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SINGULARITIES ON FLAG SPACES

Miloš Božek* — Konrad Drechsler†**

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ABSTRACT. In the paper the existence of singular points on a generalized flag space is proved if the corresponding partial order contains a crown with an additional property. For this sake the equations for the local description of generalized flag spaces in Grassmann coordinates are given. It is also shown that every generalized flag space whose partially ordered set does not contain crowns is smooth.

The classical concept of flag spaces was generalized in several manners using a partially ordered set. One possibility is to join the elements of a system of flags according to the relations of the order, see e.g. [G], [L]. Our idea was to replace the determining linear order by a partial one ([BD1]). Such new flag spaces are also algebraic varieties in a natural way, but they can contain singularities in contrast to the classical ones. Our aim is to characterize when it happens, only by properties of the partial order.

An interesting property of posets is their dismantlability ([DR]). We have already remarked in [BD1; Theorem 3], that flag spaces corresponding to dismantlable posets are smooth. We shall prove it in detail in Section 4 of the present paper. For this purpose we extend our considerations from [BD1] by constructing regular algebraic local parametrizations of those flag spaces adapted to the fibrations which are induced by doubly irreducible elements of the corresponding poset.

Duffus and Rival [DR] proved that a partially ordered set is dismantlable if and only if it contains no crown. This connection enables us to find singularities on flag spaces in Section 5. Moreover, we are sure that the following conjecture holds: A flag space has singular points if and only if its partially ordered set contains a crown. But we are not able to prove it without using some additional condition (Theorem 5.1). Nevertheless, we computed many examples in which that condition is not fulfilled but the flag space is singular, and we have no counterexample. So it could be a good conjecture.

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1. Generalized flag spaces

Let $(M, \sqsubset)$ be a finite partially ordered set, let $(\hat{M}, \sqsubset)$ be its closure by an additional lower and upper bound $\mathbf{0}$ and $\mathbf{1}$ and let

$$\delta\colon (\hat{M}, \sqsubset) \to (\mathbb{N}, <) \tag{1}$$

be a strictly order-preserving mapping[1] into the naturally ordered set $\mathbb{N}$ of non-negative integers such that $\delta\mathbf{0} = 0$ and $n = \delta\mathbf{1}$.

Then we call $\tau = (\hat{M}, \sqsubset, \delta)$ a *type*. Let K^n be the n-dimensional vector space over a field K and let $(L, \subset)$ be the lattice of its subspaces.

DEFINITION 1.1. The set of all order-preserving mappings γ from τ into $(L, \subset)$ such that δm is equal to the dimension of the subspace γm we call a *flag space* $F(\tau) = F(K, \tau)$ over K of the type $\tau = (\hat{M}, \sqsubset, \delta)$. Its elements are called *flags in* K^n.

An algebraic set F_τ corresponds to a flag space $F(\tau)$ in the product

$$G(M, \delta) = \prod_{m \in M} G^m(\delta m, n) \tag{2}$$

of *Grassmann* manifolds $G^m(\delta m, n)$ of δm-subspaces of the vector space K^n in a natural way: the equations defining F_τ are given by the pairs coming from the relation $\sqsubset$. We shall not make any distinction between F_τ and $F(\tau)$.

Let a type $\omega = (\hat{M}, \prec, \delta)$ be a refinement of $\tau = (\hat{M}, \sqsubset, \delta)$, which means that $\sqsubset$ is a subset of $\prec$. $F(\omega)$ can be considered as an algebraic subset of $F(\tau)$.

DEFINITION 1.2. We call ω a *linearization* of τ on a subset $C \subseteq M$ if the restriction $\prec_C$ of $\prec$ to $C \times C$ is a linear (total) order on C. A flag $\gamma \in F(\tau)$ is said to be *linear on* C if $\gamma \in F(\omega)$ where ω is a linearization of τ on C, and it is said to be *linear* if it is linear on M.

2. Order theoretic preliminaries

Let $-\!\sqsubset$ denote the set of all pairs of neighbours of the partially ordered set $(M, \sqsubset)$. Since M is finite, $\sqsubset$ is generated by the pairs of $-\!\sqsubset$ as the transitive closure. Let $\mathbb{Z}/s$ be the (additive factor) group of integers modulo s.

[1]In [BD1] we demanded only $(\hat{M}, \sqsubseteq) \to (\mathbb{N}, \leq)$ to be order-preserving. In [BD3] we have called the type defined in the present paper a reduced type and we have remarked there that to every flag space there exists a flag space with reduced type that is naturally equivalent to the given one. All assertions in the present paper hold for a reduced type only.

DEFINITION 2.1. Let $s \in \mathbb{N}$ and $s \geq 4$. A map $\zeta\colon \mathbb{Z}/s \to (M, \sqsubset)$ is said to be a *cycle in* $(M, \sqsubset)$ if either $\zeta i \mathrel{-\!\!\sqsubset} \zeta(i+1)$ or $\zeta(i+1) \mathrel{-\!\!\sqsubset} \zeta i$ for all $i \in \mathbb{Z}/s$.

Let $\max\zeta$ or $\min\zeta$ denote the set of all maxima or minima of ζ respectively, and let $\operatorname{ex}\zeta = \max\zeta \cup \min\zeta$. Finally, let $r_\zeta = \#(\max\zeta) = \#(\min\zeta)$ the cardinality.

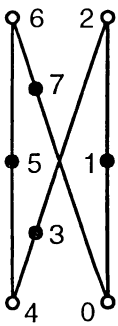
A cycle ζ determines a partial order $\sqsubset^\zeta$ on its image $\operatorname{im}\zeta$ transitively generated by the pairs of neighboured values of ζ. $\sqsubset^\zeta$ does not need to coincide with the restriction $\sqsubset_{\operatorname{im}\zeta}$.

DEFINITION 2.2. A cycle ζ in $(M, \sqsubset)$ is said to be a *crown cycle contained in* $(M, \sqsubset)$ if it fulfils the following conditions:

(i) $r_\zeta \geq 2$,
(ii) ζ is injective,
(iii) $\sqsubset^\zeta = \sqsubset_{\operatorname{im}\zeta}$.

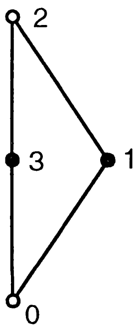
The partially ordered set $(\operatorname{ex}\zeta, \sqsubset_{\operatorname{ex}\zeta})$ is then called a *crown of order* $2r_\zeta$ *contained in* $(M, \sqsubset)$.

EXAMPLES 2.3. We shall describe the partially ordered set $(M, \sqsubset)$ by the *Hasse* diagram. By $\circ$ we denote the elements of $\operatorname{ex}\zeta$ and by $\bullet$ the remaining ones.



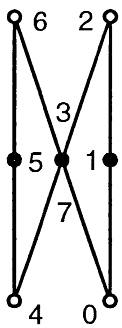
$s = 8$
$r_\zeta = 2$

FIGURE 1.



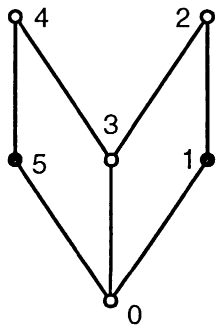
$s = 4$
$r_\zeta = 1$

FIGURE 2.



$s = 8$
$r_\zeta = 2$

FIGURE 3.



$s = 6$
$r_\zeta = 2$

FIGURE 4.

Figure 1 illustrates a crown, the other figures do not describe crowns contained in $(M, \sqsubset)$.

An element m in M is said to be *doubly irreducible in* $(M, \sqsubset)$ if m has at most one lower and at most one upper neighbour in $(M, \sqsubset)$.[2] Let $\mathcal{D}(M, \sqsubset)$ denote the set of all doubly irreducible elements in $(M, \sqsubset)$.

DEFINITION 2.4. ([DR]) $(M, \sqsubset)$ is called *dismantlable* (*by doubly irreducible elements*) if the elements of M can be labelled so that $m^\nu \in \mathcal{D}\big(M \setminus \{m^1, \dots \dots, m^{\nu-1}\}, \sqsubset\big)$ for every $\nu = 1, \dots, \#M$.

THEOREM 2.5. ([DR]) *A finite partially ordered set is dismantlable if and only if it contains no crowns.*

For the proof see [DR] and [KR]. Note that the condition for a crown to be contained in $(M, \sqsubset)$ is not so strong there as in our Definition 2.2. But we need it for the purpose of the proof of our Theorem 5.1.

3. Coordinate description of flag spaces

In the rest of the paper the field K is assumed to be infinite.

Recall that any basis $e_1, \dots, e_n$ of the vector space K^n induces local coordinates $g \longleftrightarrow (g_{\nu\mu} : \nu = 1, \dots, k;\ \mu = k+1, \dots, n)$ in the *Grassmann* manifold $G(k,n)$. They are defined on the *Zariski*-open set $\tilde{G}(k,n) \subset G(k,n)$ consisting of all $g \in G(k,n)$ which are transversal to the subspace of K^n spanned by the last $n-k$ vectors of the basis. The coordinates $g_{\nu\mu}$ are determined by the condition

$$e_\nu + \sum_{\mu=k+1}^{n} g_{\nu\mu} e_\mu \in g\,. \tag{3}$$

Using this coordinates we can identify $\tilde{G}(k,n)$ with $K^{k(n-k)}$.

The affine *Grassmann* coordinates mentioned above induce in a natural way local coordinates $(m_{\nu\mu} : m \in M;\ \nu = 1, \dots, \delta m;\ \mu = \delta m + 1, \dots, n)$ in $G(M, \delta)$ defined on the *Zariski*-open set

$$\tilde{G}(M, \delta) = \prod_{m \in M} \tilde{G}^m(\delta m, n)$$

for a flag space $F(\tau)$. Thus, varying the basis in K^n we get a covering of the flag space by *Zariski*-open subsets $\tilde{F}(\tau) = F(\tau) \cap \tilde{G}(M, \delta)$ which are imbedded in the corresponding affine space $\tilde{G}(M, \delta) = K^d$, where $d = \sum\limits_{m \in M} \delta m (n - \delta m)$.

[2]This notion comes from lattice theory: m is irreducible with respect to both lattice operations.

To each relation $m' \sqsubset m$ of $(M, \sqsubset)$ there belongs the inclusion $\gamma m' \subset \gamma m$ of subspaces of K^n for all $\gamma \in F(\tau)$. Thus, using the relations (3), we get a set of polynomials

$$f^{\nu\mu}_{m'm}(\gamma) = m_{\nu\mu} - m'_{\nu\mu} + \sum_{\lambda=\delta m'+1}^{\delta m} m'_{\nu\lambda} m_{\lambda\mu}\,, \qquad \nu = 1, \ldots, \delta m'\,; \qquad \mu = \delta m + 1, \ldots, n\,, \tag{4}$$

in the ideal $I(\tau)$ corresponding to $F(\tau)$ of the ring

$$R = K[m_{\nu\mu} : \ \nu = 1, \ldots, \delta m; \ \ \mu = \delta m + 1, \ldots, n; \ \ m \in M]$$

of coordinates on $G(M, \delta)$. $I(\tau)$ is generated by the polynomials $f^{\nu\mu}_{m'm}(\gamma)$ for $m' \mathrel{-\!\!\sqsubset} m$ only.

Let $m'' \sqsubset m' \sqsubset m$. It is easy to check that

$$f^{\nu\mu}_{m''m}(\gamma) = f^{\nu\mu}_{m''m'}(\gamma) + f^{\nu\mu}_{m'm}(\gamma) + \sum_{\lambda=\delta m''+1}^{\delta m'} m''_{\nu\lambda} f^{\lambda\mu}_{m'm}(\gamma) - \sum_{\lambda=\delta m'+1}^{\delta m} f^{\nu\lambda}_{m''m'}(\gamma) m_{\lambda\mu}$$
$$\text{for all} \quad \nu = 1, \ldots, \delta m''\,; \ \ \mu = \delta m + 1, \ldots, n \qquad \text{and for all} \quad \gamma \in F(\tau)\,. \tag{5}$$

We also use the notation

$$f^{\nu\mu}_{mm'}(\gamma) = -f^{\nu\mu}_{m'm}(\gamma)\,. \tag{6}$$

4. Smooth flag spaces

Let $\bar{m} \in M$ and let $\pi \colon F(\hat{M}, \sqsubset, \delta) \to F\big(\hat{M} \setminus \{\bar{m}\}, \sqsubset, \delta\big)$ be the canonical projection.

LEMMA 4.1. *If $\bar{m}$ is doubly irreducible in $(M, \sqsubset)$ and if u and o are the neighbours of $\bar{m}$ in $(\hat{M}, \sqsubset)$, then to every flag $\gamma^0 \in F(\hat{M}, \sqsubset, \delta)$ there exists a Zariski-neighbourhood $\tilde{F}$ and a Zariski-open set $\tilde{G} \subseteq G(\delta\bar{m} - \delta u, \delta o - \delta u)$ so that $\pi\tilde{F}$ is Zariski-open and $\tilde{F}$ is algebraically isomorphic to $\pi\tilde{F} \times \tilde{G}$.*

COROLLARY 4.2. *If $\bar{m}$ is doubly irreducible in $(M, \sqsubset)$ and if $F\big(\hat{M} \setminus \{\bar{m}\}, \sqsubset, \delta\big)$ is smooth, then $F(\hat{M}, \sqsubset, \delta)$ is smooth, too.*

This yields the following:

THEOREM 4.3. *Let K be infinite. If $(M, \sqsubset)$ is dismantlable, then the flag space $F(\hat{M}, \sqsubset, \delta)$ is smooth for all δ.*

Remark 4.4. With some more effort it can be shown that under the assumptions of Lemma 4.1, π is an algebraic locally trivial fibration with the fibre

$G(\delta m-\delta u, \delta o-\delta u)$. This is a strengthening of Theorem 1 in [BD1] from which we could obtain the above Corollary 4.2 and the Theorem 4.3 as a consequence.

Proof of Lemma 4.1. Let $\gamma^0 \in F(\hat{M},\sqsubset,\delta)$ be arbitrary. Choose a basis in K^n such that $\gamma^0 \in \tilde{F}(\hat{M},\sqsubset,\delta)$ and put $\tilde{F}=\tilde{F}(\hat{M},\sqsubset,\delta)$ and $\tilde{F}'=\tilde{F}(\hat{M}\setminus\{\bar{m}\},\sqsubset,\delta)$. Finally, choose a basis in $K^{\delta o-\delta u}$ and put $\tilde{G}=\tilde{G}(\delta\bar{m}-\delta u, \delta o-\delta u)$. The sets $\tilde{F}$, $\tilde{F}'$ and $\tilde{G}$ are *Zariski*-open, and $\pi\tilde{F}=\tilde{F}'$ because of the assumption of Lemma 4.1.

Let $\varphi\colon \tilde{F}\to\tilde{G}$ be the map defined by coordinates

$$(\varphi\gamma)_{\nu\mu}=(\gamma\bar{m})_{\delta u+\nu\,\delta u+\mu} \qquad \nu=1,\dots,\delta\bar{m}-\delta u; \qquad \mu=\delta\bar{m}-\delta u+1,\dots,\delta o-\delta u. \tag{7}$$

Then

$$\Phi\colon \tilde{F}\to\tilde{F}'\times\tilde{G}, \qquad \gamma\mapsto(\pi\gamma,\varphi\gamma) \tag{8}$$

is an algebraic map.

We shall construct $\Psi=\Phi^{-1}$ and shall show that it is an algebraic map. To every pair $(\gamma',g')\in\tilde{F}'\times\tilde{G}$ with coordinates

$$\gamma'\longleftrightarrow\big(m_{\nu\mu}: m\in M\setminus\{\bar{m}\};\ \nu=1,\dots,\delta m;\ \mu=\delta m+1,\dots,n\big),$$
$$g'\longleftrightarrow\big(g'_{\nu\mu}: \nu=1,\dots,\delta\bar{m}-\delta u;\ \mu=\delta\bar{m}-\delta u+1,\dots,\delta o-\delta u\big)$$

consider the linear subspace $\psi(\gamma',g')=g\in\tilde{G}(\delta\bar{m},n)$ with the *Grassmann* coordinates

$$g_{\nu\mu}=g'_{\nu-\delta u\,\mu-\delta u} \qquad \text{if}\quad \nu=\delta u+1,\dots,\delta\bar{m};\ \mu=\delta\bar{m}+1,\dots,\delta o, \tag{9}$$

$$g_{\nu\mu}=u_{\nu\mu}-\sum_{\lambda=\delta u+1}^{\delta\bar{m}}u_{\nu\lambda}g'_{\lambda-\delta u\,\mu-\delta u} \qquad \text{if}\quad \nu=1,\dots,\delta u;\ \mu=\delta\bar{m}+1,\dots,\delta o, \tag{10}$$

$$g_{\nu\mu}=o_{\nu\mu}+\sum_{\lambda=\delta\bar{m}+1}^{\delta o}g'_{\nu-\delta u\,\lambda-\delta u}o_{\lambda\mu} \qquad \text{if}\quad \nu=\delta u+1,\dots,\delta\bar{m};\ \mu=\delta o+1,\dots,n, \tag{11}$$

$$g_{\nu\mu}=o_{\nu\mu}+\sum_{\lambda=\delta\bar{m}+1}^{\delta o}\Big(u_{\nu\lambda}-\sum_{\kappa=\delta u+1}^{\delta\bar{m}}u_{\nu\kappa}g'_{\kappa-\delta u\,\lambda-\delta u}\Big)o_{\lambda\mu} \qquad \text{if}\quad \nu=1,\dots,\delta u;\ \mu=\delta o+1,\dots,n. \tag{12}$$

So we obtain an algebraic map $\Psi\colon\tilde{F}'\times\tilde{G}\to\tilde{G}(M,\delta)$ defined by

$$\big(\Psi(\gamma',g')\big)m=\begin{cases}\psi(\gamma',g') & \text{if } m=\bar{m},\\ \gamma'm & \text{otherwise.}\end{cases} \tag{13}$$

To prove that $\Psi(\tilde{F}' \times \tilde{G}) \subseteq \tilde{F}$, it suffices to check that $\gamma' u \subset \psi(\gamma', g') \subset \gamma' o$ because u and o are the only neighbours of $\bar{m}$ in $\hat{M}$. But this is fulfilled in virtue of (9)–(12). The equalities (9), together with (7), (8) and (13), imply that $\Psi = \Phi^{-1}$. □

5. Singularities on flag spaces

Theorem 5.1. *Let K be infinite and let a type $\tau = (\hat{M}, \sqsubset, \delta)$ fulfil the following conditions:*

(a) *$(M, \sqsubset)$ contains a crown cycle ζ^0.*

(b) *There exists $y \mathrel{-\!\!\sqsubset^{\zeta^0}} x^0$ with $\delta y = \min \delta\zeta^0$, so that there does not exist a sequence of injective cycles $(\zeta^j)_{j=1,\dots,k}$ with $\min \zeta^j = \{y\}$*[3] *and a sequence $(x^j)_{j=0,\dots,k}$ of distinct upper neighbours of y with $y \mathrel{-\!\!\sqsubset^{\zeta^j}} x^{j-1}$, $y \mathrel{-\!\!\sqsubset^{\zeta^j}} x^j$ for all $j = 1, \dots, k$ and with $y \mathrel{-\!\!\sqsubset^{\zeta^0}} x^k$.*

Then every linear flag γ^0 is a singular point in the corresponding flag space $F(\tau)$ for all dimension functions δ.

To prove the theorem we shall need several lemmas. We start by choosing of a basis in K^n such that γ^0 is described by the zero point of the induced local coordinate system on $G(M, \delta)$.

For every flag $\gamma \in F(\tau)$ we consider the K-vector space $D(\tau, \gamma)$ spanned by the differential forms

$$\begin{gathered} \mathrm{d}f^{\nu\mu}_{m'm}(\gamma) = \mathrm{d}m_{\nu\mu} - \mathrm{d}m'_{\nu\mu} + \sum_{\lambda=\delta m'+1}^{\delta m} \mathrm{d}m'_{\nu\lambda} m_{\lambda\mu} + \sum_{\lambda=\delta m'+1}^{\delta m} m'_{\nu\lambda}\, \mathrm{d}m_{\lambda\mu}, \\ \nu = 1, \dots, \delta m'; \qquad \mu = \delta m + 1, \dots, n; \qquad m' \mathrel{-\!\!\sqsubset} m. \end{gathered} \tag{14}$$

From (5) it follows that for each cycle ζ with $r_\zeta = 1$ and for all $\nu = 1, \dots, \min \delta\zeta$; $\mu = \max \delta\zeta + 1, \dots, n$ each of the $f^{\nu\mu}_{\zeta i \zeta(i+1)}(\gamma)$, $i \in \mathbb{Z}$, can be expressed in the ring R by the remaining ones belonging to ζ. Therefore we have:

Lemma 5.2. *If $r_\zeta = 1$ then each of the $\mathrm{d}f^{\nu\mu}_{\zeta i \zeta(i+1)}(\gamma)$, $\nu \le \delta m$; $\mu > \delta m$, depends linearly of the remaining ones for all $\gamma \in F(\tau)$.*

This is true also for any cycle and for the flag $\gamma = \gamma^0$ by the following:

[3] That means $r_{\zeta^j} = 1$ and ζ^j is not a crown cycle.

LEMMA 5.3. *For every cycle $\zeta\colon \mathbb{Z}/s \to (M,\sqsubset)$ and for every pair (ν,μ) satisfying $\nu \le \delta m$ and $\mu > \delta m$ for all $m \in \operatorname{im}\zeta$ it holds that*

$$\sum_{i\in\mathbb{Z}/s} \mathrm{d}f^{\nu\mu}_{\zeta i\,\zeta(i+1)}(\gamma^0) = 0\,.$$

Proof. Because $\nu \le \delta m$, $\mu > \delta m$ and $m \in \operatorname{im}\zeta$ the pair (ν,μ) occurs in (4) for every $m' \mathrel{-\!\!\sqsubset^{\zeta}} m$. The equalities (6) and (14) yield

$$\mathrm{d}f^{\nu\mu}_{\zeta i\,\zeta(i+1)}(\gamma^0) = \mathrm{d}\big(\zeta(i+1)\big)_{\nu\mu} - \mathrm{d}(\zeta i)_{\nu\mu}$$

for all $i \in \mathbb{Z}/s$. □

According to the *Jacobi* criterion γ^0 is a singular point of $F(\tau)$ if there exists a flag $\gamma^1(t) \in F(\tau)$, t transcendent over K, with $\gamma^1(t) \to \gamma^0$ for $t \to 0$ such that

$$\dim_K D(\tau,\gamma^0) < \dim_{K(t)} D\big(\tau,\gamma^1(t)\big)\,. \tag{15}$$

First, we shall construct such a suitable flag $\gamma^1(t)$. Let U_x be the set of all $u \in M$ so that $x \sqsubseteq u$ and let U^0 be the connected component of $\bigcup_{y \mathrel{-\!\!\sqsubset} x} U_x$ which contains x^0. Let P^0 be the set of all $p \in M\setminus U^0$ for which there exists a $u \in U^0$ so that $p \sqsubset u$ and $\delta p > \delta y$.

DEFINITION 5.4. Let $\gamma^1(t)$ be the point of $G(M,\delta)$ given by its coordinates:

(i)	$u_{\delta y+1\,n} = t^{n-\delta y-1}$	for all $u \in U^0$,
(ii)	$u_{\delta p+1\,n} = -t^{n-\delta p-1}$	for all $u \in U^0$ and $p \in P^0$ with $\delta p + 1 \le \delta u$,
(iii)	$p_{\delta y+1\,\delta p+1} = t^{\delta p-\delta y}$	for all $p \in P^0$,
(iv)	$p_{\delta q+1\,\delta p+1} = -t^{\delta p-\delta q}$	for all $q,p \in P^0$, $\delta q < \delta p$,
(v)	$m_{\nu\mu} = 0$	otherwise.

LEMMA 5.5. *The point $\gamma^1(t)$ is a flag of $F(\tau)$.*

Proof. We have to check that all $f^{\nu\mu}_{m'm}\big(\gamma^1(t)\big)$ vanish identically in t.

Case 1: $m' \mathrel{-\!\!\sqsubset} m = u' \mathrel{-\!\!\sqsubset} u$; $u' \in U^0$, $u \in U^0$:

The equality (4) is interesting for $\mu = n$ only. The sum in (4) vanishes because $\lambda \ne n$. So, it remains $u_{\nu n} - u'_{\nu n}$, $\nu = 1,\dots,\delta u'$, which equals zero because of (i), (ii) and (v).

Case 2: $m' \mathrel{-\!\!\sqsubset} m = m' \mathrel{-\!\!\sqsubset} u$; $u \in U^0$, $m' \in M\setminus U^0$:

If $\mu \ne n$ then all $u_{\nu\mu}$ vanish and so does the sum, and also $m'_{\nu\mu}$ because $\delta m'+1 < \delta u+1 \le \mu$. If $\delta m' \le \delta y$, then $m'_{\nu\mu}$ equals 0 for all $\mu = m'+1,\dots,n$, and so does the sum. Also $u_{\nu\mu} = 0$ because $\nu \le \delta m' < \delta y + 1 < \delta p + 1$ for all $p \in P^0$. If $\mu = n$ and $\delta y < \delta m' < \delta u$, which means that $m' \in P^0$, then it remains from (4) $u_{\nu n} - m'_{\nu n} - m'_{\nu\,\delta m'+1}t^{n-\delta m'-1}$ only according to (ii) and (v). It holds $m'_{\nu n} = 0$ for all ν because $\delta m' + 1 < n$. If $\nu = \delta y + 1$ then $u_{\nu n} = t^{n-\delta y-1}$ and $m'_{\nu\,\delta m'+1} = t^{\delta m'-\delta y}$ in virtue of (i) and (iii). If $\nu = \delta q + 1$, where $q \in P^0$, $\delta q < \delta m'$, then $u_{\nu n} = -t^{n-\delta q-1}$ and $m'_{\nu\,\delta m'+1} = -t^{\delta m'-\delta q}$ because of (ii) and (iv).

Case 3: $m' \mathrel{-\!\!\sqsubset} m = m' \mathrel{-\!\!\sqsubset} p$; $p \in P^0$:

The equality (4) is interesting only for $\mu = \delta p+1$, and then it equals $p_{\nu\,\delta p+1} - m'_{\nu\,\delta m'+1}t^{\delta p-\delta m'}$. If there is an $r \in P^0$ such that $\nu = \delta r + 1$ then $p_{\nu\,\delta p+1} = -t^{\delta p-\delta r}$ and $m'_{\nu\,\delta m'+1} = -t^{\delta m'-\delta r}$ because of (iv). If $\nu = \delta y+1$ then $p_{\nu\,\delta p+1} = t^{\delta p-\delta y}$ and $m'_{\nu\,\delta m'+1} = t^{\delta m'-\delta y}$ because of (iii). Otherwise $p_{\nu\,\delta p+1} = m'_{\nu\,\delta m'+1} = 0$ according to (v).

Case 4: $m' \mathrel{-\!\!\sqsubset} m = p \mathrel{-\!\!\sqsubset} m$; $p \in P^0$, $m \in M \setminus (P^0 \cup U^0)$:

The condition (v) implies $m_{\nu\mu} = 0$ for all $\nu = 1, \ldots, \delta m$, $\mu = \delta m+1, \cdots, n$. But then it remains only $-p_{\nu\mu}$ from (4). It holds $\mu > \delta m \geq \delta p+1$, thus $p_{\nu\mu} = 0$, again according to (v).

In all other cases the equality (4) is trivial because of the condition (v). □

LEMMA 5.6. *Let $x \in U^0$ with $y \mathrel{-\!\!\sqsubset} x$ but $x \neq x^0$. Then, independently of the choice of γ, there exists a set of generators of $D(\tau, \gamma)$ which does not contain* $\mathrm{d}f^{\delta y\, n}_{yx}(\gamma)$.

P r o o f. Because of the definition of U^0 there exists a sequence of injective cycles $(\zeta^j)_{j=1,\ldots,k}$ with $\min \zeta^j = \{y\}$ and there exists a sequence $(x^j)_{j=0,\ldots,k}$ of distinct upper neighbours of y with $x^k = x$ and with $y \mathrel{-\!\!\sqsubset}^{\zeta^j} x^{j-1}$, $y \mathrel{-\!\!\sqsubset}^{\zeta^j} x^j$ for all $j = 1, \ldots, k$. By Lemma 5.2 we can successively remove the differential forms $\mathrm{d}f^{\delta y\, n}_{yx^j}(\gamma)$ if $j \neq 0$. □

ASSUMPTION 5.7. In the following considerations we assume that the set of generators of $D(\tau, \gamma)$ does not contain $\mathrm{d}f^{\delta y\, n}_{yx}(\gamma)$ for $x \in U^0$ unless $x \in \operatorname{im} \zeta^0$.

Now we consider the differential form

$$\mathrm{d}f^{\delta y\, n}_{yx^0}(\gamma) = \mathrm{d}x_{\delta y\, n} - \mathrm{d}y_{\delta y\, n} + \sum_{\lambda=\delta y+1}^{\delta x^0} \mathrm{d}y_{\delta y\,\lambda} x^0_{\lambda n} + \sum_{\lambda=\delta y+1}^{\delta x^0} y_{\delta y\,\lambda}\, \mathrm{d}x^0_{\lambda n}\,. \tag{16}$$

Because $\delta y = \min \delta\zeta^0$, the conditions of Lemma 5.3 are fulfilled. Therefore, by Lemma 5.3 and Assumption 5.7, the differential form (16) for $\gamma = \gamma^0$ is linearly dependent on the remaining generators of $D(\tau, \gamma^0)$.

LEMMA 5.8. *Let the type of the flag space satisfy the conditions of Theorem* 5.1. *Then the differential form* (16) *for $\gamma = \gamma^1(t)$ is linearly independent over $K(t)$ on the remaining generators of $D\big(\tau, \gamma^1(t)\big)$.*

Proof. The differential $\mathrm{d}y_{\delta y\,\delta y+1}$ occurs in $\mathrm{d}f^{\delta y\,n}_{yx^0}\big(\gamma^1(t)\big)$ in the first sum only and there with the coefficient $x^0_{\delta y+1\,n} = t^{n-\delta y-1}$ because of Definition 5.4(i).

We shall show that $\mathrm{d}y_{\delta y\,\delta y+1}$ occurs in the remaining generators of $D\big(\tau,\gamma^1(t)\big)$ only with a coefficient which vanishes identically in t. The differential $\mathrm{d}y_{\delta y\,\delta y+1}$ generally occurs in (14) if $m = y$ or if $m' = y$.

In the first case it occurs in the second sum because $\delta m' < \delta y$. Its coefficient there is $m'_{\nu\,\delta y}$. But m' is not in $U^0 \cup P^0$ for the same reason. Thus, $m'_{\nu\,\delta y} = 0$ because of Definition 5.4(v).

In the second case it occurs in the first sum because $\delta y + 1 \le \delta m$. Its coefficient is $m_{\delta y+1\,\mu}$. It is zero unless $m \in U^0 \cup P^0$. But m cannot sit in P^0 because $y \mathrel{-\!\!\sqsubset} m \sqsubset u$ for some $u \in U^0$. Therefore, $m = x \in U^0$.

By Definition 5.4 we have $x_{\delta y+1\,\mu} \ne 0$ only if $\mu = n$. Then the considered differential form (14) is $\mathrm{d}f^{\delta y\,n}_{yx}(\gamma)$. By Lemma 5.6 and by Assumption 5.7 it should not occur in the set of the generators unless $x \in \operatorname{im}\zeta^0$. Then either $\mathrm{d}f^{\delta y\,n}_{yx}(\gamma) = \mathrm{d}f^{\delta y\,n}_{yx^0}(\gamma)$ or the condition (b) of Theorem 5.1 is not fulfilled by virtue of the definition of U^0. $\square$

Let $D^*(\tau,\gamma)$ be the subspace of $D(\tau,\gamma)$ generated by all differential forms (14) other than (16). Because $\gamma^1(0) = \gamma^0$ it holds that

$$\dim_{K(t)} D^*\big(\tau,\gamma^1(t)\big) \ge \dim_K D^*(\tau,\gamma^0)\,. \tag{17}$$

Lemma 5.3 implies that

$$\dim_K D(\tau,\gamma^0) = \dim_K D^*(\tau,\gamma^0) \tag{18}$$

and from Lemma 5.8 it follows that

$$\dim_{K(t)} D\big(\tau,\gamma^1(t)\big) = \dim_{K(t)} D^*\big(\tau,\gamma^1(t)\big) + 1\,. \tag{19}$$

Therefore, (15) is true and Theorem 5.1 is proved.

Remark 5.9. It could seem that we did not need the condition (a) in the proof. But notice this fact: If there is no crown cycle but a cycle in $(M,\sqsubset)$, then condition (b) cannot be fulfilled. See Figures 2, 3 and 4.

Remark 5.10. We conjecture that in Theorem 5.1 it suffices to suppose the flag γ^0 to be linear on $\operatorname{im}\zeta^0$ only.

EXAMPLE 5.11. Let $n \ge 4$. The type $\tau = (M,\sqsubset,\delta)$ given by Figure 5 fulfils condition (a) but does not fulfil condition (b) of Theorem 5.1.

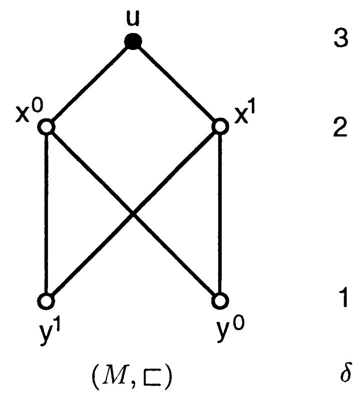


FIGURE 5.

$F(\tau)$ consists of two irreducible algebraic components $F(\tau_1)$ and $F(\tau_2)$ both of which are flag spaces too. The types τ_i can be obtained from τ by identifying $x^0 = x^1$ and $y^0 = y^1$ respectively. The intersection $F(\tau_1) \cap F(\tau_2)$ is the space of all linear flags. Therefore, every linear flag is singular on $F(\tau)$.

The foregoing geometric considerations can be replaced by algebraic ones in terms of the polynomials (4).

There are many other examples in which (a) is fulfilled but (b) is not. In each of them, proved in a special way, the linear flags are singular. Nevertheless, a general proof still remains to be done.

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