Ilja Martišovitš

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EXISTENCE OF POSITIVE SOLUTIONS TO VECTOR BOUNDARY VALUE PROBLEMS I

Ilja Martišovitš

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ABSTRACT. We show that the question about the existence of a positive solution to certain n-dimensional differential system of second order with Dirichlet boundary condition can be answered by multiple (step-by-step) solving of differential equations of the first order.

1. Introduction

In [2], M. Fečkan has dealt with the existence of a solution of the problem:

$$\begin{aligned} -u'' &= \big(f_a(x) + g(u)\big) \cdot u - s(u) \cdot v\,, \\ -v'' &= \big(a + r(u)\big) \cdot v - v^2\,, \\ u(0) &= u(\pi) = v(0) = v(\pi) = 0\,, \\ u(x) &> 0\,, \quad v(x) > 0 \qquad \text{for all} \quad x \in (0,\pi)\,, \end{aligned} \tag{1.0.1}$$

where the functions f, g, r, s fulfil the following conditions:

$$\begin{aligned} &f_{(\cdot)}(\cdot) \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})\,, \qquad g,s,r \in C^1(\mathbb{R},\mathbb{R})\,, \\ &\frac{\partial}{\partial a} f_a(\cdot) > 0\,, \qquad f_a(\cdot) \ge 2\,, \\ &g(0) = g'(0) = 0\,, \qquad g'(u) < 0 \quad \text{for } u > 0\,, \\ &r(0) = r'(0) = 0\,, \qquad s(0) = s'(0) = 0\,, \\ &r/\langle 0,\infty) \le 1\,, \qquad r'/(0,\infty) > 0\,, \qquad s/(0,\infty) \ge 0\,, \\ &\lim g = -\infty \qquad \text{for} \quad x \to \infty\,. \end{aligned}$$

Using the bifurcation method he found a necessary and sufficient condition for the parameter a that problem (1.0.1) may have at least one positive solution u, v. Attention to similar problems has been paid in papers [7], [4] where

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solutions in a cone have been studied. Another problems with similar formulation or with similar method of solution (degree theory) were studied in papers [3], [1].

In this paper we shall investigate the existence of a positive solution depending on definition intervals which are determined by the Dirichlet boundary conditions for single components of the solution. We shall consider the second-order n-dimensional vector differential system, $n \geq 2$, (see (3.0.1)). In this paper the question about the existence of solution to n-dimensional differential system can be answered by multiple (step-by-step) solving of differential equations of the first order. This can be considered as the contribution of this paper. The whole paper is divided into two parts which will be published in this journal separately. In the first part some auxiliary lemmas are stated which will be proved in the second part of this paper. These lemmas and the Brouwer degree of the mapping will be used to prove Theorem 6.1 at the end of this part. This theorem gives a sufficient condition for definition intervals that guarantees the existence of a positive solution to problem (3.0.1) when some other assumptions on the form of right sides are fulfilled. This theorem is the first main result. In the second part of this paper all auxiliary lemmas will be proved which were applied in this part. Then we will introduce and prove the second main result of this paper — Theorem 7.3, which gives a necessary condition on definition intervals for the existence of a positive solution to problem (3.0.1) under some assumptions on the form of the right sides of that problem. The last main result in the second part is Theorem 8.1, which gives simple conditions on the right sides of problem (3.0.1). This result gives a necessary and sufficient condition for the existence of a positive solution to our problem.

2. Auxiliary lemmas

In this section auxiliary lemmas are stated which are necessary for the main section of this part of work. These lemmas will be proved in the next part of the paper.

LEMMA 2.3. *Let the functions $f(x,u_1,u_2)$, $g(x,v_1,v_2)$ satisfy locally Carathéodory's conditions on the set $\big(\langle a,b\rangle \times \mathbb{R}_0^+ \times \mathbb{R}\big)$ and the conditions*

(1)

$$f(x,0,0) \equiv 0 \qquad \text{for all} \quad x \in \langle a,b\rangle\,, \tag{2.3.1}$$

(2)

$$\begin{gathered} f(x,\alpha\cdot u_1,\alpha\cdot u_2) \geq \alpha\cdot f(x,u_1,u_2) \\ \text{for all}\ \ (x,u_1,u_2) \in \big(\langle a,b\rangle \times \mathbb{R}_0^+ \times \mathbb{R}\big) \quad \text{and for all}\ \ \alpha \geq 1\,, \end{gathered} \tag{2.3.2}$$

(3) *the function* f *satisfies locally Lipschitz's condition*

$$|f(x,u_1,u_2) - f(x,v_1,v_2)| \le L_{\mathrm{loc}} \cdot \big(|u_1 - v_1| + |u_2 - v_2|\big)\,, \tag{2.3.3}$$

(4)

$$\begin{gathered} g(x,u_1,u_2) \ge f(x,u_1,u_2) \\ \textit{for all } (x,u_1,u_2) \in \big(\langle a,b\rangle \times \mathbb{R}_0^+ \times \mathbb{R}\big)\,. \end{gathered} \tag{2.3.4}$$

Let the functions $u(\cdot), v(\cdot) \in AC^1\big(\langle a,b\rangle, \mathbb{R}_0^+\big)$ *be solutions of the equations*

$$\left.\begin{aligned} u''(x) &= f\big(x,u(x),u'(x)\big) \\ v''(x) &= g\big(x,v(x),v'(x)\big) \end{aligned}\right\} \textit{ for almost all } x \in \langle a,b\rangle\,, \tag{2.3.5}$$

which satisfy

$$u(x) > 0 \qquad \textit{for all} \quad x \in (a,b)\,, \tag{2.3.6}$$

$$v(a) \le u(a)\,, \qquad v(b) \le u(b)\,. \tag{2.3.7}$$

Then at least one of two following assertions is true

(1)

$$\textit{Simultaneously} \begin{cases} v(x) < u(x) \qquad \textit{for all} \quad x \in (a,b)\,, \\ \big(|v(a) - u(a)| + |v'(a) - u'(a)|\big) > 0\,, \\ \big(|v(b) - u(b)| + |v'(b) - u'(b)|\big) > 0\,. \end{cases} \tag{2.3.8}$$

(2)

$$\exists\, \alpha \ge 1 \quad \forall x \in \langle a,b\rangle \qquad v(x) = \alpha \cdot u(x)\,. \tag{2.3.9}$$

P r o o f . In the second part of this paper. □

LEMMA 2.4. *Let the functions* $f(x,u)$, $g(x,v)$ *satisfy locally Carathéodory's conditions on the set* $\big(\langle 0,a\rangle \times \mathbb{R}_0^+\big)$ *and all assumptions* (2.3.1), (2.3.2), (2.3.3) *and* (2.3.4) *from Lemma 2.3, where* f, g *do not depend on arguments* u_2, v_2. *Let now* $v(\cdot) \in AC^1\langle 0,a\rangle$ *be a solution of the equation*

$$\begin{gathered} v''(x) = g\big(x,v(x)\big)\,, \\ v(x) > 0 \textit{ for all } x \in (0,a)\,, \quad v(0) = 0\,, \quad v'(0) > 0\,, \quad v(a) = 0\,. \end{gathered} \tag{2.4.1}$$

Then the solution $u(\cdot)$ *of the equation*

$$u''(x) = f\big(x,u(x)\big)\,, \qquad u(0) = 0\,, \quad u'(0) = v'(0) \tag{2.4.2}$$

has another zero in the interval $(0,a\rangle$.

P r o o f . In the second part of this paper. □

3. Preliminaries

Throughout the paper we shall use the following notations

$E_n \overset{\text{def}}{=} \underbrace{\langle 0,\infty) \times \langle 0,\infty) \times \cdots \times \langle 0,\infty)}_{n \text{ times}}.$

E_n^* is defined as the compactification of topological space E_n by adding point ∞ and

defining its base of neighbourhoods $O_k \overset{\text{def}}{=} \{\vec{x} \in E_n\,;\ \|\vec{x}\| > k\} \cup \{\infty\}$.

$E_{n,o} \overset{\text{def}}{=} \{\vec{x} \in E_n\,;\ x_i = 0 \text{ for some } i \in \{1,2,\cdots,n\}\}$.

$E_{n,+} \overset{\text{def}}{=} E_n \setminus E_{n,o}$.

$E_{n,o}^* \overset{\text{def}}{=} E_{n,o} \cup \{\infty\}$.

$E_{n,+}^* \overset{\text{def}}{=} E_{n,+} \cup \{\infty\}$.

In the paper we will study the problem

$$\vec{u}\,''(x) \overset{\text{a.e.}}{=} \vec{F}\big(x,\vec{u}(x)\big) \overset{\text{def}}{\iff} \begin{cases} u_1''(x) \overset{\text{a.e.}}{=} F_1\big(x,u_1(x),u_2(x),\dots,u_n(x)\big)\,, \\ u_2''(x) \overset{\text{a.e.}}{=} F_2\big(x,u_1(x),u_2(x),\dots,u_n(x)\big)\,, \\ \qquad\vdots \\ u_n''(x) \overset{\text{a.e.}}{=} F_n\big(x,u_1(x),u_2(x),\dots,u_n(x)\big) \end{cases} \tag{3.0.1}$$

with the boundary conditions

$$u_i(0) = u_i(T_i) = 0\,, \qquad \forall\, x \in (0,T_i):\ u_i(x) > 0 \text{ for } i = 1,2,\dots,n\,.$$

In the sequel we shall assume some of assumptions:

$$\begin{aligned} &\forall\, k \in \{1,2,\dots,n\} \quad \forall\, x \in \mathbb{R}_0^+ \quad \forall\, u_i \in \mathbb{R} \\ &\qquad F_k(x,u_1,\dots,u_{k-1},0,u_{k+1},\dots,u_n) = 0\,. \end{aligned} \tag{3.0.2}$$

$$\begin{aligned} &\forall\, k \in \{1,2,\dots,n\} \quad \forall\, x \in \mathbb{R}_0^+ \quad \forall\, u_i \in \mathbb{R} \\ &\qquad F_k(x,u_1,\dots,u_n) = F_k\Big(x,\tfrac{u_1+|u_1|}{2},\tfrac{u_2+|u_2|}{2},\dots,\tfrac{u_n+|u_n|}{2}\Big)\,. \end{aligned} \tag{3.0.3}$$

$$\begin{aligned} &\forall\, k \in \{1,2,\dots,n-2\} \quad \forall\, x \in \mathbb{R}_0^+ \\ &\forall\, u_i \in \mathbb{R} \text{ which fulfil } u_{k+1}\cdot u_{k+2}\cdots u_n = 0 \\ &\qquad F_k(x,u_1,u_2,\dots,u_n) = F_k(x,u_1,\dots,u_k,\underbrace{0,0,\dots,0}_{n-k \text{ times}})\,. \end{aligned} \tag{3.0.4}$$

$\forall k \in \{1,2,\dots,n-1\} \quad \forall x \in \mathbb{R}_0^+ \quad \forall u_i \in \mathbb{R}$

$$\frac{\partial F_k}{\partial u_k^+}(x,u_1,\dots,u_{k-1},0,u_{k+1},\dots,u_n) = \frac{\partial F_k}{\partial u_k^+}(x,u_1,\dots,u_{k-1},0,\underbrace{0,\dots,0}_{n-k \text{ times}}). \tag{3.0.5}$$

(3.0.6) The functions $F_k(x,u_1,u_2,\dots,u_n)$ and $\frac{\partial F_k}{\partial u_i^+}(x,u_1,u_2,\dots,u_n)$ are continuous in $(u_1,u_2,\dots,u_n)$ on the set E_n for any fixed $x \in \mathbb{R}_0^+$ and for all $k,i \in \{1,2,\dots,n\}$,
and $F_k(x,u_1,u_2,\dots,u_n)$ are measurable in $x \in \langle 0,\infty)$ for each fixed $(u_1,\dots,u_n) \in E_n$ and for all $k \in \{1,2,\dots,n\}$.

(3.0.7) $\frac{\partial F_k}{\partial u_i^+}(x,u_1,u_2,\dots,u_n)$ is locally bounded on the set $\mathbb{R}_0^+ \times E_n$ for all $k,i \in \{1,2,\dots,n\}$.

(3.0.8) For all $T>0$ there exist continuous functions $c_1(\lambda),\dots,c_n(\lambda)$; $c_i(\cdot)\colon \langle 0,\infty) \to (0,\infty)$ such that $\lim_{\lambda\to\infty} c_i(\lambda) = \infty$ for all $i \in \{1,2,\dots,n\}$ and
$\forall k \in \{1,2,\dots,n\} \quad \forall x \in \langle 0,T\rangle \quad \forall \lambda \ge 0$
$\forall \vec{u} \in \{\vec{u};\ u_k = c_k(\lambda) \text{ and } 0 \le u_i \le c_i(\lambda) \text{ for all } i \in \{1,\dots,n\},\ i \ne k\}$

$$F_k(x,u_1,u_2,\dots,u_n) \ge 0.$$

$\forall k \in \{1,2,\dots,n-1\} \quad \forall x \in \mathbb{R}_0^+ \quad \forall u_i \in \mathbb{R}_0^+,\ u_k > 0$

$$\frac{\partial F_k}{\partial u_k^+}(x,u_1,\dots,u_k,\underbrace{0,\dots,0}_{n-k \text{ times}}) > \frac{1}{u_k}F_k(x,u_1,\dots,u_k,\underbrace{0,\dots,0}_{n-k \text{ times}}). \tag{3.0.9}$$

We will study the question when problem (3.0.1) has at least one that solution. We shall apply the shooting method and therefore the following definition of the mapping $\vec{T}(\vec{\alpha})$ will be used.

Definition 3.1. Let $\vec{\alpha} \overset{\text{def}}{=} (\alpha_1,\dots,\alpha_n) \in E_{n,+}$. Let $\vec{u}$ be the solution of the following problem

$$\begin{aligned} \vec{u}\,''(x) &\overset{\text{a.e.}}{=} \vec{F}\big(x,\vec{u}(x)\big), \\ \vec{u}(0) = \vec{0}, &\quad \vec{u}\,'(0) = \vec{\alpha}. \end{aligned} \tag{3.1.1}$$

If for each component u_i, $i=1,2,\dots,n$, of solution $\vec{u}$ there exists a point T_i such that

$$0 < T_i < \infty, \quad u_i(T_i) = 0,$$
$$u_i(x) > 0 \qquad \text{for all } x,\ 0 < x < T_i,$$

then we define $\vec{T}(\vec{\alpha}) \stackrel{\text{def}}{=} (T_1, T_2, \dots, T_n)$.

In the case that at least one component $u_i(\cdot)$ is positive on the whole interval $\left(0, T_{\max}(\vec{\alpha})\right)$, where $\langle 0, T_{\max}(\vec{\alpha})\big)$ is the maximal interval where $\vec{u}$ is defined, then we put $\vec{T}(\vec{\alpha}) \stackrel{\text{def}}{=} \infty \in E_n^*$.

In the following definition the domain of $\vec{T}$ will be extended from E_n^+ to E_n. For this purpose we use the functions $\hat{u}_i(x) \stackrel{\text{def}}{=} \frac{u_i(x)}{\alpha_i} = \frac{u_i(x)}{u_i'(0)}$.

DEFINITION 3.2. Put

$$G_i(x, \hat{u}_1, \dots, \hat{u}_n, \alpha_1, \dots, \alpha_n) \stackrel{\text{def}}{=} \frac{F_i(x, \hat{u}_1 \cdot \alpha_1, \dots, \hat{u}_n \cdot \alpha_n)}{\alpha_i} \qquad \text{for} \quad \alpha_i > 0$$

and (3.2.1)

$$G_i(x, \hat{u}_1, \dots, \hat{u}_n, \alpha_1, \dots, \alpha_n) \stackrel{\text{def}}{=} \frac{\partial F_i}{\partial u_i^+}(x, \hat{u}_1 \cdot \alpha_1, \dots, \hat{u}_{i-1} \cdot \alpha_{i-1}, 0, \hat{u}_{i+1} \cdot \alpha_{i+1}, \dots, \hat{u}_n \cdot \alpha_n) \cdot \frac{\hat{u}_i + |\hat{u}_i|}{2} \qquad \text{for} \quad \alpha_i = 0\,.$$

Let now $\vec{\alpha} \in E_n^*$.

(1) If $\vec{\alpha} = \infty$, then we define $\vec{T}(\vec{\alpha}) = \infty$.

(2) If $\vec{\alpha} \neq \infty$ then we shall consider the solution $\hat{\vec{u}}$ of the following problem

$$\begin{aligned} \hat{\vec{u}}''(x) &\stackrel{\text{a.e.}}{=} \vec{G}\big(x, \hat{\vec{u}}(x), \vec{\alpha}\big)\,, \\ \hat{\vec{u}}(0) = \vec{0}, \quad \hat{\vec{u}}'(0) &= (\underbrace{1, 1, \dots, 1}_{n \text{ times}})\,. \end{aligned} \tag{3.2.2}$$

Let $\langle 0, \hat{T}_{\max}(\vec{\alpha})\big)$ be the maximal interval where the solution $\hat{\vec{u}}$ can be defined.

(a) If $\exists i \in \{1, \dots, n\} \ \ \forall x \in \big(0, \hat{T}_{\max}(\vec{\alpha})\big) \quad \hat{u}_i(x) > 0$, then we define
$$\vec{T}(\vec{\alpha}) \stackrel{\text{def}}{=} \infty\,.$$

(b) Otherwise let T_i be the zero point of $\hat{u}_i$ for $i \in \{1, 2, \dots, n\}$ such that $\hat{u}_i(T_i) = 0$, $T_i \in \big(0, \hat{T}_{\max}(\alpha)\big)$ and $\hat{u}_i(x) > 0$ for all $x \in (0, T_i)$. Then we define
$$\vec{T}(\vec{\alpha}) \stackrel{\text{def}}{=} (T_1, T_2, \dots, T_n)\,.$$

In the following lemmas we shall show the correctness of previous definitions as well as the relation between $\vec{u}$, $T_{\max}(\vec{\alpha})$ and $\hat{\vec{u}}$, $\hat{T}_{\max}(\vec{\alpha})$.

LEMMA 3.3. *Let $\vec{F}$ fulfil* (3.0.2), (3.0.3), (3.0.6) *and* (3.0.7). *Then $\vec{G}(x,\hat{\vec{u}},\vec{\alpha})$ defined in Definition* 3.2 *satisfies Carathéodory's conditions.*

Proof. In the second part of the paper. □

LEMMA 3.4. *Let $\vec{F}$ fulfil* (3.0.2), (3.0.3), (3.0.6) *and* (3.0.7). *Then the problem* (3.2.2) *has the property of global uniqueness.*

Proof. In the second part of the paper. □

LEMMA 3.5. *Let $\vec{F}$ fulfil conditions* (3.0.2), (3.0.3), (3.0.6), (3.0.7) *and* (3.0.8). *Let $T > 0$ be a fixed number. Let $c_1(\lambda),\dots,c_n(\lambda)$ be some functions satisfying* (3.0.8). *Then the following assertion is true:*

Let $\vec{\alpha} \in E_n$ be fixed. Let $\vec{u}(\cdot)$ be the maximal solution of problem (3.1.1) *which is defined on the interval $\langle 0, T_{\max}(\vec{\alpha})\big)$, that is*

$$\begin{aligned} \vec{u}\,''(x) &\overset{\text{a.e.}}{=} \vec{F}\big(x,\vec{u}(x)\big)\,, \\ \vec{u}(0) = \vec{0}, &\quad \vec{u}\,'(0) = \vec{\alpha}\,. \end{aligned} \tag{3.5.1}$$

If

$$T < T_{\max}(\vec{\alpha}) \tag{3.5.2}$$

and

$$u_i(T) \le 0 \qquad \text{for all} \quad i \in \{1,2,\dots,n\}\,, \tag{3.5.3}$$

then $u_i(x) < c_i(0)$ for all $i \in \{1,2,\dots,n\}$ and for all $x \in \langle 0, T_{\max}(\vec{\alpha})\big)$.

Proof. In the second part of this paper. □

The following lemma deals with the relation between solutions $\vec{u}$ and $\hat{\vec{u}}$ of problems (3.1.1) and (3.2.2), respectively.

LEMMA 3.6. *Let the function $\vec{F}$ fulfil* (3.0.2), (3.0.3), (3.0.6) *and* (3.0.7). *Let $\vec{\alpha} \in E_n$. Let $\vec{u}(\cdot)$ and $\hat{\vec{u}}(\cdot)$ be the maximal solutions of problems* (3.1.1) *and* (3.2.2), *respectively, which are defined on the intervals $\langle 0,T)$ and $\langle 0,\hat{T})$ respectively. Then $T = \hat{T}$ and $u_i(\cdot) \equiv \alpha_i \cdot \hat{u}_i(\cdot)$ for all $i \in \{1,\dots,n\}$.*

Proof. In the next part of the paper. □

4. Continuity of the mapping $\vec{T}$

The following theorem has fundamental meaning for us.

THEOREM 4.1. *Let $\vec{F}$ fulfil* (3.0.2), (3.0.3), (3.0.6), (3.0.7) *and* (3.0.8). *Then the mapping $\vec{T}\colon E_n^* \to E_n^*$, defined in Definition* 3.2, *is continuous.*

Proof. In the next part of the paper. □

5. Definition of the set Ω_n^o

Here we shall define the set Ω_n^o and the mappings a_k by means of which the set Ω_n^o can be easily defined. We shall also show later that for all $(T_1, T_2, \dots, T_n) \in \Omega_n^o$ problem (3.0.1) has at least one positive solution when some conditions are fulfilled.

DEFINITION 5.1. Let $F_1, \dots, F_n$ fulfil (3.0.2), (3.0.3), (3.0.6) and (3.0.7). Let the functions $G_1, \dots, G_n$ be defined as in (3.2.1). Let $k \in \{1, \dots, n\}$. Let the mapping $\vec{R}_k \colon {E_k}^* \to {E_k}^*$ be defined by the following method:

Let $\vec{\alpha} \in E_k^*$:

(1) If $\vec{\alpha} = \infty$, then we define $\vec{R}_k(\vec{\alpha}) = \infty$.

(2) If $\vec{\alpha} \neq \infty$ then we shall consider the following system

$$\begin{aligned} u_1''(x) &\overset{\text{a.e.}}{=} G_1\big(x, u_1(x), \underbrace{0, \dots, 0}_{n-1 \text{ times}}, \alpha_1, \underbrace{0, \dots, 0}_{n-1 \text{ times}}\big), \\ u_2''(x) &\overset{\text{a.e.}}{=} G_2\big(x, u_1(x), u_2(x), \underbrace{0, \dots, 0}_{n-2 \text{ times}}, \alpha_1, \alpha_2, \underbrace{0, \dots, 0}_{n-2 \text{ times}}\big), \\ &\vdots \\ u_k''(x) &\overset{\text{a.e.}}{=} G_k\big(x, u_1(x), \dots, u_k(x), \underbrace{0, \dots, 0}_{n-k \text{ times}}, \alpha_1, \dots, \alpha_k, \underbrace{0, \dots, 0}_{n-k \text{ times}}\big), \\ u_i(0) &= 0, \quad {u_i}'(0) = 1 \qquad \text{for all} \quad i \in \{1, 2, \dots, k\}. \end{aligned} \tag{5.1.1}$$

Let $\big\langle 0, R_{\max}^k(\vec{\alpha})\big)$ be the maximal interval of definition of the solution $\big(u_1(\cdot), \dots, u_k(\cdot)\big)$.

(a) If $\exists\, i \in \{1, \dots, k\}$ $\forall x \in \big(0, R_{\max}^k(\vec{\alpha})\big)$ $\quad u_i(x) > 0$, then we put

$$\vec{R}_k(\vec{\alpha}) \overset{\text{def}}{=} \infty.$$

(b) Otherwise we put R_i to be the zero point of u_i for $i \in \{1, 2, \dots, k\}$ such that $u_i(R_i) = 0$, $R_i \in \big(0, R_{\max}^k(\alpha)\big)$ and $u_i(x) > 0$ for all $x \in (0, R_i)$ and we put

$$\vec{R}_k(\vec{\alpha}) \overset{\text{def}}{=} (R_1, R_2, \dots, R_k).$$

LEMMA 5.2. *Let $F_1, \dots, F_n$ fulfil* (3.0.2), (3.0.3), (3.0.6), (3.0.7) *and* (3.0.8). *Then the mappings $\vec{R}_k \colon E_k^* \to E_k^*$ defined in Definition* 5.1 *for all $k \in \{1, \dots, n\}$ are continuous.*

P r o o f . Let $k \in \{1, 2, \dots, n\}$ be arbitrary, but fixed. Now we can use Theorem 4.1, in which we put k in place of n which we write $n \sim k$, similarly

$F_i(x,u_1,\dots,u_n) \sim F_i(x,u_1,\dots,u_i,0,\dots,0)$ for $i \in \{1,\dots,k\}$,
$G_i(x,u_1,\dots,u_n,\alpha_1,\dots,\alpha_n) \sim G_i(x,u_1,\dots,u_i,0,\dots,0,\alpha_1,\dots,\alpha_i,0,\dots,0)$ for $i \in \{1,\dots,k\}$ and
$\vec{T}(\cdot)\colon E_n^* \to E_n^* \sim \vec{R}_k(\cdot)\colon E_k^* \to E_k^*$.

Then lemma follows from Theorem 4.1. □

LEMMA 5.3. *Let $F_1,\dots,F_n$ fulfil* (3.0.2), (3.0.3), (3.0.6), (3.0.7) *and* (3.0.9). *Then $\forall\, k \in \{1,\dots,n-1\}$ the mapping $\vec{R}_k\colon E_k^* \to E_k^*$ defined in Definition* 5.1 *is injective when we restrict this mapping to the set $\big(\vec{R}_k\big)^{-1}(E_k)$.*

P r o o f . By contradiction. Let

$$\exists\, \vec{\alpha}^1,\vec{\alpha}^2 \in E_k \qquad \vec{R}_k(\vec{\alpha}^1) = \vec{R}_k(\vec{\alpha}^2) \in E_k \ \text{ while } \vec{\alpha}^1 \neq \vec{\alpha}^2 . \tag{5.3.1}$$

Then we can choose $i_0 \in \{1,\dots,k\}$ such that

$$\forall\, i \in \{1,\dots,i_0-1\} \qquad \alpha_i^1 = \alpha_i^2 \quad \text{and} \quad \alpha_{i_0}^1 \neq \alpha_{i_0}^2 . \tag{5.3.2}$$

From condition (5.3.1) using (3.0.2), (3.0.3) and Definition 5.1 it follows that the solutions $\vec{u}^1(\cdot)$, $\vec{u}^2(\cdot)$ of problem (5.1.1) for $\vec{\alpha}^1$, $\vec{\alpha}^2$ are defined on the whole interval $\langle 0,\infty)$. Using (5.3.2) and the diagonal structure of problem (5.1.1) we can easy show by (3.2.1) and (3.0.7) that

$$\forall\, i \in \{1,\dots,i_0-1\} \qquad u_i^1(\cdot) \equiv u_i^2(\cdot) \quad \text{on } \langle 0,\infty) . \tag{5.3.3}$$

Putting $u_i(\cdot) \stackrel{\text{def}}{=} u_i^1(\cdot) = u_i^2(\cdot)$, $\alpha_i \stackrel{\text{def}}{=} \alpha_i^1 = \alpha_i^2$, $R_{i_0} \stackrel{\text{def}}{=} R_{i_0}^1 = R_{i_0}^2$ for $i \in \{1,\dots,i_0-1\}$, by (5.3.3) we get that the functions $u_{i_0}^j$ for $j \in \{1,2\}$ are solutions of the following equation on the interval $\langle 0,R_{i_0}\rangle$

$$\begin{aligned} &u_{i_0}^{j\,\prime\prime}(x) \stackrel{\text{a.e.}}{=} p(x,\alpha_{i_0}^j,u_{i_0}^j(x))\,, \quad u_{i_0}^j(0)=0\,, \quad u_{i_0}^{j\,\prime}(0)=1\,, \\ &u_{i_0}^j(R_{i_0})=0 \qquad \text{and} \qquad u_{i_0}^j(x)>0 \quad \text{for all } x \in (0,R_{i_0})\,, \end{aligned} \tag{5.3.4}$$

where we have used function p which is defined for $\alpha,u \geq 0$

$$p(x,\alpha,u) \stackrel{\text{def}}{=} G_{i_0}\big(x,u_1(x),\dots,u_{i_0-1}(x),u,\overbrace{0,\dots,0}^{n-i_0\text{ times}},\alpha_1,\dots,\alpha_{i_0-1},\alpha,\overbrace{0,\dots,0}^{n-i_0\text{ times}}\big) \tag{5.3.5}$$

$$\stackrel{(3.0.2),(3.2.1)}{=} u\cdot\Bigg(\frac{\partial F_{i_0}}{\partial u_{i_0}^+}\big(x,\alpha_1\cdot u_1(x),\dots,\alpha_{i_0-1}\cdot u_{i_0-1}(x),0,\overbrace{0,\dots,0}^{n-i_0\text{ times}}\big) + \int_0^{\alpha\cdot u}\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\frac{F_{i_0}\big(x,\alpha_1\cdot u_1(x),\dots,\alpha_{i_0-1}\cdot u_{i_0-1}(x),\beta,\overbrace{0,\dots,0}^{n-i_0\text{ times}}\big)}{\beta}\right]\mathrm{d}\beta\Bigg). \tag{5.3.6}$$

We can estimate the expression in the last integral. For all $\beta > 0$

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\frac{F_{i_0}(x,\alpha_1\cdot u_1(x),\dots,\alpha_{i_0-1}\cdot u_{i_0-1}(x),\beta,\overbrace{0,\dots,0}^{n-i_0\text{ times}})}{\beta}\right]$$

$$=\frac{1}{\beta}\left(\frac{\partial F_{i_0}}{\partial u_{i_0}^+}(x,\alpha_1\cdot u_1(x),\dots,\alpha_{i_0-1}\cdot u_{i_0-1}(x),\beta,\overbrace{0,\dots,0}^{n-i_0\text{ times}})\right.$$

$$\left.-\frac{F_{i_0}(x,\alpha_1\cdot u_1(x),\dots,\alpha_{i_0-1}\cdot u_{i_0-1}(x),\beta,\overbrace{0,\dots,0}^{n-i_0\text{ times}})}{\beta}\right)\overset{(3.0.9)}{>}0\,.$$

Without losing generality we can consider that $0 \le \alpha_{i_0}^1 < \alpha_{i_0}^2$. If this is used in (5.3.6), then the following relations hold

$$\forall\, x\in\langle 0,R_{i_0}\rangle \quad \forall\, u>0: \quad p(x,\alpha_{i_0}^1,u) < p(x,\alpha_{i_0}^2,u)\,, \tag{5.3.7}$$

$$\forall\, x\in\langle 0,R_{i_0}\rangle \quad \forall\, u,\alpha\ge 0 \quad \forall\, c\ge 1: \quad p(x,\alpha,c\cdot u)\ge c\cdot p(x,\alpha,u)\,. \tag{5.3.8}$$

Now we can verify the assumptions in Lemma 2.3, where we put

$$\langle a,b\rangle=\langle 0,R_{i_0}\rangle\,,\quad f(x,u_1,u_2)=p(x,\alpha_{i_0}^1,u_1)\,,\quad g(x,v_1,v_2)=p(x,\alpha_{i_0}^2,v_1)\,,$$
$$u(\cdot)=u_{i_0}^1(\cdot)\,,\quad v(\cdot)=u_{i_0}^2(\cdot)\,.$$

By (5.3.7), (5.3.8) we get conditions (2.3.4), (2.3.2). From (5.3.5) and (3.2.1) using (3.0.7), (3.0.6) we get that $f(x,\cdot,\cdot)$ fulfil Lipschitz's condition (2.3.3) and also (2.3.1). (5.3.4) implies (2.3.5), (2.3.6) and (2.3.7). Hence Lemma 2.3 can be applied. Statement (2.3.8) cannot be true because (5.3.4) contradicts its second condition. Thus statement (2.3.9) must be true and then from $u_{i_0}^{1\,\prime}(0) = u_{i_0}^{2\,\prime}(0) = 1$ we get

$$u_{i_0}(x) \overset{\text{def}}{=} u_{i_0}^1(x) = u_{i_0}^2(x) \qquad \text{for all} \quad x\in\langle 0,R_{i_0}\rangle\,.$$

From this by (5.3.4) we get

$$p(x,\alpha_{i_0}^1,u_{i_0}(x)) \overset{\text{a.e.}}{=} p(x,\alpha_{i_0}^2,u_{i_0}(x))$$

which contradicts (5.3.7). □

DEFINITION 5.4. Let $F_1,\dots,F_n$ fulfil assumptions (3.0.2), (3.0.3), (3.0.6) and (3.0.7). Let $\vec{R}_k(\cdot)$ be the mapping defined in Definition 5.1 for $k\in\{1,\dots,n\}$. Then we put $\Omega_k \overset{\text{def}}{=} \vec{R}_k(E_k^*)\setminus\{\infty\}$. Evidently $\Omega_k\subset E_k$ for $k\in\{1,\dots,n\}$.

LEMMA 5.5. *Let $F_1, \dots, F_n$ fulfil assumptions* (3.0.2), (3.0.3), (3.0.6), (3.0.7), (3.0.8) *and* (3.0.9). *Let $\vec{R}_k(\cdot)$ be the mapping defined in Definition* 5.1 *for $k \in \{1, \dots, n-1\}$. Let $\Omega_k \subset E_k$ be defined as in Definition* 5.4. *Then the inverse mapping $\vec{R}_k^{-1}(\cdot)\colon \Omega_k \to E_k$ is continuous.*

Proof. The existence of the inverse mapping follows from Theorem 5.3. Now, by contradiction, we shall show its continuity. Let the sequence $\{\vec{T}_i\}_{i=1}^{\infty} \subset \Omega_k$ be such that

$$\vec{T}_i \to \vec{T}_0 \in \Omega_k \qquad \text{for} \quad i \to \infty$$

and in opposition to continuity,

$$\forall i \in \mathrm{N}: \quad \|\vec{\alpha}_i - \vec{\alpha}_0\| \geq \varepsilon > 0 \tag{5.5.1}$$

where

$$\vec{\alpha}_i \stackrel{\text{def}}{=} \vec{R}_k^{-1}(\vec{T}_i) \in E_k \qquad \text{for} \quad i \in \mathrm{N}_0 .$$

The space E_k^* is compact, and therefore there exists a subsequence $\{\vec{\alpha}_{i_j}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \vec{\alpha}_{i_j} = \hat{\vec{\alpha}}_0 \in E_k^* . \tag{5.5.2}$$

Because the mapping $\vec{R}_k(\cdot)$ is continuous (see Theorem 5.2) we get

$$\vec{R}_k(\hat{\vec{\alpha}}_0) = \vec{R}_k\Big(\lim_{j\to\infty} \vec{\alpha}_{i_j}\Big) = \lim_{j\to\infty} \vec{R}_k(\vec{\alpha}_{i_j}) = \lim_{j\to\infty} \vec{T}_{i_j} = \vec{T}_0 = \vec{R}_k(\vec{\alpha}_0) \in \Omega_k \subset E_k .$$

The mapping $\vec{R}_k(\cdot)$ is injective on the set $\vec{R}_k^{-1}(E_k)$ and therefore

$$\hat{\vec{\alpha}}_0 = \vec{\alpha}_0$$

and the contradiction can be obtained from (5.5.1) and (5.5.2). □

DEFINITION 5.6. Let $F_1, \dots, F_n$ fulfil assumptions (3.0.2), (3.0.3), (3.0.6), (3.0.7) and (3.0.9). Then we can define the mappings $a_k(\cdot)\colon \Omega_{k-1} \to E_1^*$ for $2 \leq k \leq n$ by the following way:

For any $\vec{T} = (T_1, \dots, T_{k-1}) \in \Omega_{k-1}$ we put $(\alpha_1, \dots, \alpha_{k-1}) \stackrel{\text{def}}{=} (\vec{R}_{k-1})^{-1}(\vec{T})$. Let the functions $u_1(\cdot), \dots, u_{k-1}(\cdot)$ be defined as solutions of the problem (5.1.1). (Using $\vec{R}_{k-1}(\vec{\alpha}) = \vec{T} \in E_{k-1}$ we can show by assumptions (3.0.3), (3.0.2) and by definition of functions $G_i(\cdot)$ (3.2.1) that the functions $u_i(\cdot)$ are defined on the whole interval $\langle 0, \infty)$.) Let the function $v(\cdot)$ be defined as the solution of the following problem

$$\begin{aligned} v''(x) &\stackrel{\text{a.e.}}{=} \frac{\partial F_k}{\partial u_k^+}\big(x, \alpha_1 \cdot u_1(x), \dots, \alpha_{k-1} \cdot u_{k-1}(x), 0, \overbrace{0, \dots, 0}^{n-k \text{ times}}\big) \cdot \frac{v(x) + |v(x)|}{2} \\ &= G_k\big(x, u_1(x), \dots, u_{k-1}(x), v(x), \underbrace{0, \dots, 0}_{n-k \text{ times}}, \alpha_1, \dots, \alpha_{k-1}, 0, \underbrace{0, \dots, 0}_{n-k \text{ times}}\big), \end{aligned}$$

$$v(0) = 1, \quad v'(0) = 1 . \tag{5.6.1}$$

By assumptions (3.0.6), (3.0.7) we can easily obtain that $v(\cdot)$ is uniquely defined also on the whole interval $\langle 0,\infty)$.

(1) If solution $v(\cdot)$ is positive on the interval $(0,\infty)$, then we put

$$a_k(T_1,\dots,T_{k-1}) \stackrel{\text{def}}{=} \infty \in E_1^* .$$

(2) Otherwise, let T be the zero point of solution $v(\cdot)$ on the interval $(0,\infty)$ such that $v(T)=0$ and $v(x)>0$ for all $x\in(0,T)$. Then we put

$$a_k(T_1,\dots,T_{k-1}) \stackrel{\text{def}}{=} T \in E_1^* .$$

LEMMA 5.7. *Let* $F_1,\dots,F_n$ *fulfil assumptions* (3.0.2), (3.0.3), (3.0.6), (3.0.7), (3.0.8) *and* (3.0.9). *Then mappings* $a_k(\cdot)$ *for* $2\le k\le n$ *defined in Definition* 5.6 *are continuous.*

P r o o f . From the definition of mappings $a_k(\cdot)$ and $\vec{R}_k(\cdot)$ we can easily obtain the following identity

$$(T_1,\dots,T_{k-1},a_k(T_1,\dots,T_{k-1})) = \vec{R}_k\Big(\underbrace{\overbrace{(\vec{R}_{k-1})^{-1}(T_1,\dots,T_{k-1})}^{\in E_{k-1}},0}_{\in E_k}\Big) \tag{5.7.1}$$

where we shall assume that if $a_k(T_1,\dots,T_{k-1})=\infty$, then we put the vector on the left side equal to $\infty\in E_k^*$. From Lemmas 5.2 and 5.5 it follows that the right side in identity (5.7.1) is continuous in $(T_1,\dots,T_{k-1})\in\Omega_{k-1}\subset E_{k-1}$ so also the left side is continuous and we can easily obtain that $a_k(\cdot)\colon \Omega_{k-1}\to E_1^*$ is continuous, too. □

The following lemma gives us a recurrent formula between sets Ω_{k-1} and Ω_k using mappings $a_k(\cdot)$.

LEMMA 5.8. *Let* $F_1,\dots,F_n$ *fulfil* (3.0.2), (3.0.3), (3.0.6), (3.0.7), (3.0.8) *and* (3.0.9). *Let* $2\le k\le n$. *Let sets* Ω_{k-1} *and* Ω_k *be defined such as in Definition* 5.4 *and let* $a_k(\cdot)$ *be defined in Definition* 5.6. *Then the formula*

$$\Omega_k=\big\{(T_1,\dots,T_{k-1},T_k)\in E_k\,;\quad (T_1,\dots,T_{k-1})\in\Omega_{k-1},\ a_k(T_1,\dots,T_{k-1})\le T_k<\infty\big\} \tag{5.8.1}$$

is true.

P r o o f . At first we show the inclusion $\subset$ in (5.8.1). Let $(T_1,\dots,T_k)\in\Omega_k$. According to Definition 5.4 and Definition 5.1 there exists $(\alpha_1,\dots,\alpha_k)\in E_k$ such that the following system is fulfilled

$$u_i''(x) \stackrel{\text{a.e.}}{=} G_i\big(x,u_1(x),\dots,u_i(x),\overbrace{0,\dots,0}^{n-i \text{ times}},\alpha_1,\dots,\alpha_i,\overbrace{0,\dots,0}^{n-i \text{ times}}\big) \qquad \text{for } i\in\{1,\dots,k\} \tag{5.8.2}$$

with boundary conditions

$$\left.\begin{aligned} u_i(0)=0\,, \quad u_i{}'(0)=1\,, \qquad u_i(T_i)=0\,,\\ u_i(x)>0 \quad \text{for all } \ x\in(0,T_i) \end{aligned}\right\} \ \text{for all } \ i\in\{1,2,\dots,k\} \tag{5.8.3}$$

where $u_i(\cdot)$ are defined on the whole interval $\langle 0,\infty)$. From system (5.8.2) for $i\in\{1,\dots,k-1\}$ we get $(T_1,\dots,T_{k-1})=\vec{R}_k(\alpha_1,\dots,\alpha_{k-1})\in\Omega_{k-1}$. In addition we must prove the inequality $a_k(T_1,\dots,T_{k-1})\le T_k$. We shall consider the solution $v(\cdot)$ of the following problem similarly as in the definition of $a_k(\cdot)$ (see Definition 5.6)

$$v''(x)=G_k\big(x,u_1(x),\dots,u_{k-1}(x),v(x),\overbrace{0,\dots,0}^{n-k \text{ times}},\alpha_1,\dots,\alpha_{k-1},0,\overbrace{0,\dots,0}^{n-k \text{ times}}\big)\,, \tag{5.8.4}$$

$$\begin{gathered} v(0)=0\,, \quad v'(0)=1\,, \qquad v\big(a_k(T_1,\dots,T_{k-1})\big)=0\,,\\ v(x)>0 \qquad \text{for all} \quad x\in\big(0,a_k(T_1,\dots,T_{k-1})\big)\,. \end{gathered}$$

If we define the function $p(x,\alpha,v)$ similarly as in (5.3.5)

$$p(x,\alpha,v)\stackrel{\text{def}}{=}G_k\big(x,u_1(x),\dots,u_{k-1}(x),v,\overbrace{0,\dots,0}^{n-k \text{ times}},\alpha_1,\dots,\alpha_{k-1},\alpha,\overbrace{0,\dots,0}^{n-k \text{ times}}\big)\,, \tag{5.8.5}$$

then in the same way as we have got formula (5.3.7), we get

$$\forall\, x\in\langle 0,T_k\rangle \quad \forall\, u\ge 0 \qquad p(x,0,u)\le p(x,\alpha_k,u)\,. \tag{5.8.6}$$

Since the function $p(x,0,v)$ is linear in v for $v\ge 0$, we get

$$\forall\, x\in\langle 0,T_k\rangle \quad \forall\, u\ge 0 \quad \forall\, c\ge 1 \qquad p(x,0,c\cdot u)=c\cdot p(x,0,u)\,. \tag{5.8.7}$$

Using this new function p the equations for $u_k(\cdot)$, $v(\cdot)$ can be rewritten in the form

$$\begin{aligned} u_k''(x)&=p\big(x,\alpha_k,u_k(x)\big) \quad \& \quad \text{boundary conditions}\,,\\ v''(x)&=p\big(x,0,v(x)\big) \qquad \& \quad \text{boundary conditions}\,. \end{aligned} \tag{5.8.8}$$

Now Lemma 2.4 can be used, where in our context we put $\langle 0,a\rangle=\langle 0,T_k\rangle$, $f(x,u)=p(x,0,u)$, $g(x,v)=p(x,\alpha_k,v)$, $u(\cdot)=v(\cdot)$ and $v(\cdot)=u_k(\cdot)$. From (5.8.5), (5.8.6) and (5.8.7) the assumptions of Lemma 2.4 follow which we put on f, g. From (5.8.2), (5.8.3), (5.8.4), (5.8.5) and (5.8.8) assumptions (2.4.1), (2.4.2) follow. According to Lemma 2.4 we get that $a_k(T_1,\dots,T_{k-1})\le T_k$ and this is what we needed.

Further we show the inclusion $\supset$ in (5.8.1). Let $(T_1,\dots,T_k)\in E_k$ fulfil

$$(T_1,\dots,T_{k-1})\in\Omega_{k-1} \qquad \text{and} \qquad a_k(T_1,\dots,T_{k-1})\le T_k<\infty\,. \tag{5.8.9}$$

Let $\alpha \in \mathbb{R}_0^+$ be a free variable in the following expression

$$\vec{R}_k\big(\underbrace{\overbrace{(\vec{R}_{k-1})^{-1}(T_1,\ldots,T_{k-1})}^{\in E_{k-1}},\alpha}_{\in E_k}\big)\,.$$

This expression is a function from $\mathbb{R}_0^+ \to E_k^*$, which, by Lemmas 5.2, 5.5, is continuous. Simultaneously we easily see from definition of $\vec{R}_k(\cdot)$ that first $k-1$ components in the image (if it is not just $\infty \in E_k^*$) are again $T_1,\ldots,T_{k-1}$. Using (5.7.1) we know that if we put $\alpha = 0$, then the last component in the image is $a_k(T_1,\ldots,T_{k-1})$. If α is increasing, then only the last component in image will being changed (continuously). So if α increase to ∞, then from $\vec{R}_k(\infty) = \infty$ it follows that the last component of image must reach all numbers from the interval $\langle a_k(T_1,\ldots,T_{k-1}),\infty)$ so also T_k will be reached and therefore $(T_1,\ldots,T_k) \in \vec{R}_k(E_k) \cap E_k = \Omega_k$. Thus also the second inclusion is showed, and so (5.8.1) is proved. □

DEFINITION 5.9. Let $F_1,\ldots,F_n$ fulfil (3.0.2), (3.0.3), (3.0.6), (3.0.7) and (3.0.9). Let $v(\cdot)$ be the solution of the following problem

$$v''(x) = \frac{\partial F_1}{\partial u_1^+}(x,\underbrace{0,\ldots,0}_{n\ \text{times}})\cdot\frac{v(x)+|v(x)|}{2} = G_1\big(x,v(x),\underbrace{0,\ldots,0}_{n-1\ \text{times}},0,\underbrace{0,\ldots,0}_{n-1\ \text{times}}\big)\,,$$

$$v(0)=1\,,\qquad v'(0)=1\,.$$

The right-hand side fulfils locally Carathéodory's conditions and locally Lipschitz's condition because conditions (3.0.6) and (3.0.7) are satisfied. From this, using linearity of right-hand side in v for $v \ge 0$ and condition (3.0.7), we obtain that $v(\cdot)$ is uniquely defined on the whole interval $\langle 0,\infty)$. We will now define $a_1 \in E_1^*$:

(1) If solution $v(\cdot)$ is positive on the whole interval $\langle 0,\infty)$, then we put

$$a_1 \stackrel{\text{def}}{=} \infty \in E_1^*\,.$$

(2) Otherwise let T be the zero point of solution $v(\cdot)$ such that $v(T)=0$ and $v(x)>0$ for all $x \in \langle 0,T)$. Then we put

$$a_1 \stackrel{\text{def}}{=} T \in E_1^*\,.$$

LEMMA 5.10. *Let $F_1,\ldots,F_n$ fulfil* (3.0.2), (3.0.3), (3.0.6), (3.0.7), (3.0.8) *and* (3.0.9). *Let the number a_1 be defined as in Definition* 5.9. *Let the set Ω_1 be defined as in Definition* 5.4. *Then*

$$\Omega_1 = \{T\,;\ a_1 \le T < \infty\}\,.$$

Proof. Analogically as in proof of Lemma 5.8. □

LEMMA 5.11. *Let* $F_1, \dots, F_n$ *fulfil* (3.0.2), (3.0.3), (3.0.6), (3.0.7), (3.0.8) *and* (3.0.9). *Let* $1 \le k \le n$. *Let* Ω_k *be defined as in Definition* 5.4. *Let* $a_1, a_2(\cdot), \dots, a_k(\cdot)$ *be defined in the same way as in Definition* 5.6 *and Definition* 5.9. *Then*

$$\Omega_k = \Big\{(T_1, \dots, T_k) \in E_k \ \text{ such that the following conditions step-by-step hold}$$
$$a_1 \le T_1 < \infty\,,$$
$$a_2(T_1) \le T_2 < \infty\,,$$
$$\vdots$$
$$a_k(T_1, \dots, T_{k-1}) \le T_k < \infty\Big\}\,.$$

P r o o f . By mathematical induction. The first step follows from Lemma 5.10. The second step is based on Lemma 5.8. □

Now we are prepared for the following definition of the domain Ω_n^o.

DEFINITION 5.12. Let $F_1, \dots, F_n$ fulfil (3.0.2), (3.0.3), (3.0.6), (3.0.7), (3.0.8) and (3.0.9). Let $a_1, a_2(\cdot), \dots, a_n(\cdot)$ be defined as in Definition 5.6 and Definition 5.9. Let us put

$$\Omega_n^o \stackrel{\text{def}}{=} \Big\{(T_1, \dots, T_n) \in E_n \ \text{ such that gradually the following conditions hold}$$
$$a_1 < T_1 < \infty\,,$$
$$a_2(T_1) < T_2 < \infty\,,$$
$$a_3(T_1, T_2) < T_3 < \infty\,,$$
$$\vdots$$
$$a_n(T_1, \dots, T_{n-1}) < T_n < \infty\Big\}\,.$$

Note. $(T_1, \dots, T_{i-1})$ belongs to the domain of functions $a_i(\cdot)$ (for $i \in \{2, 3, \dots \dots, n\}$) and it follows gradually from Lemma 5.11.

Note. We also show an easy algorithm for verifying if $(T_1, \dots, T_n) \in \Omega_n^0$. The whole algorithm follows from facts, which we already have shown in this part. We put on functions $F_1, \dots, F_n$ the same assumptions as we have put in Definition 5.12. Let us have some $\vec{T} = (T_1, \dots, T_n) \in E_n$.

1*st step.* Let $v(\cdot)$ be the solution of the following equation

$$v''(x) \stackrel{\text{a.e.}}{=} \frac{\partial F_1}{\partial u_1^+}(x, 0, \underbrace{0, \dots, 0}_{n-1 \text{ times}}) \cdot \frac{v(x) + |v(x)|}{2}\,,$$
$$v(0) = 1\,, \quad v'(0) = 1\,.$$

From all assumptions it follows that solution $v(\cdot)$ is defined on the whole interval $\langle 0,\infty)$. If $v(\cdot)$ does not have any zero point in the interval $(0,T_1)$ then we need not continue and we know that $\vec{T}\notin\Omega_n^o$. Otherwise we have verified the 1st step.

kth step. Let us assume that we have verified $k-1$ steps of verifying ($2\le k\le n$) and so we have constructed $k-2$ functions $u_1(\cdot),\dots,u_{k-2}(\cdot)$, which fulfil the following system of equations (for $i=1,\dots,k-2$)

$$u_i''(x)\overset{\text{a.e.}}{=}F_i\big(x,u_1(x),\dots,u_i(x),\underbrace{0,\dots,0}_{n-i\ \text{times}}\big)\qquad\text{on interval}\quad\langle 0,\infty)\,,$$

$$u_i(0)=0\,,\qquad u_i(T_i)=0,\qquad \forall\,x\in(0,T_i):\ \ u_i(x)>0\,.$$

Because we have verified the $(k-1)$st step, we know how to construct $u_{k-1}(\cdot)$, in order that it fulfil the system for $i\in\{1,\dots,k-1\}$. Let $v(\cdot)$ be the solution of the following equation

$$v''(x)\overset{\text{a.e.}}{=}\frac{\partial F_k}{\partial u_k^+}\big(x,u_1(x),\dots,u_{k-1}(x),0,\underbrace{0,\dots,0}_{n-k\ \text{times}}\big)\cdot\frac{v(x)+|v(x)|}{2}\,,$$

$$v(0)=1\,,\quad v'(0)=1\,.$$

It follows from our assumptions that $v(\cdot)$ will be defined on the whole interval $\langle 0,\infty)$. If $v(\cdot)$ has no zero point in the interval $(0,T_k)$, then we need not continue and we know that $\vec{T}\notin\Omega_n^o$. Otherwise we have verified the kth step.

If we verify gradually all n steps, then $\vec{T}\in\Omega_n^o$. Otherwise $\vec{T}\notin\Omega_n^o$.

6. Sufficient condition for the existence of a solution

Purpose of this part is to prove the following theorem.

THEOREM 6.1. *Let $F_1,\dots,F_n$ fulfil assumptions* (3.0.2) *to* (3.0.9). *Let Ω_n^o be the domain which has been defined in Definition* 5.12. *Then problem* (3.0.1) *has at least one positive solution for $\forall\,(T_1,\dots,T_n)\in\Omega_n^o$.*

Proof. We shall reduce it to Theorem 6.7, which will be proved in the end of this part and which has stronger assumptions than this theorem. Let $T_{\text{def}}\in(0,\infty)$ be arbitrary but fixed and let us redefine the functions $F_1,\dots,F_n$ in the interval (T_{def},∞) by the following way. Let $c_1(\cdot),\dots,c_n(\cdot)$ be the functions whose existence is guaranteed by assumption (3.0.8) for $T=T_{\text{def}}$. Let Min >0 is chosen such small that the following condition holds

$$\forall\,i\in\{1,\dots,n\}\quad\forall\,\lambda\in\langle 0,\infty)\qquad c_i(\lambda)>\text{Min}\,.$$

Let us now redefine the functions F_i in the interval for all $x \in (T_{\text{def}}, \infty)$ by the following way

$$F_i(x, u_1, \dots, u_i, \dots, u_n) \stackrel{\text{def}}{=} \begin{cases} u_i \cdot (u_i - \text{Min}) & \text{for } u_i \geq 0\,, \\ 0 & \text{for } u_i \leq 0\,. \end{cases}$$

We can easily see that the new redefined functions fulfil not only assumptions (3.0.2) to (3.0.9), but also the following two conditions:

(6.1.1) There exist functions $c_1(\lambda), \dots, c_n(\lambda)$, $c_i(\cdot)\colon \langle 0,\infty) \to (0,\infty)$ such that $\lim\limits_{\lambda\to\infty} c_i(\lambda) = \infty$ for all $i \in \{1, 2, \dots, n\}$
and
$\forall\, k \in \{1,2,\dots,n\}$ $\forall\, x \in \langle 0,\infty)$ $\forall\, \lambda \geq 0$
$\forall\, \vec{u} \in \{\vec{u};\ u_k = c_k(\lambda) \text{ and } 0 \leq u_i \leq c_i(\lambda) \text{ for all } i \in \{1,\dots,n\},\ i \neq k\}$

$$F_k(x, u_1, u_2, \dots, u_n) \geq 0\,.$$

(6.1.2) $\exists$ Min > 0 $\exists\, T_{\text{def}} > 0$
$\forall\, k \in \{1,\dots,n\}$ $\forall\, x > T_{\text{def}}$ $\forall\, u_i \in \mathbb{R}$ $(1 \leq i \leq k-1)$

$$\frac{\partial F_k}{\partial u_k^+}(x, u_1, \dots, u_{k-1}, 0, \underbrace{0,\dots,0}_{n-k \text{ times}}) \leq -\,\text{Min}\,.$$

In Theorem 6.7 just these two assumptions will be added to the set of assumptions which are considered in this theorem and assumption (3.0.8) will be replaced by new assumption (6.1.1). We shall now define two domains Ω_n^o and $\Omega_{n,T_{\text{def}}}^o$ such as in Definition 5.12, where we will use the old functions $F_1, \dots, F_n$ or the new redefined functions $F_1, \dots, F_n$ defined in this part respectively. From definition it follows that

$$\Omega_n^o \cap \langle 0, T_{\text{def}}\rangle^n = \Omega_{n,T_{\text{def}}}^o \cap \langle 0, T_{\text{def}}\rangle^n\,.$$

Let now $(T_1, \dots, T_n) \in \Omega_n^o$ be chosen arbitrarily. If we choose $T_{\text{def}} > \max\limits_{1\leq i\leq n} T_i$, then according to the last identity $(T_1, \dots, T_n) \in \Omega_{n,T_{\text{def}}}^o$, and then by Theorem 6.7 problem (3.0.1) with redefined functions F_i has a positive solution which will be also a solution of the original problem with the old functions because functions F_i have been redefined only in the interval (T_{def}, ∞). So the reduction of this theorem on Theorem 6.7 is done. □

DEFINITION 6.2. Let functions $F_1, \dots, F_n$ fulfil assumptions (3.0.2) to (3.0.7), (3.0.9), (6.1.1) and (6.1.2). Let $a_1, a_2(\cdot), \dots, a_n(\cdot)$ be defined according to Definition 5.6 and Definition 5.9. Then we define the following mapping $\vec{B}\colon E_n^* \to E_n^*$. Let $\vec{T} = (T_1, \dots, T_n) \in E_n^*$.

(1) If $\vec{T} = \infty$, then we put $\vec{B}(\vec{T}) \stackrel{\text{def}}{=} \infty$.

(2) Otherwise $\vec{T} \in E_n$, and step-by-step we shall define

$$
\begin{aligned}
B_1 &:= \delta(T_1 - a_1)\,, & d_1 &:= a_1 + B_1\,,\\
B_2 &:= \delta\big(T_2 - a_2(d_1)\big)\,, & d_2 &:= a_2(d_1) + B_2\,,\\
&\;\vdots & &\;\vdots\\
B_{n-1} &:= \delta\big(T_{n-1} - a_{n-1}(d_1,\dots,d_{n-2})\big), & d_{n-1} &:= a_{n-1}(d_1,\dots,d_{n-2}) + B_{n-1},\\
B_n &:= \delta\big(T_n - a_n(d_1,\dots,d_{n-1})\big)\,,
\end{aligned}
$$

where $\delta(s) \overset{\text{def}}{=} \max\{s,0\}$. And then we put: $\vec{B}(\vec{T}) \overset{\text{def}}{=} (B_1,\dots,B_n)$.

Lemma 6.3. *Let $F_1,\dots,F_n$ fulfil the same assumptions as in Definition* 6.2. *Then the mapping $\vec{B}\colon E_n^* \to E_n^*$ from that definition is correctly defined and is continuous.*

Proof. By definition of mapping $\vec{B}$ and by Lemma 5.11 step-by-step we see that for $i \in \{1,\dots,n-1\}$: $(d_1,\dots,d_i) \in \Omega_i$, which implies correctness of definition $a_{i+1}(d_1,\dots,d_i)$. We just need to eliminate the case that some $a_{k+1}(d_1,\dots,d_k)$ reach value ∞. We even show that $a_{k+1}(d_1,\dots,d_k)$ are bounded on the domain Ω_k for $1 \le k \le n-1$. (Finiteness of a_1 would be shown by simple adaptation of the following method.) So let $(T_1,\dots,T_k) \in \Omega_k$ be arbitrary and let $T_{k+1} \overset{\text{def}}{=} a_{k+1}(T_1,\dots,T_k) \in E_1^*$. Using Definition 5.6 and Definition 5.1 we get the existence of functions $u_i(\cdot)$ $(i \in \{1,\dots,k\})$, $v(\cdot)$ and numbers α_i $(i \in \{1,\dots,k\})$, which fulfil the following system

$$
\begin{aligned}
u_1''(x) &\overset{\text{a.e.}}{=} G_1\big(x, u_1(x), \underbrace{0,\dots,0}_{n-1 \text{ times}}, \alpha_1, \underbrace{0,\dots,0}_{n-1 \text{ times}}\big)\,,\\
u_2''(x) &\overset{\text{a.e.}}{=} G_2\big(x, u_1(x), u_2(x), \underbrace{0,\dots,0}_{n-2 \text{ times}}, \alpha_1, \alpha_2, \underbrace{0,\dots,0}_{n-2 \text{ times}}\big)\,,\\
&\;\vdots \\
u_k''(x) &\overset{\text{a.e.}}{=} G_k\big(x, u_1(x),\dots,u_k(x), \underbrace{0,\dots,0}_{n-k \text{ times}}, \alpha_1,\dots,\alpha_k, \underbrace{0,\dots,0}_{n-k \text{ times}}\big)\,,\\
&u_i(0) = u_i(T_i) = 0\,,\quad u_i'(0) = 1 \qquad \text{for all} \quad i \in \{1,2,\dots,k\}\,,\\
&u_i(x) > 0 \qquad \text{for all} \quad x \in (0,T_i)\,,\\
v''(x) &\overset{\text{a.e.}}{=} \frac{\partial F_{k+1}}{\partial u_{k+1}^+}\big(x, \alpha_1\cdot u_1(x),\dots,\alpha_k\cdot u_k(x), \overbrace{0,\dots,0}^{n-k \text{ times}}\big)\cdot \frac{v(x)+|v(x)|}{2}\,,\\
&v(0) = 0\,,\quad v'(0) = 1\,, \qquad \forall\, x \in (0,T_{k+1}):\ v(x) > 0\,.
\end{aligned}
\tag{6.3.1}
$$

If we define new functions $v_i(\cdot)$ by the following way

$$v_i(\cdot) \stackrel{\text{def}}{=} \alpha_i \cdot u_i(\cdot) \qquad \text{for} \quad i \in \{1,\dots,k\}\,,$$
$$v_i(\cdot) \stackrel{\text{def}}{=} 0\,, \quad \alpha_i \stackrel{\text{def}}{=} 0 \qquad \text{for} \quad i \in \{k+1,\dots,n\}\,,$$

then using the definition (3.2.1) of functions $G_i(\cdot)$ and assumptions (3.0.4) and (3.0.5) we get that system (6.3.1) is transformed to

$$\begin{aligned} v_1''(x) &\stackrel{\text{a.e.}}{=} F_1\big(x, v_1(x),\dots,v_n(x)\big)\,,\\ &\ \vdots\\ v_k''(x) &\stackrel{\text{a.e.}}{=} F_k\big(x, v_1(x),\dots,v_n(x)\big)\,,\\ v_{k+1}''(x) &\stackrel{\text{a.e.}}{=} F_{k+1}\big(x, v_1(x),\dots,v_n(x)\big)\,,\\ &\ \vdots\\ v_n''(x) &\stackrel{\text{a.e.}}{=} F_n\big(x, v_1(x),\dots,v_n(x)\big)\,,\\ \vec{v}(0) &= \vec{0}\,, \quad \vec{v}\,'(0) = \vec{\alpha}\,,\\ v_i(T) \le 0 \qquad &\text{for all} \quad i \in \{1,\dots,n\} \quad \text{where } T \stackrel{\text{def}}{=} \max_{1\le i\le k}(T_i)\,, \end{aligned} \tag{6.3.2}$$

$$\begin{aligned} v''(x) &\stackrel{\text{a.e.}}{=} \frac{\partial F_{k+1}}{\partial u_{k+1}^+}\big(x, v_1(x),\dots,v_n(x)\big) \cdot \frac{v(x)+|v(x)|}{2}\,,\\ v(0) &= 0\,, \quad v'(0) = 1\,, \qquad \forall\, x \in (0, T_{k+1}):\ v(x) > 0\,. \end{aligned} \tag{6.3.3}$$

Now let us apply Lemma 3.5 to system (6.3.2). If we realize that functions $c_1(\cdot),\dots,c_n(\cdot)$, whose existence follows from assumption (6.1.1), fulfil assumption (3.0.8) for arbitrary T, then we get from Lemma 3.5

$$\forall\, x \in \langle 0, \infty) \quad \forall\, i \in \{1,\dots,n\} \qquad v_i(x) < c_i(0)\,. \tag{6.3.4}$$

Let T_{def} be defined by assumption (6.1.2). If we now define the following compact

$$\tilde{K} \stackrel{\text{def}}{=} \big\{(x,\vec{u}) \in (\mathbb{R}_0^+ \times E_n)\,;\ 0 \le x \le T_{\text{def}}\,,\ 0 \le u_i \le c_i(0) \text{ for all } i \in \{1,\dots,n\}\big\} \tag{6.3.5}$$

then according to (3.0.7) we can choose such a sufficiently great constant M that

$$\forall\,(x,\vec{u}) \in \tilde{K}: \qquad \left|\frac{\partial F_{k+1}}{\partial u_{k+1}^+}(x,\vec{u})\right| \le M\,.$$

From this and from (6.3.5) and (6.3.4) we obtain

$$\left|\frac{\partial F_{k+1}}{\partial u_{k+1}^+}\big(x, v_1(x),\dots,v_n(x)\big)\right| \le M \qquad \text{for all} \quad x \in \langle 0, T_{\text{def}}\rangle\,. \tag{6.3.6}$$

Let Min > 0 is chosen according to assumption (6.1.2). Let us define $z(\cdot)$ as the solution of the following equation

$$z''(x) = r(x) \cdot z(x)\,, \quad z(0) = 0\,, \quad z'(0) = 1\,,$$

where (6.3.7)

$$r(x) \stackrel{\text{def}}{=} M \qquad \text{for all } x \in \langle 0, T_{\text{def}} \rangle\,,$$

$$r(x) \stackrel{\text{def}}{=} -\text{Min} \qquad \text{for all } x > T_{\text{def}}\,.$$

By the form of this equation it follows that $z(\cdot)$ has on the interval $(0,\infty)$ a zero-point $\tilde{T}$ such that

$$z(\tilde{T}) = 0 \qquad \text{and} \qquad z(x) > 0 \quad \text{for all } x \in (0, \tilde{T})\,. \tag{6.3.8}$$

Now by contradiction we shall show validity of the following estimate from which the estimation of $a_{k+1}(\cdot)$ follows

$$T_{k+1} = a_{k+1}(T_1, \dots, T_k) \le \tilde{T}\,. \tag{6.3.9}$$

Let us prove it. If

$$\tilde{T} < T_{k+1}\,, \tag{6.3.10}$$

then according to (6.3.3) $v(\cdot)$ fulfils on the interval $\langle 0, \tilde{T} \rangle$

$$\begin{gathered} v''(x) \stackrel{\text{a.e.}}{=} p(x) \cdot v(x) \qquad \text{on} \quad \langle 0, \tilde{T} \rangle\,, \\ v(0) = 0\,, \qquad v'(0) = 1\,, \qquad v(x) > 0 \quad \text{for all } x \in (0, \tilde{T} \rangle\,, \\ \text{where} \qquad p(x) \stackrel{\text{def}}{=} \frac{\partial F_{k+1}}{\partial u_{k+1}^{+}} (x, v_1(x), \dots, v_n(x))\,. \end{gathered} \tag{6.3.11}$$

From estimate (6.3.6), definition (6.3.7) of $r(\cdot)$ and assumption (6.1.2) the inequality

$$p(x) \le r(x) \qquad \text{for all} \quad x \in \langle 0, \tilde{T} \rangle \tag{6.3.12}$$

follows. Now we shall use Lemma 2.4, where we put $f(x,u) = p(x) \cdot u$, $g(x,v) = r(x) \cdot v$, $u(\cdot) = v(\cdot)$ and $v(\cdot) = z(\cdot)$. By (6.3.12), (6.3.7), (6.3.8) and (6.3.11) then assumptions of Lemma 2.4 follow. Then Lemma 2.4 implies, that $v(\cdot)$ has a zero-point on the interval $(0, \tilde{T} \rangle$, what gives us the contradiction with (6.3.10). So assertion (6.3.9) holds, from which the correctness of Definition 6.2 follows. We should verify else the continuity of the mapping $\vec{B}(\cdot)$.

(1) Continuity at points $\vec{T} \in E_n$ follows from its correct definition and from continuity of mappings $a_k(\cdot)$, which is proved in Lemma 5.7.

(2) Continuity at point $\vec{T} = \infty \in E_n^*$ easily follows from definition of mapping $\vec{B}(\cdot)$ and from proved global estimation of mappings $a_k(\cdot)$.

□

Lemma 6.4. *Let $F_1, \dots, F_n$ fulfil* (3.0.2) *to* (3.0.7), (3.0.9), (6.1.1) *and* (6.1.2). *Let the mappings $\vec{T}(\cdot)$ and $\vec{B}(\cdot)$ be defined as in Definition* 3.2 *and Definition* 6.2, *respectively. Then the mapping $\vec{B}\circ\vec{T}\colon E_n^* \to E_n^*$ is continuous and has the following properties:*

(1) $\vec{B}\circ\vec{T}(\infty) = \infty$.
(2) *If $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in E_n$ and if i_0 exists such that $\alpha_{i_0} = 0$, then either $\vec{B}\circ\vec{T}(\vec{\alpha}) = \infty$ or if $\vec{B}\circ\vec{T}(\vec{\alpha}) =: (B_1, \dots, B_n) \in E_n$, then $B_{i_0} = 0$.*

Proof. The continuity of the mapping follows from Theorem 4.1 and from Lemma 6.3. The first property follows similarly from definitions of both mappings. To prove the second property let us assume

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in E_n, \qquad \vec{T}(\vec{\alpha}) =: \vec{T} = (T_1, \dots, T_n) \in E_n,$$
$$\vec{B}(\vec{T}) =: \vec{B} = (B_1, \dots, B_n) \in E_n.$$

Let, according to assumptions, $\exists k \in \{0, 1, \dots, n-1\}$ such that $\alpha_{k+1} = 0$. Now we shall show that $B_{k+1} = 0$. Assumption $\vec{T}(\vec{\alpha}) = \vec{T}$ implies according to Definition 3.2 that $u(\cdot)$ fulfils the system

$$\begin{gathered} u_i''(x) \overset{\text{a.e.}}{=} G_i\big(x, u_1(x), \dots, u_n(x), \alpha_1, \dots, \alpha_n\big) \qquad \text{for} \quad i \in \{1, \dots, n\}, \\ u_i(0) = 0, \qquad u_i'(0) = 1, \qquad u_i(T_i) = 0, u_i(x) > 0 \quad \text{for all } x \in (0, T_i). \end{gathered} \tag{6.4.1}$$

If we use $\alpha_{k+1} = 0$, the $(k+1)$th equation transforms by using (3.2.1) and (3.0.5) to the following form

$$\begin{gathered} u_{k+1}''(x) \overset{\text{a.e.}}{=} \frac{\partial F_{k+1}}{\partial u_{k+1}^+}\big(x, \alpha_1 \cdot u_1(x), \dots, \alpha_k \cdot u_k(x), \underbrace{0, \dots, 0}_{n-k \text{ times}}\big) \cdot \frac{u_{k+1}(x) + |u_{k+1}(x)|}{2}, \\ u_{k+1}(0) = 0, \qquad u_{k+1}'(0) = 1, \qquad u_{k+1}(T_{k+1}) = 0, \\ u_{k+1}(x) > 0 \qquad \text{for all} \quad x \in (0, T_{k+1}). \end{gathered} \tag{6.4.2}$$

When we use $\alpha_{k+1} = 0$ and (3.2.1) and (3.0.4), we obtain for all i, $1 \le i \le k$, and for all $\beta > 0$ the following identity

$$\begin{aligned} &G_i(x, u_1, \dots, u_n, \alpha_1, \dots, \alpha_{i-1}, \beta, \alpha_{i+1}, \dots, \alpha_n) \\ &\qquad = G_i(x, u_1, \dots, u_i, \underbrace{0, \dots, 0}_{n-i \text{ times}}, \alpha_1, \dots, \alpha_{i-1}, \beta, \underbrace{0, \dots, 0}_{n-i \text{ times}}). \end{aligned}$$

Lemma 3.3 implies continuity of the mapping G_i in β, so the previous identity holds also for $\forall \beta \ge 0$, what we can obtain when $\beta \to 0^+$ in the last identity.

Then from (6.4.1) it follows that

$$u_i''(x) \overset{\text{a.e.}}{=} G_i\big(x, u_1(x), \dots, u_i(x), \overbrace{0, \dots, 0}^{n-i \text{ times}}, \alpha_1, \dots, \alpha_i, \overbrace{0, \dots, 0}^{n-i \text{ times}}\big) \quad \text{for} \quad i \in \{1, \dots, k\}, \tag{6.4.3}$$

$$u_i(0) = 0, \qquad u_i'(0) = 1, \qquad u_i(T_i) = 0, \qquad u_i(x) > 0 \quad \text{for all } x \in (0, T_i).$$

Let us now consider two following cases:

1. If possibility $k \geq 1$ occurs, then from Definition 5.1 equation (6.4.3) implies that

$$\vec{R}_k(\alpha_1, \dots, \alpha_k) = (T_1, \dots, T_k). \tag{6.4.4}$$

Using this, from Definition 5.6 and from equation (6.4.2) we get that

$$a_{k+1}(T_1, \dots, T_k) = T_{k+1}. \tag{6.4.5}$$

Now from (6.4.4) it follows by Definition 5.4 that $(T_1, \dots, T_k) \in \Omega_k$, what, together with Lemma 5.11, implies

$$\begin{aligned} a_1 &\leq T_1, \\ a_2(T_1) &\leq T_2, \\ &\vdots \\ a_k(T_1, \dots, T_{k-1}) &\leq T_k. \end{aligned}$$

When we use these inequalities in the definition of the mapping $\vec{B}(\cdot)$ (Definition 6.2), step-by-step we get that $d_1 = T_1$, $d_2 = T_2$, …, $d_k = T_k$ from what it follows that

$$B_{k+1} = \delta\big(T_{k+1} - a_{k+1}(d_1, \dots, d_k)\big) = \delta\big(T_{k+1} - a_{k+1}(T_1, \dots, T_k)\big) \overset{(6.4.5)}{=} 0$$

what we needed to prove.

2. If possibility $k = 0$ occurs, then from (6.4.2) and from Definition 5.9 we get that $T_1 = a_1$, what by Definition 6.2 implies that $B_1 = \delta(T_1 - a_1) = 0$. □

Lemma 6.5. *A mapping $\mathcal{M}\colon E_n^* \to \mathbb{R}^n$ exists such that it fulfils the following conditions:*

(1) *$\mathcal{M}$ is one-to-one mapping from E_n^* to $B_n(0,1) \overset{\text{def}}{=} \{\vec{x} \in \mathbb{R}^n ;\ \|\vec{x}\| \leq 1\}$.*
(2) *$\mathcal{M}\colon E_n^* \to B_n(0,1)$ is continuous.*
(3) *$\mathcal{M}^{-1}\colon B_n(0,1) \to E_n^*$ is continuous.*
(4) *$\mathcal{M}(E_{n,0}^*) = S_n(0,1) \overset{\text{def}}{=} \{\vec{x} \in \mathbb{R}^n ;\ \|\vec{x}\| = 1\}$.*
(5) *If $\vec{z}, \vec{z}\,' \in S_n(0,1)$ $\vec{z} = -\vec{z}\,'$ and $\{\vec{z}, \vec{z}\,'\} \neq \{\mathcal{M}(0), \mathcal{M}(\infty)\}$ then if we put $\vec{x} := \mathcal{M}^{-1}(\vec{z})$ and $\vec{x}\,' := \mathcal{M}^{-1}(\vec{z}\,')$, for all $i \in \{1, \dots, n\}$ it holds that $|x_i| + |x_i{}'| > 0$.*

P r o o f . We can define $\mathcal{M}$ for example by the following way.

(1) $\mathcal{M}(\underbrace{0,\dots,0}_{n \text{ times}}) \stackrel{\text{def}}{=} (\underbrace{0,\dots,0}_{n-1 \text{ times}}, -1)$.

(2) $\mathcal{M}(\infty) \stackrel{\text{def}}{=} (\underbrace{0,\dots,0}_{n-1 \text{ times}}, 1)$.

(3) Let $\vec{x} = (x_1,\dots,x_n) \in E_n$, $\vec{x} \neq 0$. We want to define $\mathcal{M}(\vec{x}) =: (z_1,\dots,z_n)$. Let us define gradually $s := \frac{x_1+x_2+\cdots+x_n}{n}$ (evidently $s > 0$), $z_n := \frac{\operatorname{arctg} s}{\pi/4} - 1$, $y_i := s - x_i$ for all $i \in \{1,\dots,n\}$ (evidently $\sum_{i=1}^{n} y_i = 0$).

   (a) If $y_1 = y_2 = \cdots = y_{n-1} = 0$ (what implies $y_n = 0$), then we put $z_1 := z_2 := \cdots := z_{n-1} := 0$.

   (b) Otherwise, if at least one $y_i \neq 0$, then we put $r := \frac{\max\{y_1,\dots,y_n\}}{s}$ (evidently $\sum_{i=1}^{n} y_i = 0$ and $s > 0 \implies r > 0$). $z_i := r \cdot \frac{y_i}{\sqrt{y_1{}^2+y_2{}^2+\cdots+y_{n-1}{}^2}} \cdot \sqrt{1 - z_n{}^2}$ for all $i \in \{1,\dots,n-1\}$.

   And now we put $\mathcal{M}(\vec{x}) \stackrel{\text{def}}{=} \vec{z} = (z_1,\dots,z_n)$.

It is easy to check that so defined mapping fulfils all conditions we put on it. □

LEMMA 6.6. *Let $F_1,\dots,F_n$ fulfil* (3.0.2) *to* (3.0.7), (3.0.9), (6.1.1) *and* (6.1.2). *Let $\vec{T}$, $\vec{B}$ be defined, by Definitions* 3.2, 6.2, *respectively. Then $\vec{B} \circ \vec{T}(E_{n,+}) \supset E_{n,+}$ holds.*

P r o o f . According to Lemma 6.5 it shall be sufficient for us to prove the following assertions:

(6.6.1) $\mathcal{M} \circ \vec{B} \circ \vec{T} \circ \mathcal{M}^{-1}(G_n(0,1)) \supset G_n(0,1)$
where $G_n(0,1) \stackrel{\text{def}}{=} \{\vec{z} \in \mathbb{R}^n ;\ \|\vec{z}\| < 1\}$ and where $\mathcal{M}$ is defined in Lemma 6.5.

Let us define for this purpose the following mapping $\mathcal{F}\colon B_n(0,1) \to B_n(0,1)$

$$\mathcal{F}(\vec{z}) \stackrel{\text{def}}{=} \mathcal{M} \circ \vec{B} \circ \vec{T} \circ \mathcal{M}^{-1}(\vec{z}) \qquad \text{for} \quad \vec{z} \in B_n(0,1). \tag{6.6.2}$$

Now we shall prove the following properties of the mapping $\mathcal{F}$:

(6.6.3) $\mathcal{F}\colon B_n(0,1) \to B_n(0,1)$ is continuous.
(6.6.4) $\mathcal{F}(S_n(0,1)) \subset S_n(0,1)$.
(6.6.5) $\forall \vec{z} \in S_n(0,1):\quad \mathcal{F}(\vec{z}) \neq -\vec{z}$.

Let us prove it:

(1) From Lemmas 6.4 and 6.5 (points 1, 2 and 3) we obtain that $\mathcal{F}$ as composition of continuous mappings is continuous, too.

(2) Property (6.6.4) also follows from Lemmas 6.4 and 6.5 (statement 4).

(3) Let us prove the last property (6.6.5). We shall divide the proof to the following cases:

(a) If $\vec{z} = \mathcal{M}(\infty) = (\overbrace{0, \dots, 0}^{n-1 \text{ times}}, 1)$, then from Lemma 6.4 we obtain $\mathcal{F}(\vec{z}) = \mathcal{M} \circ \vec{B} \circ \vec{T} \circ \mathcal{M}^{-1}(\mathcal{M}(\infty)) = \mathcal{M}(\infty) = \vec{z} \neq -\vec{z}$ what had to be proved.

(b) If $\vec{z} = \mathcal{M}(\vec{0}) = (\overbrace{0, \dots, 0}^{n-1 \text{ times}}, -1)$, then the proof of condition $\mathcal{F}(\vec{z}) \neq -\vec{z} = \mathcal{M}(\infty)$ ($\iff \vec{B} \circ \vec{T}(\vec{0}) \neq \infty \in E_n^*$) is sufficient in this case. And for this it is sufficient to show condition $\vec{T}(\vec{0}) \neq \infty$, because from definition of the mapping $\vec{B}$ (Definition 6.2) it follows that $\infty \notin \vec{B}(E_{n,0})$. Let us prove it: According to Definition 3.2 we obtain that the system (3.2.2) for $\vec{\alpha} = \vec{0}$ transforms to

$$u_i''(x) = \frac{\partial F_i}{\partial u_i^+}(x, \overbrace{0, \dots, 0}^{n \text{ times}}) \cdot \frac{u_i(x) + |u_i(x)|}{2},$$
$$u_i(0) = 0, \quad u_i'(0) = 1 \qquad \text{for all} \quad i \in \{1, \dots, n\}.$$

By assumption (6.1.2) it follows that some numbers Min, T_{def} exist such that

$$\forall x > T_{\text{def}} \qquad \frac{\partial F_i}{\partial u_i^+}(x, \underbrace{0, \dots, 0}_{n \text{ times}}) \leq \text{Min} < 0.$$

From this it follows by use of comparison theorems that all $u_i(\cdot)$ must have a zero point somewhere on the interval $(0, \infty)$ and therefore $\vec{T}(\vec{0}) \neq \infty$, what we had to prove.

(c) If $\vec{z} \notin \{\mathcal{M}(\vec{0}), \mathcal{M}(\infty)\}$, then validity of (6.6.5) will be shown by contradiction. Let $\mathcal{F}(\vec{z}) = -\vec{z}$. Let us define $\vec{x} := \mathcal{M}^{-1}(\vec{z})$ $\vec{x}\,' := \mathcal{M}^{-1}(\mathcal{F}(\vec{z})) = \mathcal{M}^{-1}(-\vec{z})$. (It is evident that $\vec{x}, \vec{x}\,' \in E_{n,0}$). Then $\vec{B} \circ \vec{T}(\vec{x}) = \mathcal{M}^{-1}(\mathcal{F}(\vec{z})) = \vec{x}\,'$. Because $\vec{x} \in E_{n,0} \implies \exists k \in \{1, \dots, n\} : \ x_k = 0$, and therefore from previous assertions it follows, when we use Lemma 6.4, that $x_k' = 0$, what together with

$x_k = 0$ by using previous definitions of $\vec{x}$, $\vec{x}\,'$ and Lemma 6.5 (statement 5), gives us the contradiction.

So property (6.6.5) is showed, too.

Let us return to the proof of (6.6.1). Here we shall use the Brouwer degree theory as it is given in [6]. Let d be the Brouwer degree of mapping. Let us define the first homotopy $h_1(t,\vec{z})\colon \langle 0,1\rangle \times B_n(0,1) \to B_n(0,1)$ by the following way

$$h_1(t,\vec{z}) \stackrel{\text{def}}{=} t\cdot \mathcal{F}(\vec{z}) + (1-t)\cdot \vec{z}\,.$$

This homotopy connects $\mathcal{F}$ (for $t=1$) with identity mapping for $t=0$. Its continuity follows from (6.6.3). Let us verify that

$$\forall\, t \in \langle 0,1\rangle \quad \forall\, \vec{z} \in S_n(0,1): \qquad h_1(t,\vec{z}) \neq \vec{0}\,.$$

By contradiction let us assume that for some t, $\vec{z}$ it holds that $t\cdot\mathcal{F}(\vec{z}) = (t-1)\cdot\vec{z}$. When we use (6.6.4), we obtain $t = |t-1| = 1-t \implies t = 1/2$ and therefore we obtain $1/2\cdot\mathcal{F}(\vec{z}) = -1/2\cdot\vec{z}$, what by (6.6.5) implies contradiction. Now by the homotopy property we obtain

$$\mathrm{d}\big(\mathcal{F},\vec{0},G_n(0,1)\big) = \mathrm{d}\big(I,\vec{0},G_n(0,1)\big) = 1\,. \tag{6.6.6}$$

Let us choose an arbitrary point $\vec{z}_0 \in G_n(0,1)$ ($\|\vec{z}_0\| < 1$). Let us define the second homotopy $h_2(t,\vec{z})\colon \langle 0,1\rangle \times B_n(0,1) \to B_n(0,2)$

$$h_2(t,\vec{z}) \stackrel{\text{def}}{=} \mathcal{F}(\vec{z}) - t\cdot\vec{z}_0\,.$$

Its continuity follows from (6.6.3). Let us verify that

$$\forall\, t \in \langle 0,1\rangle \quad \forall\, \vec{z} \in S_n(0,1): \qquad h_2(t,\vec{z}) \neq \vec{0}\,.$$

By the contradiction we can obtain for some t, $\vec{z}$ that

$$\mathcal{F}(\vec{z}) = t\cdot\vec{z}_0\,.$$

Using (6.6.4), we get

$$1 = \|\mathcal{F}(\vec{z})\| = t\cdot\|\vec{z}_0\| \le \|\vec{z}_0\| < 1$$

what gives us the required contradiction. So, when we use second homotopy, we get

$$\mathrm{d}\big(\mathcal{F}(\cdot),\vec{z}_0,G_n(0,1)\big) = \mathrm{d}\big(\mathcal{F}(\cdot)-\vec{z}_0,\vec{0},G_n(0,1)\big) = \mathrm{d}\big(\mathcal{F}(\cdot),\vec{0},G_n(0,1)\big) \stackrel{(6.6.6)}{=} 1\,.$$

By the properties of the degree, we obtain that $\vec{z}_0 \in \mathcal{F}\big(G_n(0,1)\big)$. Because $\vec{z}_0$ was arbitrary, we get $G_n(0,1) \subset \mathcal{F}\big(G_n(0,1)\big)$, from what required (6.6.1) follows. □

THEOREM 6.7. *Let $F_1, \dots, F_n$ fulfil* (3.0.2) *to* (3.0.7), (3.0.9), (6.1.1) *and* (6.1.2). *Let Ω_n^o be the domain defined in Definition* 5.12. *Then problem* (3.0.1) *has for all $(T_1, \dots, T_n) \in \Omega_n^o$ at least one positive solution.*

Proof. Let $\hat{\vec{T}} = (\hat{T}_1, \dots, \hat{T}_n) \in \Omega_n^o$ be arbitrary, but fixed. From Definition 5.12 it follows that the following sequence of definitions is correct.

$$\begin{aligned} \hat{B}_1 &:= \hat{T}_1 - a_1 , \\ \hat{B}_2 &:= \hat{T}_2 - a_2(\hat{T}_1) , \\ &\vdots \\ \hat{B}_n &:= \hat{T}_n - a_n(\hat{T}_1, \dots, \hat{T}_{n-1}) . \end{aligned} \tag{6.7.1}$$

By Definition 5.12 we also obtain that $\forall i \in \{1, \dots, n\}$: $\hat{B}_i > 0 \implies \hat{\vec{B}} := (\hat{B}_1, \dots, \hat{B}_n) \in E_{n,+}$. Then from Lemma 6.6 we get $\exists \vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in E_{n,+}$ such that $\vec{B} \circ \vec{T}(\vec{\alpha}) = \hat{\vec{B}}$. If we now put $\vec{T} = (T_1, \dots, T_n) := \vec{T}(\vec{\alpha})$, then when we rewrite the identity $\hat{\vec{B}} = \vec{B}(\vec{T})$ with help of Definition 6.2, we gradually obtain

$$\begin{array}{ll} 0 < \hat{B}_1 = \delta(T_1 - a_1) = T_1 - a_1 & d_1 = a_1 + \hat{B}_1 = T_1 \\ 0 < \hat{B}_2 = \delta(T_2 - a_2(T_1)) = T_2 - a_2(T_1) & d_2 = T_2 \\ \vdots & \vdots \\ 0 < \hat{B}_{n-1} = \delta(T_{n-1} - a_{n-1}(T_1, \dots, T_{n-2})) = T_{n-1} - a_{n-1}(T_1, \dots, T_{n-2}) & d_{n-1} = T_{n-1} \\ 0 < \hat{B}_n = \delta(T_n - a_n(T_1, \dots, T_{n-1})) = T_n - a_n(T_1, \dots, T_{n-1}) . & \end{array}$$

From this and from Definition (6.7.1) we successively obtain

$$\begin{aligned} \hat{T}_1 &= \hat{B}_1 + a_1 = (T_1 - a_1) + a_1 = T_1 , \\ \hat{T}_2 &= \hat{B}_2 + a_2(\hat{T}_1) = \big(T_2 - a_2(T_1)\big) + a_2(T_1) = T_2 , \\ &\vdots \\ \hat{T}_n &= \hat{B}_n + a_n(\hat{T}_1, \dots, \hat{T}_{n-1}) = \big(T_n - a_n(T_1, \dots, T_{n-1})\big) + a_n(T_1, \dots, T_{n-1}) = T_n . \end{aligned}$$

So $\vec{T}(\vec{\alpha}) = \vec{T} = \hat{\vec{T}}$, what from Definition 3.2 implies, that problem (3.0.1) has for $T_i = \hat{T}_i$ solution, what was to be proved. □

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Department of Mathematical Analysis
Comenius University
Mlynská dolina
SK-842 15 Bratislava
SLOVAKIA
E-mail: Ilja.Martisovits@fmph.uniba.sk