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REGULARITY OF MINIMA OF VARIATIONAL INTEGRALS

JOSEF DANĚČEK* — EUGEN VISZUS**

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ABSTRACT. The BMO-regularity of the gradient of a local minimum for a nonlinear functional is shown.

1. Introduction

In this paper we shall consider the problem of the regularity of the derivatives of functions minimizing the variational integral

$$F(u;\Omega)=\int_\Omega f(x,u,\mathrm{D}u)\;\mathrm{d}x \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$, $n>1$, is an open set, $u\colon\Omega\to\mathbb{R}^N$, $N>1$, $\mathrm{D}u=\{\mathrm{D}_\alpha u^i\}$, $\alpha=1,\dots,n$, $i=1,\dots,N$ and $f\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}$ will be stated below. A local minimum for the functional F is a function $u\in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ such that for every $\varphi\in W^{1,2}(\Omega,\mathbb{R}^N)$ with $\operatorname{supp}\varphi\Subset\Omega$ we have

$$F(u;\operatorname{supp}\varphi)\le F(u+\varphi;\operatorname{supp}\varphi)\,.$$

In his article [2], the first author has proved the $\mathcal{L}^{2,n}_{\mathrm{loc}}$-regularity for the gradient of solutions of some nonlinear elliptic system. This article and [3] motived us to investigate the $\mathcal{L}^{2,n}_{\mathrm{loc}}$-regularity for the gradient of the functional (1.1).

2. Preliminary results and definitions

We shall consider a local minima of the functional

$$F(u;\Omega)=\int_\Omega A^{\alpha\beta}_{ij}(x)\,\mathrm{D}_\alpha u^i\,\mathrm{D}_\beta u^j\;\mathrm{d}x+\int_\Omega g(x,u,\mathrm{D}u)\;\mathrm{d}x \tag{2.1}$$

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where the coefficients $A_{ij}^{\alpha\beta}$ are continuous in $\overline{\Omega}$ and satisfy the Legendre-Hadamard condition:

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \geqq \nu|\xi|^2|\eta|^2\,, \qquad x\in\Omega\,,\ \ \xi\in\mathbb{R}^n\,,\ \ \eta\in\mathbb{R}^N\,;\ \ \nu>0\,. \tag{2.2}$$

Here and in what follows we use the summation convention over repeated indices.

Concerning the function g we suppose that for almost $x\in\Omega$ and all $(u,z)\in\mathbb{R}^N\times\mathbb{R}^{nN}$ the following condition holds:

$$-f(x)-l\big(|u|^\delta+|z|^\gamma\big)\le g(x,u,z)\le f(x)+l\big(|u|^\delta+|z|^\gamma\big) \tag{2.3}$$

where $l\ge 0$, $1\le\delta/2<n/(n-2)$ for $n>2$, $\delta\ge 0$ for $n\le 2$, $0\le\gamma<2$ and $f\in L^p(\Omega)$, $p>1$. From these assumptions it follows that our functionals (2.1) are, in general, non differentiable and therefore that $u\notin W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$.

Let $u\in W^{1,2}(\Omega,\mathbb{R}^N)$ be a local minimum for the functional (2.1). We shall get estimates for the derivatives of u in the spaces $L^{2,\lambda}$ and $\mathcal{L}^{2,\lambda}$. For detailed information see [1], [4], [5] and [7]. The proofs will be given in detail only for $n>2$.

Definition 2.4. The *Zygmund class* $\Lambda^1(\overline{\Omega},\mathbb{R}^N)$ is the subspace of those functions $u\in\mathbb{C}^0(\overline{\Omega},\mathbb{R}^N)$ for which $[u]_{\Lambda^1(\overline{\Omega},\mathbb{R}^N)}=\sup\Big\{\frac{|u(x)+u(y)-2u((x+y)/2)|}{|x-y|}:\ x,y,(x+y)/2\in\overline{\Omega}\Big\}<\infty$.

Proposition 2.5. *For a domain of the class $C^{0,1}$ we have the following*

(i) $\mathcal{L}^{2,\lambda}(\Omega,\mathbb{R}^N)$ *is isomorphic to* $\mathbb{C}^{0,\frac{\lambda-n}{2}}(\overline{\Omega},\mathbb{R}^N)$, *for* $n<\lambda\le n+2$.

(ii) $L^{2,\lambda}(\Omega,\mathbb{R}^N)$ *is isomorphic to* $\mathcal{L}^{2,\lambda}(\Omega,\mathbb{R}^N)$, $\lambda\in[0,n)$.

(iii) $\mathcal{L}^{2,n+2}_1(\Omega,\mathbb{R}^N)$ *is isomorphic to* $\Lambda^1(\overline{\Omega},\mathbb{R}^N)$.

(iv) $\mathbb{C}^{0,1}(\overline{\Omega},\mathbb{R}^N)\subsetneq\Lambda^1(\overline{\Omega},\mathbb{R}^N)\subsetneq\bigcap_{0<\alpha<1}\mathbb{C}^{0,\alpha}(\overline{\Omega},\mathbb{R}^N)$,
where $\mathbb{C}^{0,\alpha}(\overline{\Omega},\mathbb{R}^N)$, $\alpha\in(0,1]$ *is the Hőlder-Lipschitz space.*

(v) *If* $u\in W^{1,2}(\Omega,\mathbb{R}^N)$ *and* $\mathrm{D}u\in\mathcal{L}^{2,\lambda}(\Omega,\mathbb{R}^{nN})$, $n-2<\lambda<n$, *then* $u\in\mathbb{C}^{0,\alpha}(\Omega,\mathbb{R}^N)$, $\alpha=(\lambda+2-n)/2$.

We recall some results needed for the next paragraph.

Lemma 2.6. ([1]) *Let $B_R(x_0)\subset\Omega$ be arbitrary and let $v\in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ be a solution of the Dirichlet problem*

$$\begin{gathered}\int_{B_R(x_0)} A_{ij}^{\alpha\beta}(x)\,\mathrm{D}_\alpha v^i\,\mathrm{D}_\beta v^j\,\mathrm{d}x\ \longrightarrow\ \min\\ u-v\in W^{1,2}_0\big(B_R(x_0),R^N\big)\,,\qquad u\in W^{1,2}_{\mathrm{loc}}\big(\Omega,R^N\big)\end{gathered} \tag{2.7}$$

with coefficients $A_{ij}^{\alpha\beta} \in C^{0,\mu}(\Omega)$ *satisfying* (2.2). *Then there exists a constant* c_1 *such that for all* $t \in (0,1]$ *the following estimate holds:*

$$\begin{aligned} &\int_{B_{tR}(x_0)} \left|\mathrm{D}v - (\mathrm{D}v)_{x_0,tR}\right|^2 \, \mathrm{d}x \\ &\leq c_1 \left(t^{n+2} \int_{B_R(x_0)} \left|\mathrm{D}v - (\mathrm{D}v)_{x_0,R}\right|^2 \, \mathrm{d}x + R^{2\mu} \int_{B_R(x_0)} |\,\mathrm{D}v|^2 \, \mathrm{d}x \right). \end{aligned} \tag{2.8}$$

LEMMA 2.9. ([6]) *Let* $\Phi = \Phi(R)$, $R \in (0,d]$, $d > 0$, *be a nonnegative function and let* A, B, C, a, b *be nonnegative constants. Suppose that for all* $t \in (0,1]$ *and all* $R \in (0,d]$

$$\Phi(tR) \leq (At^a + B)\Phi(R) + CR^b \tag{2.10}$$

hold. Further let $K \in (0,1)$ *be such that* $\varepsilon = AK^{a-b} + BK^{-b} < 1$. *Then*

$$\Phi(R) \leq cR^b, \qquad R \in (0,d] \tag{2.11}$$

where $c = \max\left\{C/K(1-\varepsilon), \sup_{R\in[Kd,d]} \Phi(R)/R^b\right\}$.

LEMMA 2.12. ([1]) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded convex domain,* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *with* $\mathrm{D}u \in L^{2,\tau}(\Omega, \mathbb{R}^{nN})$, $\tau \in (0,n)$. *If* $\tau < n-2$, *then* $u \in L^{2^*,\tau 2^*/2}(\Omega, \mathbb{R}^N)$, *where* $2^* = 2n/(n-2)$, *and for all* $x \in \Omega$, $R \leq \operatorname{diam}\Omega$ *we have*

$$\int_{\Omega(x,R)} |u(y)|^{2^*} \, \mathrm{d}y \leq c_2 M^{2^*} R^{\tau 2^*/2}. \tag{2.13}$$

If $\tau \geqq n-2$ *then* $u \in L^\infty(\Omega)$ *and*

$$\|u\|_{\infty,\Omega} \leq c_3 M, \tag{2.14}$$

where $M = \|u\|_{W^{1,2}(\Omega,\mathbb{R}^N)} + \|Du\|_{L^{2,\tau}(\Omega,\mathbb{R}^{nN})}$ *and* c_2, c_3 *depend on* $\operatorname{diam}\Omega$.

In the following we shall use:

LEMMA 2.15. *Let* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, $\mathrm{D}u \in L^{2,\tau}(\Omega, \mathbb{R}^{nN})$, $\tau \in [0,n)$ *and*

$$|g(x,u,z)| \leq f(x) + l\big(|u|^\delta + |z|^\gamma\big) \tag{2.16}$$

where f, l, δ, γ *are defined by* (2.3). *Then for each* $\varepsilon \in (0,1)$ *and all* $B_R(x_0) \subset \Omega$

$$\left| \int_{B_R(x_0)} g(x,u,\mathrm{D}u) \, \mathrm{d}x \right| \leq \frac{\gamma l}{2}\varepsilon \int_{B_R(x_0)} |\,\mathrm{D}u|^2 \, \mathrm{d}x + c_4 R^\lambda. \tag{2.17}$$

Here $\lambda = \min\{n(1-1/p),\, n+(\tau+2-n)\delta/2\}$, $c_4 = c_4(\|f\|_{L^p(\Omega)}, M, \varepsilon, \delta, \gamma, l)$ *and* M *is the constant from Lemma* 2.12.

P r o o f . According to (2.16) it follows that

$$\Bigg|\int_{B_R(x_0)} g(x,u,\mathrm{D}u)\ \mathrm{d}x\Bigg| \le c\|f\|_{L^p(\Omega)} R^{n(1-1/p)} + l\int_{B_R(x_0)} |u|^\delta\ \mathrm{d}x + l\int_{B_R(x_0)} |\mathrm{D}u|^\gamma\ \mathrm{d}x\,. \tag{2.18}$$

From Hölder's inequality we get

$$\int_{B_R(x_0)} |u|^\delta\ \mathrm{d}x \le c\Bigg(\int_{B_R(x_0)} |u|^{2^*}\ \mathrm{d}x\Bigg)^{\delta/2^*} R^{n(1-\delta/2^*)}\,. \tag{2.19}$$

Using Lemma 2.12 (and in the case $\tau = 0$, the Sobolev imbedding theorem) we have

$$\int_{B_R(x_0)} |u|^\delta\ \mathrm{d}x \le cM^\delta R^{\tau\delta/2+n(1-\delta/2^*)}\,. \tag{2.20}$$

By Young's inequality we obtain

$$\int_{B_R(x_0)} |Du|^\gamma \le \frac{\gamma}{2}\varepsilon\int_{B_R(x_0)} |Du|^2\ \mathrm{d}x + c(n,\varepsilon,\gamma)R^n, \qquad \varepsilon > 0\,. \tag{2.21}$$

From (2.18), (2.20), (2.21) we obtain (2.17). □

3. Main results

The following theorem may be seen as a generalization of Theorem 4.1 in [3].

THEOREM 3.1. *Let* $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ *be a local minimum of the functional* (2.1) *and let* (2.2), (2.3) *be satisfied. Then* $\mathrm{D}u \in L^{2,n(1-1/p)}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN})$.

P r o o f . Let $B_R(x_0) \Subset \Omega$ and v be a minimum of the functional

$$F^0(v;B_R(x_0)) = \int_{B_R(x_0)} A^{\alpha\beta}_{ij}(x_0)\,\mathrm{D}_\alpha v^i\,\mathrm{D}_\beta v^j\ \mathrm{d}x \tag{3.2}$$

among all the functions in $W^{1,2}(B_R(x_0),\mathbb{R}^N)$ taking the value u on $\partial B_R(x_0)$. It is known that v is smooth in $B_R(x_0)$ and we have (see [1])

$$\int_{B_{tR}(x_0)} |\mathrm{D}v|^2\ \mathrm{d}x \le c_1 t^n \int_{B_R(x_0)} |\mathrm{D}v|^2\ \mathrm{d}x\,, \qquad t\in(0,1]\,. \tag{3.3}$$

Put $w = u - v$. We have $w \in W_0^{1,2}(B_R(x_0), \mathbb{R}^N)$. We can suppose that there exists $\tilde{R} > 0$ such that $\int_{B_R(x_0)} |Dw|^2 \, dx < 1$ [1)] for all $R \leq \tilde{R}$. By standard arguments we obtain, using (3.3),

$$\int_{B_{tR}(x_0)} |Du|^2 \, dx \leq c_2 \left\{ t^n \int_{B_R(x_0)} |Du|^2 \, dx + \int_{B_R(x_0)} |Dw|^2 \, dx \right\}. \tag{3.4}$$

In the following we shall estimate the last integral on the right hand side of (3.4).

From [3; Lemma 2.1] we have

$$\begin{aligned} \nu \int_{B_R(x_0)} |Dw|^2 \, dx &\leq c_3 \Big(F^0\big(u; B_R(x_0)\big) - F^0\big(v; B_R(x_0)\big) \Big) \\ &= c_3 \Bigg\{ \int_{B_R(x_0)} \big(A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x)\big) D_\alpha u^i D_\beta u^j \, dx \\ &\quad + \int_{B_R(x_0)} \big(A_{ij}^{\alpha\beta}(x) - A_{ij}^{\alpha\beta}(x_0)\big) D_\alpha v^i D_\beta v^j \, dx \\ &\quad + \int_{B_R(x_0)} \big(-g(x, u, Du)\big) \, dx + \int_{B_R(x_0)} g(x, v, Dv) \, dx \\ &\quad + F\big(u; B_R(x_0)\big) - F\big(v; B_R(x_0)\big) \Bigg\} \\ &= c_3 \big\{ I + II + III + IV + F\big(u; B_R(x_0)\big) - F\big(v; B_R(x_0)\big) \big\} \\ &\leq c_3 \{ I + II + III + IV \}. \end{aligned} \tag{3.5 }$$

Notice that $F\big(u; B_R(x_0)\big) - F\big(v; B_R(x_0)\big) \leq 0$, since u is a minimizer.

Taking into account the properties of continuity modulus of $A_{ij}^{\alpha\beta}$ and Hőlder's inequality we obtain

$$|I| \leq \omega(R) \int_{B_R(x_0)} |Du|^2 \, dx, \tag{3.6}$$

$$|II| \leq 2\omega(R) \left(\int_{B_R(x_0)} |Du|^2 \, dx + \int_{B_R(x_0)} |Dw|^2 \, dx \right) \tag{3.7}$$

1) This follows from the absolute continuity of the integrals, the definition of w and the fact, that v is the solution of the Dirichlet problem for a linear elliptic system (see [8; p. 185]).

where $\omega(s) \searrow 0$ as $s \searrow 0$. From Lemma 2.15 (for $\tau = 0$) we have

$$III \le \frac{\gamma l}{2}\varepsilon \int\limits_{B_R(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x + c_4 R^\lambda \tag{3.8}$$

where $\lambda = \min\{n(1-1/p), n+(2-n)\delta/2\}$. From (2.3) we obtain

$$IV \le \int\limits_{B_R(x_0)} |f(x)| \,\mathrm{d}x + l \int\limits_{B_R(x_0)} |v|^\delta \,\mathrm{d}x + l \int\limits_{B_R(x_0)} |\mathrm{D}v|^\gamma \,\mathrm{d}x\,.$$

By methods similar to the proof of Lemma 2.15 we obtain

$$\int\limits_{B_R(x_0)} |f(x)| \,\mathrm{d}x \le c_5 R^{n(1-1/p)}\,,$$

$$\begin{aligned}\int\limits_{B_R(x_0)} |v|^\delta \,\mathrm{d}x &\le c_6 \|v\|^\delta_{W^{1,2}(B_R(x_0))} R^{n(1-\delta/2^*)} \\ &\le c_7\Big(\|u\|^\delta_{W^{1,2}(\Omega)} + \|w\|^\delta_{W^{1,2}(B_R(x_0))}\Big) R^{n(1-\delta/2^*)} \\ &\le c_8\left(\|u\|^\delta_{W^{1,2}(\Omega)} + \left(\int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x\right)^{\delta/2}\right) R^{n(1-\delta/2^*)}\,.\end{aligned}$$

Because $\int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x < 1$ for all $R \le \tilde{R}$ and $\delta/2 \ge 1$ we finally obtain the estimate

$$\int\limits_{B_R(x_0)} |v|^\delta \,\mathrm{d}x \le c_8\left(\|u\|^\delta_{W^{1,2}(\Omega)} + \int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x\right) R^{n(1-\delta/2^*)}\,,$$

$$\int\limits_{B_R(x_0)} |\mathrm{D}v|^\gamma \,\mathrm{d}x \le c_9\varepsilon\left(\int\limits_{B_R(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x + \int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x\right) + c_{10} R^n\,.$$

Combining these we have

$$IV \le c_{11}\left\{\varepsilon\left(\int\limits_{B_R(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x + \int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x\right) + R^{n(1-\delta/2^*)} \int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x + R^\lambda\right\} \tag{3.9}$$

where λ is from (3.8).

The estimates (3.5), (3.6), (3.7), (3.8) and (3.9) give

$$\nu \int_{B_R(x_0)} |\mathrm{D}w|^2 \, \mathrm{d}x$$
$$\le c_{12}(\varepsilon+\omega(R)) \int_{B_R(x_0)} |\mathrm{D}u|^2 \, \mathrm{d}x + c_{13}(\varepsilon+\omega(R)+R^{n(1-\delta/2^*)}) \int_{B_R(x_0)} |\mathrm{D}w|^2 \, \mathrm{d}x + c_{14}R^{\lambda}. \tag{3.10}$$

Now we can choose $\varepsilon_0 > 0$, $R_0 > 0$ such that $\nu - c_{13}(\varepsilon+\omega(R)+R^{n(1-\delta/2^*)})$ is positive for each $\varepsilon < \varepsilon_0$ and $R < R_0$.

Thus we have

$$\int_{B_R(x_0)} |\mathrm{D}w|^2 \, \mathrm{d}x \le c_{15}(\varepsilon+\omega(R)) \int_{B_R(x_0)} |\mathrm{D}u|^2 \, \mathrm{d}x + c_{16}R^{\lambda}. \tag{3.11}$$

Now from (3.4) and (3.11) we obtain

$$\int_{B_{tR}(x_0)} |\mathrm{D}u|^2 \, \mathrm{d}x \le c_{17}\left\{t^n + (\varepsilon+\omega(R))\right\} \int_{B_R(x_0)} |\mathrm{D}u|^2 \, \mathrm{d}x + c_{18}R^{\lambda}. \tag{3.12}$$

Using Lemma 2.9 for $\Phi(R) = \int_{B_R(x_0)} |\mathrm{D}u|^2 \, \mathrm{d}x$ we have

$$\int_{B_R(x_0)} |\mathrm{D}u|^2 \, \mathrm{d}x \le c_{19}R^{\lambda} \tag{3.13}$$

for each $0 < R < \widetilde{R} \le R_0$. From the last estimate it follows that $\mathrm{D}u \in L^{2,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$. If $\lambda = n(1-1/p)$, the proof is finished. If $\lambda < n(1-1/p)$ we use Lemma 2.15 for $\tau = \lambda$ (see [2; Theorem 2.1]) and, by repeating the previous procedure of the proof, we obtain that $\mathrm{D}u \in L^{2,\lambda'}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$, where $\lambda' > \lambda$. After a finite number of steps we obtain that $\lambda' = n(1-1/p)$. The proof is finished. □

To obtain the $\mathcal{L}^{2,n}$-regularity of $\mathrm{D}u$ we strengthen the conditions on the function g. We shall suppose that for a.e. $x, y \in \Omega$ and all $u, v \in \mathbb{R}^N$, $z, q \in \mathbb{R}^{nN}$

$$|g(x,u,z) - g(y,v,q)| \le |f(x) - f(y)| + l\big((|u|+|v|)^{\delta} + |z-q|^{\gamma}\big). \tag{3.14}$$

Now we can state the main result of this paper:

THEOREM 3.15. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a local minimum of the functional (2.1). Suppose that the conditions (2.2) with coefficients $A^{\alpha\beta}_{ij} \in C^{0,\mu}(\Omega)$ and (3.14) are satisfied for $f \in \mathcal{L}^{1,n}(\Omega)$. Then $\mathrm{D}u \in \mathcal{L}^{2,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$.*

Remark 3.16. We recall that (2.3) follows from (3.14).

Proof. From Theorem 3.1 and Proposition 2.5(ii), it follows that $Du \in \mathcal{L}^{2,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for every $n-2 < \lambda < n$ and from Proposition 2.5(v) it follows that $u \in \mathbb{C}^{0,(\lambda+2-n)/2}(\Omega, \mathbb{R}^N)$. Let $x_0 \in \Omega$ is a fixed and consider the function v of Lemma 2.6.

$$\begin{aligned}
&\int\limits_{B_{tR}(x_0)} \left|\mathrm{D}u - (\mathrm{D}u)_{x_0,tR}\right|^2 \,\mathrm{d}x \\
&\le 2 \int\limits_{B_{tR}(x_0)} \left|\mathrm{D}v - (\mathrm{D}v)_{x_0,tR}\right|^2 \mathrm{d}x + 2 \int\limits_{B_{tR}(x_0)} \left|\mathrm{D}w - (\mathrm{D}w)_{x_0,tR}\right|^2 \mathrm{d}x \\
&\le c_1 \left(t^{n+2} \int\limits_{B_R(x_0)} \left|\mathrm{D}v - (\mathrm{D}v)_{x_0,R}\right|^2 \mathrm{d}x + R^{2\mu} \int\limits_{B_R(x_0)} |\mathrm{D}v|^2 \,\mathrm{d}x + \int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x \right) \\
&\le c_2 \left(t^{n+2} \int\limits_{B_R(x_0)} \left|\mathrm{D}u - (\mathrm{D}u)_{x_0,R}\right|^2 \mathrm{d}x + R^{2\mu} \int\limits_{B_R(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x + \int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x \right).
\end{aligned} \tag{3.17}$$

We have to estimate the last integral in (3.17). Since v is a minimizer of the functional from (2.7) we have

$$\int\limits_{B_R(x_0)} A^{\alpha\beta}_{ij}(x)\, \mathrm{D}_\alpha(u^i - v^i)\, \mathrm{D}_\beta(u^j - v^j)\,\mathrm{d}x = \int\limits_{B_R(x_0)} A^{\alpha\beta}_{ij}(x)\, \mathrm{D}_\alpha(u^i - v^i)\, \mathrm{D}_\beta(u^j + v^j)\,\mathrm{d}x$$

and because u is the minimum of F we may write (putting $w = u - v$)

$$\begin{aligned}
&\nu \int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x \\
&\le \int\limits_{B_R(x_0)} A^{\alpha\beta}_{ij}(x_0)\, \mathrm{D}_\alpha w^i\, \mathrm{D}_\beta w^j \,\mathrm{d}x \\
&= \int\limits_{B_R(x_0)} \left(A^{\alpha\beta}_{ij}(x_0) - A^{\alpha\beta}_{ij}(x)\right) \mathrm{D}_\alpha w^i\, \mathrm{D}_\beta w^j \,\mathrm{d}x \\
&\quad + \int\limits_{B_R(x_0)} \left[g(x,v,\mathrm{D}v) - \left(g(x,v,\mathrm{D}v)\right)_{x_0,R}\right] \mathrm{d}x \\
&\quad + \int\limits_{B_R(x_0)} \left[\left(g(x,v,\mathrm{D}v)\right)_{x_0,R} - \left(g(x,u,\mathrm{D}u)\right)_{x_0,R}\right] \mathrm{d}x \\
&\quad + \int\limits_{B_R(x_0)} \left[\left(g(x,u,\mathrm{D}u)\right)_{x_0,R} - g(x,u,\mathrm{D}u)\right] \mathrm{d}x + F(u;B_R) - F(v;B_R)
\end{aligned}$$

$$
\begin{aligned}
&= I + II + III + IV + F(u; B_R) - F(v; B_R) \\
&\leq I + II + III + IV\,.
\end{aligned}
$$

The $F(u; B_R) - F(v; B_R) \leq 0$, because u is the minimum of F.

From the preceding considerations, the Sobolev theorem and by means of Young's inequality we have

$$
I \leq 2\omega(R) \int_{B_R(x_0)} |\mathrm{D}w|^2 \, \mathrm{d}x \leq 2R^{\mu} \int_{B_R(x_0)} |\mathrm{D}w|^2 \, \mathrm{d}x\,,
$$

$$
\begin{aligned}
II &\leq R^{-n} \int_{B_R(x_0)} \int_{B_R(x_0)} \big| g\big(x, v(x), \mathrm{D}v(x)\big) - g\big(y, v(y), \mathrm{D}v(y)\big) \big| \, \mathrm{d}y \, \mathrm{d}x \\
&\leq R^{-n} \int_{B_R(x_0)} \Bigg\{ \int_{B_R(x_0)} |f(x) - f(y)| \, \mathrm{d}y + l \int_{B_R(x_0)} |v(x) - v(y)|^{\delta} \, \mathrm{d}y \\
&\qquad\qquad + l \int_{B_R(x_0)} |\mathrm{D}v(x) - \mathrm{D}v(y)|^{\gamma} \, \mathrm{d}y \Bigg\} \, \mathrm{d}x \\
&\leq 2 \int_{B_R(x_0)} \big| f(x) - f_{x_0,R} \big| \, \mathrm{d}x + c(l, \gamma)\varepsilon \int_{B_R(x_0)} \big| \mathrm{D}v(x) - (\mathrm{D}v)_{x_0,R} \big|^2 \, \mathrm{d}x \\
&\qquad\qquad + c\Big(l, n, \varepsilon, \gamma, \delta, \|\mathrm{D}u\|_{\mathcal{L}^{2,\lambda}(B_R(x_0), \mathbb{R}^{nN})}\Big) R^n \\
&\leq c(l, \gamma)\varepsilon \int_{B_R(x_0)} \big| \mathrm{D}v(x) - (\mathrm{D}v)_{x_0,R} \big|^2 \, \mathrm{d}x \\
&\qquad\qquad + c\Big(l, n, \varepsilon, \gamma, \delta, \|f\|_{\mathcal{L}^{1,n}(\Omega)}, \|\mathrm{D}u\|_{\mathcal{L}^{2,\lambda}(B_R(x_0), \mathbb{R}^{nN})}\Big) R^n\,,
\end{aligned}
$$

$$
\begin{aligned}
III &\leq \int_{B_R(x_0)} \big| g\big(x, v(x), \mathrm{D}v(x)\big) - g\big(x, u(x), \mathrm{D}u(x)\big) \big| \, \mathrm{d}x \\
&\leq c(l, \delta) \int_{B_R(x_0)} |v(x) - u(x)|^{\delta} \, \mathrm{d}x + \int_{B_R(x_0)} |\mathrm{D}v(x) - \mathrm{D}u(x)|^{\gamma} \, \mathrm{d}x \\
&\leq \frac{l\gamma}{2}\varepsilon \int_{B_R(x_0)} |\mathrm{D}w(x)|^2 \, \mathrm{d}x + c\Big(l, n, \varepsilon, \gamma, \delta, \|\mathrm{D}u\|_{\mathcal{L}^{2,\lambda}(B_R(x_0), \mathbb{R}^{nN})}\Big) R^n\,.
\end{aligned}
$$

We may estimate the term IV in the same way as II.

$$
IV \leq R^{-n} \int_{B_R(x_0)} \int_{B_R(x_0)} \big| g\big(x, u(x), \mathrm{D}u(x)\big) - g\big(y, u(y), \mathrm{D}u(y)\big) \big| \, \mathrm{d}y \, \mathrm{d}x
$$

$$\le c(l,\gamma)\varepsilon \int\limits_{B_R(x_0)} \left|\mathrm{D}u(x) - (\mathrm{D}u)_{x_0,R}\right|^2 \mathrm{d}x$$
$$+ c\left(l, n, \varepsilon, \gamma, \delta, \|f\|_{\mathcal{L}^{1,n}(\Omega)}, \|\mathrm{D}u\|_{\mathcal{L}^{2,\lambda}(B_R(x_0),\mathbb{R}^{nN})}\right) R^n .$$

Taking into consideration the above estimates we have

$$\nu \int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x$$
$$\le \left(2R^\mu + \frac{l\gamma}{2}\varepsilon\right) \int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x + c_3\varepsilon \int\limits_{B_R(x_0)} \left|\mathrm{D}u(x) - (\mathrm{D}u)_{x_0,R}\right|^2 \mathrm{d}x + c_4 R^n$$

where $c_3 = c_3(l,\gamma)$ and $c_4 = c_4\left(l, n, \varepsilon, \gamma, \delta, \|f\|_{\mathcal{L}^{1,n}(\Omega)}, \|\mathrm{D}u\|_{\mathcal{L}^{2,\lambda}(B_R(x_0),\mathbb{R}^{nN})}\right)$.

Now we can choose $\varepsilon_0 > 0$, $R_0 > 0$ such that $\nu - (2R^\mu + l\gamma\varepsilon/2)$ is positive for each $\varepsilon < \varepsilon_0$ and $R < R_0$. Thus we have

$$\int\limits_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x \le c_5\varepsilon \int\limits_{B_R(x_0)} \left|\mathrm{D}u(x) - (\mathrm{D}u)_{x_0,R}\right|^2 \mathrm{d}x + c_6 R^n \tag{3.18}$$

where

$c_5 = c_5(l,\gamma,\mu,\nu)$ and $c_6 = c_6\left(l, n, \varepsilon, \gamma, \delta, \mu, \nu\, \|f\|_{\mathcal{L}^{1,n}(\Omega)}, \|\mathrm{D}u\|_{\mathcal{L}^{2,\lambda}(B_R(x_0),\mathbb{R}^{nN})}\right)$.

From (3.17) and (3.18) we obtain

$$\int\limits_{B_{tR}(x_0)} \left|\mathrm{D}u - (\mathrm{D}u)_{x_0,tR}\right|^2 \mathrm{d}x$$
$$\le (c_2 t^{n+2} + c_7\varepsilon) \int\limits_{B_R(x_0)} \left|\mathrm{D}u - (\mathrm{D}u)_{x_0,R}\right|^2 \mathrm{d}x + c_8 R^{2\mu} \int\limits_{B_R(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x + c_9 R^n \tag{3.19}$$
$$\le (c_2 t^{n+2} + c_7\varepsilon) \int\limits_{B_R(x_0)} \left|\mathrm{D}u - (\mathrm{D}u)_{x_0,R}\right|^2 \mathrm{d}x + c_{10} R^n$$

because $\mathrm{D}u \in L^{2,\lambda}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and $n-2 < \lambda < n$ is arbitrary.

Since, the inequality (3.19) holds for all $t \in (0,1]$ and $\varepsilon < \varepsilon_0$ we may use Lemma 2.14 from which we obtain

$$\int\limits_{B_R(x_0)} \left|\mathrm{D}u - (\mathrm{D}u)_{x_0,R}\right|^2 \mathrm{d}x \le c_{11} R^n \qquad \text{for all} \quad R < \min\{R_0, \operatorname{dist}(x_0,\Omega)\} . \tag{3.20}$$

Now let Ω_0 be an arbitrary domain such that $\Omega_0 \Subset \Omega$. Since (3.20) holds for every $x_0 \in \Omega_0$ and $R < \min\{R_0, \operatorname{dist}(\Omega_0, \partial\Omega)\}$ we get

$$\int_{\Omega_0(x_0,R)} \left|\mathrm{D}u - (\mathrm{D}u)_{x_0,R}\right|^2 \mathrm{d}x \le c_{12} R^n . \tag{3.21}$$

If $\min\{R_0, \operatorname{dist}(\Omega_0, \partial\Omega)\} < \operatorname{diam}\Omega_0$ it is easy to check that for

$$\min\{R_0, \operatorname{dist}(\Omega_0, \partial\Omega)\} < R < \operatorname{diam}\Omega_0$$

we have

$$\int_{\Omega_0(x_0,R)} \left|\mathrm{D}u - (\mathrm{D}u)_{x_0,R}\right|^2 \mathrm{d}x \le c_{13} \Big(\min\{R_0, \operatorname{dist}(\Omega_0, \partial\Omega)\}\Big)^n R^n . \tag{3.22}$$

Thus we have

$$\|\mathrm{D}u\|_{\mathcal{L}^{2,n}(\Omega_0, \mathcal{R}^{nN})} \le c_{14} \|\mathrm{D}u\|_{L^2(\Omega, \mathcal{R}^{nN})} .$$

The theorem is proved. □

COROLLARY 3.23. *If the assumptions of Theorem* 3.15 *are satisfied then* $u \in \Lambda^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

P r o o f . This follows from Proposition 2.10 (iii), Poincaré's inequality and the conclusion of Theorem 3.15. □

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