binom{n-i}{k} \binom{n}{i-k-1} \left(r_n h^{(n-i+k+1)}\right)^{(n-i-k+1)} + \sum_{k=0}^{n-i-1} (-1)^k \sum_{l=0}^{k} \binom{k}{l} \binom{i+k}{i-k+l-1} \left(r_{i+k} h^{(2k-l+1)}\right)^{(l+1)},$$

$$K_{i,i-1} \overset{(B.4)}{=} (-1)^{n-i} \sum_{k=0}^{n-i} \binom{n-i}{k} \binom{n}{i-k-2} \left(r_n h^{(n-i+k+2)}\right)^{(n-i-k)} + \sum_{k=0}^{n-i-1} (-1)^k \sum_{l=0}^{k} \binom{k}{l} \binom{i+k}{i-k+l-2} \left(r_{i+k} h^{(2k-l+2)}\right)^{(l)},$$

$$K_{i-1,i} \overset{(B.6)}{=} \frac{1}{h} \sum_{k=i}^{n} \left[\binom{k-1}{i-1} h^{(k-i)} K_{k,i-1} - \binom{k-1}{i-2} h^{(k-i+1)} K_{k,i} \right]$$

with $K_{k,i-1}$, $K_{k,i}$ given by (B.4).

An easy computation shows that

$$\tilde{B}_{i,j} = (H^{-1} B H^{T-1})_{i,j} = \begin{cases} (r_n h^2)^{-1} & \text{if } i = j = n, \\ 0 & \text{else.} \end{cases}$$

The proof is now complete. □

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