Mathematica Slovaca

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Mathematica Slovaca, Vol. 49 (1999), No. 4, 425--431

Persistent URL: http://dml.cz/dmlcz/136758

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Math. Slovaca, **49** (1999), No. 4, 425–431

Mathematica Slovaca
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SPECHT MODULES FOR FINITE GROUPS

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(*Communicated by Tibor Katriňák*)

ABSTRACT. The construction of all the irreducible modules of the symmetric groups over an arbitrary field is given by G. D. James. The present author and Morris describe a possible extension of James' work for Weyl groups in general, where Young tableaux are interpreted in terms of root systems. In this paper, we further develop the theory and give a possible extension of this construction for finite groups in general.

1. Introduction

There are well-known constructions of the irreducible representations and of the irreducible (Specht) modules, for the symmetric groups S_n which are based on elegant combinatorial concepts connected with Young tableaux and tabloids (see, e.g. [5]). Al-Aamily, Morris and Peel [1] showed how this construction could be extended to deal with the Weyl groups of type B_n. In [6] Morris described a possible extension of James' work for Weyl groups in general. An alternative and improved approach was given by the present author and Morris [3], where Young tableaux and tabloids are interpreted in terms of root systems. Recently, an extension of this construction for finite reflection groups has been given in [4]. In this paper, we further develop the theory and describe a possible extension of this construction for finite groups in general.

2. Specht modules for finite groups

Let $\mathcal{G}$ be a finite group and $\mathcal{H}$ be a subgroup of $\mathcal{G}$ such that $[\mathcal{G} : \mathcal{H}] = 2$. If $g \in \mathcal{G}$, then the sign of g

$$s(g) = \begin{cases} -1 & \text{if } g \notin \mathcal{H}\,, \\ 1 & \text{if } g \in \mathcal{H}\,. \end{cases}$$

AMS Subject Classification (1991): Primary 20C30, 05E10.
Key words: Specht module, tableau, tabloid, finite group.

Let $\mathcal{W}$ be a subgroup of $\mathcal{G}$. Define equivalence relation on the elements of $\mathcal{G}$:

$$g \sim g' \iff g^{-1}g' \in \mathcal{W} \text{ for } g, g' \in \mathcal{G}.$$

Let $g \in \mathcal{G}$. The equivalence class $\{g\mathcal{W}\}$ is called a $\mathcal{W}$*-tabloid.* Let $\tau_{\mathcal{W}}$ be the set of all $\mathcal{W}$-tabloids. It is clear that the number of distinct elements in $\tau_{\mathcal{W}}$ is $[\mathcal{G} : \mathcal{W}]$. If $L_{\mathcal{W}}$ is the set of left coset representatives of $\mathcal{W}$ in $\mathcal{G}$, then we have

$$\tau_{\mathcal{W}} = \left\{\{g\mathcal{W}\} \mid g \in L_{\mathcal{W}}\right\}.$$

The group $\mathcal{G}$ acts on $\tau_{\mathcal{W}}$ according to

$$g'\{g\mathcal{W}\} = \{g'g\mathcal{W}\} \qquad \text{for all} \quad g' \in \mathcal{G}.$$

This action is easily seen to be well defined.

Now if K is arbitrary field, let $M^{\mathcal{W}}$ be the K-space whose basis elements are $\mathcal{W}$-tabloids. Extend the action of $\mathcal{G}$ on $\tau_{\mathcal{W}}$ linearly on $M^{\mathcal{W}}$, then $M^{\mathcal{W}}$ becomes $K\mathcal{G}$-module. Then we have the following theorem.

THEOREM 2.1. *The $K\mathcal{G}$-module $M^{\mathcal{W}}$ is the permutation module on the subgroup $\mathcal{W}$. $M^{\mathcal{W}}$ is a cyclic $K\mathcal{G}$-module generated by any one $\mathcal{W}$-tabloid and* $\dim_K M^{\mathcal{W}} = [\mathcal{G} : \mathcal{W}]$.

Proof. If $m \in M^{\mathcal{W}}$, then

$$m = \sum_{d \in L_{\mathcal{W}}} \alpha_d \{d\mathcal{W}\}, \qquad \text{where} \quad \alpha_d \in K.$$

But

$$m = \Big(\sum_{d \in L_{\mathcal{W}}} \alpha_d d\Big)\{\mathcal{W}\}.$$

Then we have $M^{\mathcal{W}} = KG\{\mathcal{W}\}$. □

If $\mathcal{W}_1$ and $\mathcal{W}_2$ are conjugate subgroups of $\mathcal{G}$, then $[\mathcal{G} : \mathcal{W}_1] = [\mathcal{G} : \mathcal{W}_2]$ and $M^{\mathcal{W}_1} \cong M^{\mathcal{W}_2}$.

Now we proceed to consider the possibility of constructing a $K\mathcal{G}$-module $S^{\mathcal{W},\mathcal{W}'}$ which generalizes the Specht modules in the case of symmetric groups. In this direction we first define a useful dual of a subgroup $\mathcal{W}$.

DEFINITION 2.2. A *useful dual* of $\mathcal{W}$ is a subgroup $\mathcal{W}'$ of $\mathcal{G}$ which satisfies $\mathcal{W} \cap \mathcal{W}' = \langle e \rangle$.

Then we have the following lemma.

LEMMA 2.3. *If $\mathcal{W}'$ is a useful dual of $\mathcal{W}$, then $w\mathcal{W}'w^{-1}$ is also a useful dual of $\mathcal{W}$ for all $w \in \mathcal{W}$.*

Proof. If $w \in \mathcal{W}$ and $x \in \mathcal{W} \cap w\mathcal{W}'w^{-1}$, then $x \in \mathcal{W}$ and $x \in w\mathcal{W}'w^{-1}$. So $x = wx'w^{-1}$, which implies $x \in \mathcal{W} \cap \mathcal{W}'$ and so $x = e$. □

DEFINITION 2.4. Let $\mathcal{W}'$ be a useful dual of $\mathcal{W}$. Let

$$\kappa_{\mathcal{W}'} = \sum_{\sigma \in \mathcal{W}'} s(\sigma)\sigma \qquad \text{and} \qquad e_{\mathcal{W},\mathcal{W}'} = \kappa_{\mathcal{W}'}\{\mathcal{W}\}.$$

Then $e_{\mathcal{W},\mathcal{W}'}$ is called a *generalized $\mathcal{W}$-polytabloid.*

Let $S^{\mathcal{W},\mathcal{W}'}$ be the subspace of $M^{\mathcal{W}}$ spanned by all the generalized $\mathcal{W}$-polytabloids $ge_{\mathcal{W},\mathcal{W}'}$, $g \in \mathcal{G}$. By the same method as in [3], $S^{\mathcal{W},\mathcal{W}'}$ is a $K\mathcal{G}$-submodule of $M^{\mathcal{W}}$, which is called a *generalized Specht module.*

It is clear that if $\mathcal{W}' = \langle e \rangle$, then $S^{\mathcal{W},\mathcal{W}'} \cong M^{\mathcal{W}}$.

THEOREM 2.5. *$S^{\mathcal{W},\mathcal{W}'}$ is a cyclic submodule generated by any $\mathcal{W}$-polytabloid.*

A Specht module is spanned by the $ge_{\mathcal{W},\mathcal{W}'}$ for all $g \in \mathcal{G}$; the next lemma shows that we need only consider a certain subset of $\mathcal{G}$.

LEMMA 2.6. *Let $\mathcal{W}$ be a subgroup of $\mathcal{G}$ and let $\mathcal{W}'$ be a useful dual of $\mathcal{W}$. Let g be a left coset representative of $\mathcal{W}'$ in $\mathcal{G}$. Then $S^{\mathcal{W},\mathcal{W}'}$ is spanned by $e_{g\mathcal{W},g\mathcal{W}'}$.*

Proof. See [3; Lemma 3.10]. □

LEMMA 2.7. *Let $\mathcal{W}'$ be a useful dual of $\mathcal{W}$ and $g \in L_{\mathcal{W}}$. If $\{g\mathcal{W}\}$ appears in $e_{\mathcal{W},\mathcal{W}'}$ then it appears only once.*

Proof. If $\sigma, \sigma' \in \mathcal{W}'$ and suppose that $\sigma = gw$, $\sigma' = gw'$ where $w, w' \in \mathcal{W}$. Then $g = \sigma w^{-1} = \sigma' {w'}^{-1}$ and ${\sigma'}^{-1}\sigma = {w'}^{-1}w \in \mathcal{W} \cap \mathcal{W}' = \langle e \rangle$. Hence we have $w = w'$ and $\sigma = \sigma'$. Then $\{g\mathcal{W}\}$ appears in $e_{\mathcal{W},\mathcal{W}'}$ only once. □

COROLLARY 2.8. *If $\mathcal{W}'$ is a useful dual of $\mathcal{W}$, then $e_{\mathcal{W},\mathcal{W}'} \neq 0$.*

Proof. By Lemma 2.7, if $\{\sigma\mathcal{W}\}$ appears in $e_{\mathcal{W},\mathcal{W}'}$, then all the $\{\sigma\mathcal{W}\}$ are different, where $\sigma \in \mathcal{W}'$. But $\{\{\sigma\mathcal{W}\} \mid \sigma \in \mathcal{W}'\}$ is a linearly independent subset of $\{\{g\mathcal{W}\} \mid g \in L_{\mathcal{W}}\}$. If $e_{\mathcal{W},\mathcal{W}'} = 0$ then $s(\sigma) = 0$ for all $\sigma \in \mathcal{W}'$. This is a contradiction and so $e_{\mathcal{W},\mathcal{W}'} \neq 0$. □

LEMMA 2.9. *If there exists $w \in \mathcal{W} \cap \mathcal{W}'$ such that w has order 2, and $s(w) = -1$ then $e_{\mathcal{W},\mathcal{W}'} = 0$.*

Proof. If $w \in \mathcal{W} \cap \mathcal{W}'$ and w has order 2, then

$$(e - w)\{\mathcal{W}\} = \{\mathcal{W}\} - \{\mathcal{W}\} = 0$$

and also $\{e, w\}$ is a subgroup of $\mathcal{W}'$. Thus we can select signed coset representatives $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_s$ for $\{e, w\}$ in $\mathcal{W}'$ such that

$$e_{\mathcal{W},\mathcal{W}'} = \sum_{\sigma \in \mathcal{W}'} s(\sigma)\sigma\{\mathcal{W}\} = \left(\sum_{i=1}^{s} \sigma_i\right)(e - w)\{\mathcal{W}\} = 0.$$

□

LEMMA 2.10. *Let $\mathcal{W}_1'$ and $\mathcal{W}_2'$ be useful duals of $\mathcal{W}$. If $\mathcal{W}_1'$ is a subgroup of $\mathcal{W}_2'$, then $S^{\mathcal{W},\mathcal{W}_2'}$ is a $K\mathcal{G}$-submodule of $S^{\mathcal{W},\mathcal{W}_1'}$.*

Proof. Since $\mathcal{W}_1'$ is a subgroup of $\mathcal{W}_2'$, we can select left coset representatives $a_1, a_2, \dots, a_n$ for $\mathcal{W}_1'$ in $\mathcal{W}_2'$ such that $\mathcal{W}_2' = \bigcup_{i=1}^{n} a_i\mathcal{W}_1'$. Then we have

$$\begin{aligned} e_{\mathcal{W},\mathcal{W}_2'} &= \sum_{\sigma\in\mathcal{W}_2'} s(\sigma)\sigma\{\mathcal{W}\} \\ &= \sum_{i=1}^{n} a_i s(a_i) \sum_{\sigma\in\mathcal{W}_1'} s(\sigma)\sigma\{\mathcal{W}\} \\ &= \left(\sum_{i=1}^{n} a_i s(a_i)\right) e_{\mathcal{W},\mathcal{W}_1'} . \end{aligned}$$

□

Now we can consider under what conditions $S^{\mathcal{W},\mathcal{W}'}$ is irreducible.

LEMMA 2.11. *If $g \in L_{\mathcal{W}}$ and $\mathcal{W}'$ is a useful dual of $\mathcal{W}$, then the following conditions are equivalent:*

(i) *$\{g\mathcal{W}\}$ appears with non-zero coefficient in $e_{\mathcal{W},\mathcal{W}'}$.*
(ii) *There exists $\sigma \in \mathcal{W}'$ such that $\sigma\{\mathcal{W}\} = \{g\mathcal{W}\}$.*
(iii) *There exist $\rho \in \mathcal{W}$, $\sigma \in \mathcal{W}'$ such that $g = \sigma\rho$.*

Proof. The equivalence of (i) and (ii) follows directly from the definition of $e_{\mathcal{W},\mathcal{W}'}$.

(ii) $\Longrightarrow$ (iii) Suppose that there exists $\sigma \in \mathcal{W}'$ such that $\sigma\{\mathcal{W}\} = \{g\mathcal{W}\}$. Then we have $\sigma^{-1}g\{\mathcal{W}\} = \{\mathcal{W}\}$. By the definition of equivalence, $\sigma^{-1}g \in \mathcal{W}$ and there exists $\rho \in \mathcal{W}$ such that $\sigma^{-1}g = \rho$. Hence $g = \sigma\rho$, where $\sigma \in \mathcal{W}'$ and $\rho \in \mathcal{W}$.

(iii) $\Longrightarrow$ (ii) If $g = \sigma\rho$, then since $\rho \in \mathcal{W}$, $\rho\{\mathcal{W}\} = \{\mathcal{W}\}$ and $\{g\mathcal{W}\} = \{\sigma\mathcal{W}\}$.

□

DEFINITION 2.12. A useful dual $\mathcal{W}'$ of $\mathcal{W}$ is called a *good dual* of $\mathcal{W}$ if $\kappa_{\mathcal{W}'}\{g\mathcal{W}\} \neq 0$, then $\{g\mathcal{W}\}$ appears in $e_{\mathcal{W},\mathcal{W}'}$, where $g \in L_{\mathcal{W}}$.

LEMMA 2.13. *Let $\mathcal{W}'$ be a good dual of $\mathcal{W}$ and $g \in L_{\mathcal{W}}$.*

(i) *If $\{g\mathcal{W}\}$ does not appear in $e_{\mathcal{W},\mathcal{W}'}$, then $\kappa_{\mathcal{W}'}\{g\mathcal{W}\} = 0$.*
(ii) *If $\{g\mathcal{W}\}$ appears in $e_{\mathcal{W},\mathcal{W}'}$, then there exists $\sigma \in \mathcal{W}'$ such that*

$$\kappa_{\mathcal{W}'}\{g\mathcal{W}\} = s(\sigma)e_{\mathcal{W},\mathcal{W}'} .$$

Proof.

(i) It follows from the definition of good dual.

(ii) Since $\{g\mathcal{W}\}$ appears in $e_{\mathcal{W},W'}$ it follows by Lemma 2.11 that there exists $\sigma \in \mathcal{W}'$ such that $\sigma\{\mathcal{W}\} = \{g\mathcal{W}\}$. Then we have

$$\kappa_{\mathcal{W}'}\{g\mathcal{W}\} = \left(\sum_{\rho\in\mathcal{W}'} s(\rho)\rho\right)\{\sigma\mathcal{W}\} = s(\sigma)e_{\mathcal{W},W'}\,.$$

□

If $m \in M^{\mathcal{W}}$, then by the previous lemma $\kappa_{\mathcal{W}'}m$ is a multiple of $e_{\mathcal{W},W'}$.

We now define a bilinear form $\langle\,,\rangle$ on $M^{\mathcal{W}}$ by setting

$$\langle\{g_1\mathcal{W}\},\{g_2\mathcal{W}\}\rangle = \begin{cases} 1 & \text{if } g_1 = g_2\,, \\ 0 & \text{otherwise}. \end{cases}$$

This is a symmetric, non-singular, $\mathcal{G}$-invariant, bilinear form on $M^{\mathcal{W}}$.

We shall use the following trick:

For $u, v \in M^{\mathcal{W}}$

$$\begin{aligned}
\langle\kappa_{\mathcal{W}'}u, v\rangle &= \left\langle \sum_{\sigma\in\mathcal{W}'} s(\sigma)\sigma u, v\right\rangle \\
&= \sum_{\sigma\in\mathcal{W}'} \langle u, s(\sigma)\sigma^{-1}v\rangle \qquad \text{(since the form is } \mathcal{G}\text{-invariant)} \\
&= \sum_{\sigma\in\mathcal{W}'} \langle u, s(\sigma)\sigma v\rangle \\
&= \langle u, \kappa_{\mathcal{W}'}v\rangle\,.
\end{aligned}$$

Now the analogue of James' submodule theorem can be proved in this more general setting.

THEOREM 2.14. *Let $\mathcal{W}'$ be a good dual of $\mathcal{W}$ and let U be submodule of $M^{\mathcal{W}}$. Then either $S^{\mathcal{W},W'} \subseteq U$ or $U \subseteq S^{\mathcal{W},W'^{\perp}}$, where $S^{\mathcal{W},W'^{\perp}}$ is the complement of $S^{\mathcal{W},W'}$ in $M^{\mathcal{W}}$.*

Proof. If $u \in U$, then

$$\langle u, e_{\mathcal{W},W'}\rangle = \langle u, \kappa_{\mathcal{W}'}\{\mathcal{W}\}\rangle = \langle\kappa_{\mathcal{W}'}u, \{\mathcal{W}\}\rangle\,.$$

But $\kappa_{\mathcal{W}'}u = \lambda e_{\mathcal{W},W'}$ for some $\lambda \in K$. If $\lambda \neq 0$ for some $u \in U$, then $e_{\mathcal{W},W'} \in U$, that is, $S^{\mathcal{W},W'} \subseteq U$. However, if $\lambda = 0$ for all $u \in U$, then $\langle u, e_{\mathcal{W},W'}\rangle = 0$, that is, $U \subseteq S^{\mathcal{W},W'^{\perp}}$. □

We can now prove our principal result.

THEOREM 2.15. *Let $\mathcal{W}'$ be a good dual of $\mathcal{W}$. The $K\mathcal{G}$-module $D^{\mathcal{W},\mathcal{W}'} = S^{\mathcal{W},\mathcal{W}'}/S^{\mathcal{W},\mathcal{W}'} \cap S^{\mathcal{W},\mathcal{W}'^{\perp}}$ is zero or irreducible.*

Proof. If U is a submodule of $S^{\mathcal{W},\mathcal{W}'}$, then U is a submodule of $M^{\mathcal{W}}$ and by Theorem 2.14 either $S^{\mathcal{W},\mathcal{W}'} \subseteq U$ in which case $U = S^{\mathcal{W},\mathcal{W}'}$ or $U \subseteq S^{\mathcal{W},\mathcal{W}'^{\perp}}$ and $U \subseteq S^{\mathcal{W},\mathcal{W}'} \cap S^{\mathcal{W},\mathcal{W}'^{\perp}}$, which completes the proof. □

In the case of $K = \mathbb{Q}$ or any field of characteristic zero $\langle\,,\rangle$ is an inner product and $D^{\mathcal{W},\mathcal{W}'} = S^{\mathcal{W},\mathcal{W}'}$. Thus if for a subgroup $\mathcal{W}$ of $\mathcal{G}$ a good dual $\mathcal{W}'$ can be found, then we have a construction for irreducible $K\mathcal{G}$-modules. Hence it is essential to show for each subgroup that a good dual exists which satisfies Definition 2.12.

EXAMPLE 2.16. Let

$$\mathcal{G} = \langle a, b : \ a^6 = b^2 = e\,, \ \ bab = a^{-1} \rangle$$

be the dihedral group of order 12 and $\mathcal{H} = \langle a : a^6 = e \rangle$ be the subgroup of $\mathcal{G}$. The representative of conjugate classes and the character table of $\mathcal{G}$ have been given in [2]. Let $\mathcal{W} = \langle a^5 b \rangle$ and $\mathcal{W}' = \langle ab, a^4 b \rangle$. Since $\mathcal{W} \cap \mathcal{W}' = \langle e \rangle$, $\mathcal{W}'$ is a useful dual of $\mathcal{W}$. Then $\tau_{\mathcal{W}}$ contains $\mathcal{W}$-tabloids $\{\mathcal{W}\}$, $\{b\mathcal{W}\}$, $\{a^4 b\mathcal{W}\}$, $\{ab\mathcal{W}\}$, $\{a^4\mathcal{W}\}$, $\{a^3\mathcal{W}\}$. For $g = e$, $a^4 b$, ab, a^4, we have $\kappa_{\mathcal{W}'}\{g\mathcal{W}\} \neq 0$. Since

$$e_{\mathcal{W},\mathcal{W}'} = \{\mathcal{W}\} - \{a^4 b\mathcal{W}\} - \{ab\mathcal{W}\} - \{a^4\mathcal{W}\}$$

then $\mathcal{W}'$ is a good dual of $\mathcal{W}$.

Now let K be a field with $\operatorname{Char} K = 0$. Let $M^{\mathcal{W}}$ be K-space whose basis elements are the $\mathcal{W}$-tabloids. Let $S^{\mathcal{W},\mathcal{W}'}$ be the corresponding $K\mathcal{G}$-submodule of $M^{\mathcal{W}}$, then by definition of the Specht module we have

$$S^{\mathcal{W},\mathcal{W}'} = Sp\{e_{\mathcal{W},\mathcal{W}'}, e_{b\mathcal{W},b\mathcal{W}'}\}$$

where

$$\begin{aligned} e_{\mathcal{W},\mathcal{W}'} &= \{\mathcal{W}\} - \{a^4 b\mathcal{W}\} - \{ab\mathcal{W}\} - \{a^4\mathcal{W}\}\,, \\ e_{b\mathcal{W},b\mathcal{W}'} &= \{b\mathcal{W}\} - \{a^4 b\mathcal{W}\} - \{ab\mathcal{W}\} - \{a^3\mathcal{W}\}\,. \end{aligned}$$

Let T be the matrix representation of $\mathcal{G}$ afforded by $S^{\mathcal{W},\mathcal{W}'}$ with character ψ and let a be the representative of the conjugate class C_2. Then

$$\begin{aligned} a(e_{\mathcal{W},\mathcal{W}'}) &= e_{b\mathcal{W},b\mathcal{W}'} - e_{\mathcal{W},\mathcal{W}'}\,, \\ a(e_{b\mathcal{W},b\mathcal{W}'}) &= -e_{\mathcal{W},\mathcal{W}'}\,. \end{aligned}$$

Thus we have

$$T(a) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\psi(a) = -1$.

By a similar calculation to the above it can be shown that $\psi = \xi_6$ in [2].

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Received May 22, 1996

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