Dao Rong Tong

Torsion classes and torsion prime selectors of hl-groups

Mathematica Slovaca, Vol. 50 (2000), No. 1, 31--40

Persistent URL: http://dml.cz/dmlcz/136766

Math. Slovaca, **50** (2000), No. 1, 31–40

Mathematica Slovaca

TORSION CLASSES AND TORSION PRIME SELECTORS OF hl-GROUPS

DAO-RONG TON

(*Communicated by Stanislav Jakubec*)

ABSTRACT. In this paper we introduce two notions: A *torsion class* of hl-groups is a class closed under taking convex hl-subgroups, joins of convex hl-subgroups and hl-homomorphic images; a *torsion prime selector* of hl-groups is a function assigning to each hl-group G some subset $M(G)$ of $P(G)$. We show that there exists a complete lattice isomorphism from the family of torsion classes into the family of torsion prime selectors.

1. Introduction

M. Giraudet and F. Lucas introduced a new concept of half l-groups in [4]. The concept of half l-groups is a natural generalization of l-groups. For the definitions and standard results concerning l-groups, the reader is referred to [1], [2], [3], [5].

Let G be a group with unit e and a non-trivial ordered underlying set. Set

$$G{\uparrow} = \{g \in G \mid \ x \le y \implies gx \le gy \ \text{ for all } x, y \in G\},$$
$$G{\downarrow} = \{g \in G \mid \ x \le y \implies gx \ge gy \ \text{ for all } x, y \in G\}.$$

$G{\uparrow}$ is called the *increasing part* of G and $G{\downarrow}$ the *decreasing part* of G. G is called a *half l-group* (abbreviated: *hl-group*), if

(1) $x \le y$ implies $xg \le yg$ for all x, y and $g \in G$;
(2) $G = G{\uparrow} \cup G{\downarrow}$;
(3) $G{\uparrow}$ is an l-group.

For example, the set $M(\omega)$ of all monotonic permutations of a chain ω is an hl-group. Let $\mathcal{G}_1$ be the set of all hl-groups (and similarly for $\mathcal{G}_2$). Let G be an

1991 Mathematics Subject Classification: Primary 06F15.
Key words: hl-group, torsion class, torsion prime selector, lattice isomorphism.

The author wishes to express his appreciation to the University of Main in France for its hospitality during his visit in Le Mans when this paper was prepared.

hl-group and $G{\downarrow} \neq \emptyset$, then the index $(G, G{\uparrow}) = 2$, so $G{\uparrow}$ is normal in G. An element in $G{\uparrow}$ and an element in $G{\downarrow}$ are never comparable. $G{\uparrow}$ is isomorphic to $G{\downarrow}$ as a lattice. $G = G{\uparrow} \cup aG{\uparrow}$, where $a \in G{\downarrow}$ can be selected to be an element of order 2 ([4], [9]). Put $E(G) = \{x \in G \mid x^2 = e,\ x \neq e\}$.

A subgroup H of an hl-group G is said to be a *half l-subgroup* (abbreviated: *hl-subgroup*) if $H{\uparrow} = H \cap G{\uparrow}$ is an l-subgroup of $G{\uparrow}$. An hl-subgroup H of G is called *convex*, if $H{\uparrow}$ is convex in $G{\uparrow}$. A normal convex hl-subgroup of G is called an *hl-ideal* of G. $G{\uparrow}$ is an hl-ideal of G. We denote by $\mathcal{C}(G)$ the set of all convex hl-subgroups of G. Let $X \subseteq G$ and $a \in G$. We denote by $G(X)$ the convex hl-subgroup of G generated by X, which is the smallest convex hl-subgroup of G containing X, and $G(X, a)$ the convex hl-subgroup of G generated by $\{X, a\}$. Let H be an l-group and G an hl-group with $G{\uparrow} = H$; then G is called an *h-extension* of H.

A mapping ϕ from an hl-group G onto an hl-group G' is called an *hl-homomorphism*, if

(1) ϕ is a group homomorphism,
(2) $\phi|_{G{\uparrow}}$ is a lattice homomorphism of $G{\uparrow}$ onto $G'{\uparrow}$.

A 1−1 hl-homomorphism is called an *hl-isomorphism*. It is denoted by $G \simeq G'$. The join in a lattice L is denoted by $\vee_L$.

PROPOSITION 1.1. *Let G be an hl-group and $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{C}(G)$. Then $\bigcap_{\lambda\in\Lambda} G_\lambda$ is also a convex hl-subgroup of G; moreover, $\left(\bigcap_{\lambda\in\Lambda} G_\lambda\right){\uparrow} = \bigcap_{\lambda\in\Lambda} G_\lambda{\uparrow}$.*

The assertion of this proposition is obvious and we omit the proof.

Let G be an hl-group and $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{C}(G)$. By Proposition 1.1, we can define meets and joins in $\mathcal{C}(G)$ as follows:

$$\bigwedge_{\lambda\in\Lambda} G_\lambda = \bigcap_{\lambda\in\Lambda} G_\lambda,$$

$$\bigvee_{\lambda\in\Lambda} G_\lambda = \bigcap\Big\{K \in \mathcal{C}(G) \mid K \supseteq \bigcup_{\lambda\in\Lambda} G_\lambda\Big\}.$$

Thus, $\mathcal{C}(G)$ becomes a complete lattice. Let H be an l-group and $X \subseteq H$. We denote by $\langle X\rangle_H$ the convex l-subgroup of H generated by X.

PROPOSITION 1.2. *Let G be an hl-group and $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{C}(G)$, $G_\lambda = G_\lambda{\uparrow} \cup a_\lambda G_\lambda{\uparrow}$ with $a_\lambda \in E(G_\lambda)$ for each $\lambda \in \Lambda$. Then*

$$\Big(\bigvee_{\lambda\in\Lambda} G_\lambda\Big){\uparrow} = \Big\langle \bigcup_{\lambda\in\Lambda} G_\lambda{\uparrow} \cup \{a_\lambda a_\mu \mid \lambda, \mu \in \Lambda\}\Big\rangle_{G{\uparrow}} \tag{1.1}$$

and

$$\bigvee_{\lambda\in\Lambda} G_\lambda = \Big(\bigvee_{\lambda\in\Lambda} G_\lambda\Big)\!\uparrow \cup\, a_\lambda\Big(\bigvee_{\lambda\in\Lambda} G_\lambda\Big)\!\uparrow \qquad \textit{for any} \quad a_\lambda \in E(G_\lambda) \tag{1.2}$$

$$= \Big(\bigvee_{\lambda\in\Lambda} G_\lambda\Big)\!\uparrow \cup\, b\Big(\bigvee_{\lambda\in\Lambda} G_\lambda\Big)\!\uparrow \qquad \textit{for any} \quad b \in \bigcup_{\lambda\in\Lambda} a_\lambda G_\lambda\!\uparrow. \tag{1.3}$$

Proof. Put $H = \bigvee_{\lambda\in\Lambda} G_\lambda$. Let $C \in \mathcal{C}(G)$. Then $C\!\uparrow \supseteqq H\!\uparrow$ if and only if $C\!\uparrow \supseteqq \bigcup_{\lambda\in\Lambda} G_\lambda\!\uparrow \cup \Big(\bigcup_{\lambda,\mu\in\Lambda} a_\lambda G_\lambda\!\uparrow a_\mu G_\mu\!\uparrow\Big) = \bigcup_{\lambda\in\Lambda} G_\lambda\!\uparrow \cup \Big(\bigcup_{\lambda,\mu\in\Lambda} G_\lambda\!\uparrow a_\lambda a_\mu G_\mu\!\uparrow\Big)$, if and only if

$$cC\!\uparrow \supseteqq \Big\langle \bigcup_{\lambda\in\Lambda} G_\lambda\!\uparrow \cup \{a_\lambda a_\mu \mid \lambda,\mu\in\Lambda\}\Big\rangle_{G\uparrow}.$$

So we get (1.1). For any $\lambda,\mu \in \Lambda$,

$$a_\mu G_\mu\!\uparrow = a_\lambda a_\lambda a_\mu G_\mu\!\uparrow \subseteqq a_\lambda H\!\uparrow.$$

Hence for any $\lambda,\mu\in\Lambda$, $a_\mu H\!\uparrow = a_\lambda H\!\uparrow$. So we have (1.2) and (1.3). □

Corollary 1.3. *Let G be an hl-group and $\{G_\lambda \mid \lambda\in\Lambda\} \subseteqq \mathcal{C}(G)$, $G_\lambda = G_\lambda\!\uparrow \cup a_\lambda G_\lambda\!\uparrow$ with $a_\lambda \in E(G_\lambda)$ such that $G_\lambda\!\uparrow = H$ for any $\lambda\in\Lambda$. Then $\Big(\bigvee_{\lambda\in\Lambda} G_\lambda\Big)\!\uparrow = \bigvee_{\lambda\in\Lambda} G_\lambda\!\uparrow$ if and only if $\bigcap_{\lambda\in\Lambda} a_\lambda G_\lambda\!\uparrow \neq \emptyset$ if and only if $G_\lambda = G_\mu$ for any $\lambda,\mu\in\Lambda$.*

Proof. If $\Big(\bigvee_{\lambda\in\Lambda} G_\lambda\Big)\!\uparrow = \bigvee_{\lambda\in\Lambda} G_\lambda\!\uparrow$, then $a_\lambda, a_\mu \in \bigvee_{\lambda\in\Lambda} G_\lambda = H$ for any $\lambda,\mu\in\Lambda$ by (1.1). Since $a_\lambda \in G_\lambda\!\downarrow$, so $a_\mu \in G_\lambda\!\downarrow$. Hence $G_\mu = G_\mu\!\uparrow \cup a_\mu G_\mu\!\uparrow = H \cup a_\mu H = G_\lambda$ for any $\lambda,\mu\in\Lambda$. Hence $\bigcap_{\lambda\in\Lambda} a_\lambda G_\lambda \neq \emptyset$. Conversely, if there exists $a \in \bigcap_{\lambda\in\Lambda} a_\lambda G_\lambda\!\uparrow$, let $a' = a \vee a^{-1}$. Then $a' \in E(G_\lambda)$ and $G_\lambda = G_\lambda\!\uparrow \cup a' G_\lambda\!\uparrow = H \cup a'H$ for each $\lambda\in\Lambda$. It follows from (1.1) that $\Big(\bigvee_{\lambda\in\Lambda} G_\lambda\Big)\!\uparrow = \bigvee_{\lambda\in\Lambda} G_\lambda\!\uparrow = H$. □

2. Torsion classes of hl-groups

A family $\mathcal{R}$ of hl-groups is called a *torsion class* if it is closed under

(1) taking convex hl-subgroups,
(2) forming joins of convex hl-subgroups,
(3) taking hl-homomorphic images.

Let $\mathcal{R}$ be a torsion class of hl-groups, and G be an hl-group. Then there exists a largest convex hl-subgroup $\mathcal{R}(G)$ of G belonging to $\mathcal{R}$. $\mathcal{R}(G)$ is called a *torsion radical* of G. It is invariant under all hl-automorphisms of G, and in particular, it is an hl-ideal of G. The mapping $G \to \mathcal{R}(G)$ is called a torsion radical mapping. Let T denote the family of all torsion classes of hl-groups and T^l the complete lattice of all torsion classes of l-groups. The notion of torsion classes of hl-groups is a generalization of torsion classes of l-groups. Torsion classes of l-groups were studied by Martinez and torsion classes of hl-groups were investigated by M. Giraudet and J. Rachůnek [*Varieties of half lattice-ordered groups of monotonic permutations in chains*, Prepublication No 57, Paris 7CNRS LOGIQUE, 1996]. Let $\mathcal{R}$ be a family of hl-groups. Put

$$\mathcal{R}^l = \{H \in \mathcal{G}_2 \mid H = G\!\uparrow \text{ for some } G \in \mathcal{R}\}.$$

THEOREM 2.1. *Let $\mathcal{R}$ be a torsion class of hl-groups, and let G be an hl-group. Then*

(1) *$\mathcal{R}^l$ is a torsion class of l-groups,*

(2) *$\mathcal{R}^l(G\!\uparrow)$ has at most one h-extension in G belonging to $\mathcal{R}$,*

(3) *$\mathcal{R}(G)\!\uparrow = \mathcal{R}^l(G\!\uparrow)$.*

Proof.

(1) is clear, because $\mathcal{G}_2 \subseteqq \mathcal{G}_1$ and $G\!\uparrow \in \mathcal{C}(G)$ for any hl-group G.

(2) Let G_1 and G_2 be two hl-subgroups of G belonging to $\mathcal{R}$ such that $\mathcal{G}_1\!\uparrow = G_2\!\uparrow = \mathcal{R}^l(G\!\uparrow)$, $G_1\!\downarrow \neq \emptyset \neq G_2\!\downarrow$ and $G_1\!\downarrow \neq G_2\!\downarrow$. Then $G_1 \vee G_2 \in \mathcal{R}$. If there exists $a \in G_1\!\downarrow \cap G_2\!\downarrow$, then $G_1\!\downarrow = aG_1\!\uparrow = aG_2\!\uparrow = G_2\!\downarrow$, which is a contradiction. So $G_1\!\downarrow \cap G_2\!\downarrow = \emptyset$. Hence $(G_1 \vee G_2)\!\uparrow \supsetneqq \mathcal{R}^l(G\!\uparrow)$ by Corollary 1.3. But $(G_1 \vee G_2)\!\uparrow \in \mathcal{R}$, which is a contradiction.

(3) Since $\mathcal{R}(G)$ is the largest convex hl-subgroup of G belonging to $\mathcal{R}$, $\mathcal{R}(G) \supseteqq \mathcal{R}^l(G\!\uparrow)$ and so $\mathcal{R}(G)\!\uparrow = \mathcal{R}(G) \cap G\!\uparrow \supseteqq \mathcal{R}^l(G\!\uparrow)$. On the other hand, $\mathcal{R}(G) \in \mathcal{R}$ and $\mathcal{R}(G)\!\uparrow \in \mathcal{C}(\mathcal{R}(G))$ imply $\mathcal{R}(G)\!\uparrow \mathcal{R}^l(G\!\uparrow)$. □

Theorem 2.1 tells us that, for a torsion class $\mathcal{R}$ of hl-groups, the torsion radical $\mathcal{R}(G)$ of an hl-group G is uniquely determined by the torsion radical $\mathcal{R}^l(G\!\uparrow)$ of the increasing part $G\!\uparrow$ of G. This fact is very useful in what follows.

THEOREM 2.2. *Suppose that $\mathcal{R}$ is a torsion class of hl-groups and G is an hl-group. Then*

(I) *if $A \in \mathcal{C}(G)$, then $\mathcal{R}(A) = A \cap \mathcal{R}(G)$;*

(II) *if $\varphi\colon G \to H$ is a surjective hl-homomorphism, then $\varphi[\mathcal{R}(G)] \subseteq \mathcal{R}(H)$.*

Conversely, any mapping ϕ associating to each hl-group G an hl-ideal and satisfying properties (I) and (II) always defines a unique torsion class $\mathcal{R}$ of hl-groups such that $\mathcal{R}(G) = \phi(G)$.

P r o o f . By the above Theorem 2.1(3) and [7; Proposition 1.1] for any $A \in \mathcal{C}(G)$ we have

$$\mathcal{R}(A)\uparrow = \mathcal{R}^l(A\uparrow) = A\uparrow \cap \mathcal{R}^l(G\uparrow) = A\uparrow \cap \mathcal{R}(G)\uparrow = \big(A \cap \mathcal{R}(G)\big)\uparrow .$$

So $\mathcal{R}(A)$ and $A \cap \mathcal{R}(G)$ are all h-extensions of $\mathcal{R}^l(A\uparrow)$, and Theorem 2.1(2) implies that $\mathcal{R}(A) = A \cap \mathcal{R}(G)$.

If $\varphi\colon G \to H$ is an onto hl-homomorphism, then $\mathcal{R}(G) \in \mathcal{R}$ and so $\varphi[\mathcal{R}(G)] \in \mathcal{R}$. Hence $\varphi[\mathcal{R}(G)] \subseteqq R(H)$, because $\mathcal{R}(H)$ is the largest convex hl-subgroup belonging to $\mathcal{R}$.

Conversely, suppose that the mapping ϕ satisfies (I) and (II). Let $\mathcal{R} = \{G \in \mathcal{G}_2 \mid \phi(G) = G\}$. It is easy to show that $\mathcal{R}$ is a torsion class of hl-groups. For each hl-group G, $\phi(\phi(G)) = \phi(G)$ implies $\phi(G) \in \mathcal{R}$ and $\phi(G) \subseteqq \mathcal{R}(G)$. On the other hand, $\mathcal{R}(G) = \phi(\mathcal{R}(G)) = \mathcal{R}(G) \cap \phi(G)$. Hence $\mathcal{R}(G) = \phi(G)$.

Suppose that $\{\mathcal{R}_\lambda \mid \lambda \in \Lambda\} \subseteqq T$. Since the intersection of a family of torsion classes of hl-groups is also a torsion class, we can define

$$\bigwedge_{\lambda\in\Lambda} \mathcal{R}_\lambda = \bigcap_{\lambda\in\Lambda} \mathcal{R}_\lambda ,$$
$$\bigvee_{\lambda\in\Lambda} \mathcal{R}_\lambda = \bigcap\{\mathcal{R} \in T \mid \mathcal{R} \supseteqq \mathcal{R}_\lambda \text{ for all } \lambda \in \Lambda\} .$$

Thus, T becomes a complete lattice and we have

$$\Big(\bigvee\nolimits_{\mathrm{T}\,\lambda\in\Lambda} \mathcal{R}_\lambda\Big)^l = \bigcap\{\mathcal{R}^l \in T^l \mid \mathcal{R}^l \supseteqq \mathcal{R}\} = \bigvee\nolimits_{\mathrm{T}^l\,\lambda\in\Lambda} \mathcal{R}^l_\lambda .$$

□

THEOREM 2.3. *If $\{\mathcal{U}_\lambda \mid \lambda \in \Lambda\} \subseteqq T$. Then for any hl-group G*

$$\Big(\bigvee_{\lambda\in\Lambda} \mathcal{U}_\lambda\Big)(G) = \bigvee_{\lambda\in\Lambda} \mathcal{U}_\lambda(G) . \tag{2.2}$$

The proof is similar to that used in [7].

3. Torsion prime selectors of hl-groups

The prime subgroups are the most important subgroups of an l-group in the theory of l-groups. All representation theorems and most structure results come from properties of prime subgroups. So we want to define a similar concept in an hl-group. Let L be a lattice. An element $a \in L$ is called *meet irreducible*, if $a = \bigwedge_{\lambda\in\Lambda} a_\lambda$ implies $a = a_\lambda$ for some $\lambda \in \Lambda$; a is called *finitely meet irreducible*, if $a = \bigwedge_{i=1}^{n} a_i$ implies $a = a_k$ for some k $(1 \le k \le n)$.

A convex hl-subgroup P of an hl-group G is *prime*, if whenever $e \leq a$, $e \leq b$ and $a \vee b \in P$, then either $a \in P$ or $b \in P$. Let $P(G)$ be the set of all prime subgroups of G.

THEOREM 3.1. *Let P be a convex hl-subgroup of an hl-group G. Then the following conditions are equivalent:*

(1) *P is prime,*
(2) *$P\uparrow$ is prime in $G\uparrow$ as an l-group,*
(3) *if $g \wedge h = e$, then $g \in P$ or $h \in P$,*
(4) *if $g, h \in G^+ \setminus P$, then $g \wedge h \succ e$,*
(5) *$\{A \in \mathcal{C}(G) \mid A \supseteqq P\}$ is a chain,*
(6) *P is finitely meet irreducible in $\mathcal{C}(G)$,*
(7) *$g, h \in G^+ \setminus P$ implies $g \wedge h \in G^+ \setminus P$.*

Proof.

(1) $\iff$ (2) is evident.

It is clear that (1) $\Longrightarrow$ (3) $\Longrightarrow$ (4).

Now suppose that (4) is valid and $A, B \in \mathcal{C}(G)$, $A \supseteqq P$ and $B \supseteqq P$. If $A\uparrow$ and $B\uparrow$ are incomparable, then there exist $e \prec a \in A\uparrow \setminus B\uparrow$ and $e \prec b \in B\uparrow \setminus A\uparrow$. Then $a = a'(a \wedge b)$ and $b = b'(a \wedge b)$, where $e \prec a' \in G^+ \setminus P$ and $e \prec b \in G^+ \setminus P$ and $a' \wedge b' = e$, which is absurd. If $A\uparrow \subseteqq B\uparrow$ and $A\downarrow \subseteqq B\downarrow$, then $A \subseteqq B$. If $A\uparrow \subseteqq B\uparrow$ and $A\downarrow \supseteqq B\downarrow$, let $A\downarrow = fA\uparrow$ with $f \in A\downarrow$ and $B\downarrow = gB\uparrow$ with $g \in E(B) \subseteqq A\downarrow$. Then $A\downarrow = B\downarrow = gB\uparrow$. This implies that $A\uparrow \supseteqq B\uparrow$. Hence $A\uparrow = B\uparrow$ and $A \supseteqq B$.

(5) $\Longrightarrow$ (6) is also clear.

(6) $\Longrightarrow$ (7) is shown by the fact that $P \subseteqq G(P,g) \cap G(P,h) = [P \vee G(g)] \cap [P \vee G(h)] = P \vee G(g \wedge h) = G(P, g \wedge h)$.

For (7) $\Longrightarrow$ (1), if $e \prec a \wedge b \in P$, then clearly $a \in P$ or $b \in P$. □

Now we shall give a special kind of prime subgroups for an hl-group. Let G be an hl-group and $e \neq g \in G$. By Zorn's Lemma there exists a maximal convex hl-subgroup G_g of G not containing g. G_g is called a *value* of g and is also called a *regular subgroup* of G. The convex hl-subgroup $G(G_g, g)$ generated by $\{G_g, g\}$ is a cover of G_g. As in [1; Theorem 1.2.8] we can prove that a convex hl-subgroup P of an hl-group G is meet irreducible in $\mathcal{C}(G)$ if and only if P is regular. The proof of the following lemma is similar to that for [1; Theorem 1.2.13].

LEMMA 3.2. *Let G be an hl-group and $H \in \mathcal{C}(G)$. Then $\rho\colon P \to P' = P \cap H$ is a 1–1 correspondence from $\{P \in P(G) \mid H \not\subseteq P\}$ onto $P(H)$.*

A function M assigning to each hl-group G a subset $M(G)$ of $P(G)$ is called a *torsion prime selector* of hl-groups if the following is true:

(1) if $A \in \mathcal{C}(G)$ and $P \in P(G)$, then

$$M(A) = \{P \cap A \mid P \in M(G) \text{ and } A \not\subset P\},$$

(2) if $\varphi\colon G \to H$ is an onto hl-homomorphism, then

$$M(H) \supseteqq \{\varphi(P) \mid P \in M(G) \text{ and } P \supseteqq \operatorname{Ker}(\varphi)\}.$$

Now let M be a torsion prime selector of hl-groups. Set

$$\mathbf{R}(M) = \{G \in \mathcal{G}_1 \mid M(G) = P(G)\}.$$

THEOREM 3.3. *For each torsion prime selector M of hl-groups, $\mathbf{R}(M)$ is a torsion class of hl-groups.*

The proof is similar to that for l-groups.

Let $\mathcal{R}$ be a torsion class of hl-groups. We define a function

$$M(\mathcal{R})\colon G \to \{H \in P(G) \mid \mathcal{R}(G) \not\subset H\}.$$

THEOREM 3.4. *For each torsion class $\mathcal{R}$ of hl-groups, $M(\mathcal{R})$ is a torsion prime selector of hl-groups; moreover, for any hl-group G we have $G \in \mathcal{R}$ if and only if $M(\mathcal{R})(G) = P(G)$.*

The proof is analogous to that for l-groups.

4. Connection between torsion classes and torsion prime selectors

Let M and M^* be two torsion prime selectors of hl-groups. We define $M < M^*$ if $M(G) \subseteqq M^*(G)$ for any hl-group G. Let $\{M_i \mid i \in I\}$ be a family of torsion prime selectors of hl-groups. We define $M_1(G) = \bigcap_{i\in I} M_i(G)$ and $M_2(G) = \bigcup_{i\in I} M_i(G)$ for any hl-group G.

THEOREM 4.1. *M_1 and M_2 are all torsion prime selectors of hl-groups.*

P r o o f . We prove that M_1 and M_2 satisfy conditions (1) and (2).

(1) Let G be an hl-group and let $A \in \mathcal{C}(G)$. If $Q \in M_1(A) = \bigcap_{i\in I} M_i(A)$, then for each $i \in I$ we have $Q \in M_i(A)$, and there exists $Q_i' \in M_i(G)$ such that $A \not\subset Q_i'$ and $Q = Q_i' \cap A$. So $Q_i'\!\uparrow \in P(G\!\uparrow)$ for each $i \in I$. But

$$Q_i'\!\uparrow \cap A\!\uparrow = (Q_i' \cap A)\!\uparrow = Q\!\uparrow = (Q_j' \cap A)\!\uparrow = Q_j'\!\uparrow \cap A\!\uparrow.$$

So $Q'_i\uparrow = Q'_j\uparrow$ for any $i \neq j \in I$ by [1; Theorem 1.2.13]. Hence $Q'_i = Q'_i\uparrow \cup aQ'_i\uparrow = Q'_j\uparrow \cup aQ'_j\uparrow$ for any $i \neq j$, where $a \in Q\downarrow$. Let $Q' = Q'_i$ for any $i \in I$. Then $Q' \in \bigcap_{i\in I} M_i(G) = M_1(G)$, and so $Q \in \{P \cap A \mid P \in M_1(G) \text{ and } A \not\subseteq P\}$.

Conversely, it is clear that $\{P \cap A \mid P \in M_1(G) \text{ and } A \not\subseteq P\} \subseteqq M_1(A)$. Therefore

$$M_1(A) = \{P \cap A \mid P \in M_1(G) \text{ and } A \not\subseteq P\}.$$

We have proved that M_1 satisfies the condition (1).

(2) Suppose that φ is an hl-homomorphism of an hl-group G onto an hl-group H. Since each M_i is a torsion prime selector of hl-groups,

$$M_i(H) \supseteqq \{\varphi(P) \mid P \in M_i(G) \text{ and } P \supseteqq \operatorname{Ker}(\varphi)\}$$

for each $i \in I$, and so

$$M_i(H) \supseteqq \Big\{\varphi(P) \mid P \in \bigcap_{i\in I} M_i(G) \text{ and } P \supseteqq \operatorname{Ker}(\varphi)\Big\}$$

for each $i \in I$. Hence

$$M_1(H) = \bigcap_{i\in I} M_i(H) \supseteqq \{\varphi(P) \mid P \in M_1(G) \text{ and } P \supseteqq \operatorname{Ker}(\varphi)\}.$$

We have proved that M_1 satisfies the condition (2).

We can prove that M_2 satisfies the conditions (1) and (2) similarly. □

Now we define

$$M_1 = \bigwedge_{i\in I} M_i \qquad \text{and} \qquad M_2 = \bigvee_{i\in I} M_i .$$

Thus, the set S of all torsion prime selectors of hl-groups is a complete lattice. And we have the mappings

$$\mathbf{R}\colon S \to T \qquad \text{and} \qquad \mathbf{M}\colon T \to S .$$

A mapping φ from a lattice L_1 into a lattice L_2 is called a *complete lattice homomorphism* if, whenever $\bigvee_{\alpha\in A} a_\alpha$ and $\bigwedge_{\beta\in B} b_\beta$ exist in L_1, $\varphi\Big(\bigvee_{\alpha\in A} a_\alpha\Big) = \bigvee_{\alpha\in A} \varphi(a_\alpha)$ and $\varphi\Big(\bigwedge_{\beta\in B} b_\beta\Big) = \bigwedge_{\beta\in B} \varphi(b_\beta)$. A 1–1 complete lattice homomorphism is called a *complete lattice isomorphism.*

THEOREM 4.2. *Let U be a torsion class of hl-groups. Then* $\mathbf{R}(\mathbf{M}(U)) = U$.

Proof. By Theorem 3.4, $G \in U$ if and only if $\mathbf{M}(U(G)) = P(G)$, that is, $G \in U$ if and only if $G \in \mathbf{R}(\mathbf{M}(U))$. □

By Theorem 4.2, we see that $\mathbf{RM} = 1_T$, where 1_T is the identity mapping on T. So $\mathbf{R}$ is onto and M is 1–1.

THEOREM 4.3. *M is a complete lattice isomorphism of T into S.*

Proof. Suppose that $\{\mathcal{R}_\lambda \mid \lambda \in \Lambda\} \subseteqq T$. It is clear that for any hl-group G and $H \in P(G)$, $\bigvee_{\lambda\in\Lambda} \mathcal{R}_\lambda \not\subseteq H$ if and only if $\mathcal{R}_\lambda(G) \not\subseteq H$ for some $\lambda \in \Lambda$. By Theorem 2.3 we have

$$\Big(\bigvee_{\lambda\in\Lambda} \mathcal{R}_\lambda\Big)(G) = \bigvee_{\lambda\in\Lambda} \mathcal{R}_\lambda(G).$$

Hence $\Big\{H \in P(G) \mid \bigvee_{\lambda\in\Lambda} \mathcal{R}_\lambda(G) \not\subseteq H\Big\} = \bigcup_{\lambda\in\Lambda} \{H \in P(G) \mid \mathcal{R}_\lambda(G) \not\subseteq H\}$. That is,

$$M\Big(\bigvee_{\lambda\in\Lambda} \mathcal{R}_\lambda\Big)(G) = \bigcup_{\lambda\in\Lambda} M(\mathcal{R}_\lambda)(G)$$

for any hl-group G. So

$$M\Big(\bigvee_{\lambda\in\Lambda} \mathcal{R}_\lambda\Big) = \bigvee_{\lambda\in\Lambda} M(\mathcal{R}_\lambda)$$

and M preserves arbitrary joins.

Now consider meets. Let $\{\mathcal{R}_\lambda \mid \lambda \in \Lambda\} \subseteqq T$. Assume that $H \in P(G)$. If $\bigcap_{\lambda\in\Lambda} M(\mathcal{R}_\lambda)(G) \not\subseteq H$, then $M(\mathcal{R}_\lambda)(G) \not\subseteq H$ for $\lambda \in \Lambda$. Conversely, if $M(\mathcal{R}_\lambda)(G) \not\subseteq H$ for all $\lambda \in \Lambda$, then $\bigcap_{\lambda\in\Lambda} M(\mathcal{R}_\lambda)(G) \not\subseteq H$ by the meet irreducibility of regular subgroups in $\mathcal{C}(G)$. Hence $M\Big(\bigwedge_{\lambda\in\Lambda} \mathcal{R}_\lambda\Big)(G) = \bigcap_{\lambda\in\Lambda} M\big(\mathcal{R}_\lambda(G)\big)$ for any hl-group G. That means

$$M\Big(\bigwedge_{\lambda\in\Lambda} \mathcal{R}_\lambda\Big) = \bigwedge_{\lambda\in\Lambda} M(\mathcal{R}_\lambda),$$

and M preserves any meets. □

Note that Theorem 4.3 generalizes some results in [6], [8].

REFERENCES

[1] ANDERSON, M.—FEIL, T.: *Lattice-Ordered Groups (An Introduction)*, D. Reidel Publishing Company, Dordrecht, 1988.

[2] CONRAD, P.: *Lattice-Ordered Groups*, Tulane University, Tulane, 1970.

[3] DARNEL, M.: *Theory of Lattice-Ordered Groups*, Marcel Dekker, Inc., New York, 1995.

[4] GIRAUDET, M. LUCAS, F.: *Groupes à moité ordonnés*, Fund. Math. **139** (1991), 75 89.

[5] GLASS, A. M. W. HOLLAND, W. C.: *Lattice-Ordered Groups (Advances and Techniques)*, Kluwer Academic Publishers, Dordrecht, 1989.

[6] JAKUBÍK, J.: *Prime selectors and torsion classes of lattice ordered groups*, Czechoslovak Math. J. **31(106)** (1981), 325 337.

[7] MARTINEZ, J.: *Torsion theory for lattice ordered groups*, Czechoslovak Math. J. **25(100)** (1975), 284–299.

[8] MARTINEZ, J.: *Is the lattice of torsion classes algebraic?*, Proc. Amer. Math. Soc. **63** (1977), 9–14.

[9] SCOTT, T. J.: *Monotonic permutation of chains*, Pacific J. Math. **55** (1974), 583 594.

Received January 27, 1997

29-305, 3 Xikang Road
Nanjing, 210024
P.R. CHINA
E-mail: dao-rong@public1.ptt.js.cn