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ON THE CIRCLE PROBLEM WITH POLYNOMIAL WEIGHT

GERALD KUBA

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ABSTRACT. For arbitrary real a, b, r, $r \geq 1$, and a real polynomial p in two variables let $R(p;a,b;r) = \sum p(x,y)$ and $T(p;a,b;r) = \iint p(x,y)\,\mathrm{d}(x,y)$, where the disk $(x-a)^2+(y-b)^2 \leq r^2$ is the summation and integration domain, respectively. We give an upper bound for the "weighted" lattice rest $R(p;a,b;r) - T(p;a,b;r)$.

1. Introduction and statement of results

Let p be a polynomial in two variables with real coefficients, and let a, b, r, $r \geq 1$, be arbitrary real numbers. Furthermore, let $R(p;a,b;r)$ be the number of lattice points (of the standard lattice $\mathbb{Z}^2$) that lie within a circle with center (a,b) and radius r, each lattice point (x,y) being counted with *weight* $p(x,y)$. According to the headline of the present paper we are interested in an asymptotic evaluation of the function $R(p;a,b;r)$ in terms of the coefficients of the polynomial and the three circle parameters.

In a previous article [5] we studied the special case when the polynomial is *linear*. Now we are going to analyse the general case. The aim of this paper is to prove the following theorem.

THEOREM. *For arbitrary $a,b,r \in \mathbb{R}$, $r \geq 1$, and $p \in \mathbb{R}[X,Y] \setminus \mathbb{R}$,*

$$p(X,Y) = \sum_{(m,n)\in I} \alpha_{m,n} X^m Y^n$$

$$(\alpha_{m,n} \in \mathbb{R}\setminus\{0\},\ I \subset \mathbb{N}_0^2,\ I \setminus \{(0,0)\} \neq \emptyset \text{ finite}),$$

let

$$R(p;a,b;r) := \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x-a)^2+(y-b)^2 \leq r^2}} p(x,y).$$

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Then

$$R(p;a,b;r) = T(p;a,b;r) + \Delta(p;a,b;r)$$

where

$$T(p;a,b;r) = \iint\limits_{(x-a)^2+(y-b)^2\le r^2} p(x,y)\,\mathrm{d}(x,y)$$
$$= r^2\pi\Big(\sum_{(m,n)\in I}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}\alpha_{m,n}\omega_{k,l}a^{m-k}b^{n-l}r^{k+l}\Big),$$

with

$$\omega_{k,l} = \frac{\big(1+(-1)^k\big)\big(1+(-1)^l\big)}{4}\,\frac{\prod_{i=1}^{k/2-1}(1+2i)\prod_{j=1}^{l/2-1}(1+2j)}{2^{(k+l)/2}\big(\frac{k+l}{2}+1\big)!},$$

and the error term $\Delta(p;a,b;r)$ *may be estimated as follows.*

Let $d = \max\{m+n \mid (m,n)\in I\} \ge 1$ *be the degree and let* $h = \max\{|\alpha_{m,n}| \mid (m,n)\in I\}$ *be the height of the polynomial* p. *Furthermore assume w.l.o.g. that* $|a|\ge|b|$. *Then*

$$\Delta(p;a,b;r) = d^2 2^d h\Big(O\big((1+|a|^d)r^{\frac{46}{73}}(\log r)^{\frac{315}{146}}\big) + O\big(|a|^{d-1}r^{\frac{5}{3}}\big) + O\big(r^{d+\frac{2}{3}}\big)\Big).$$

The second and the third O*-term may be omitted if either* $r \ll |a|^{\frac{219}{227}}$, *or* $2a, 2b\in\mathbb{Z}$ *and* $r\ll|a|^{\frac{219}{223}}$.

All O*-constants are absolute (provided that the* $\ll$*-constants are absolute).*

2. Number theoretic applications

In this section we apply the Theorem to summatory functions where the arithmetic function $r(n)$ is involved.[1] As usual,

$$r(n) = \#\{(x,y)\in\mathbb{Z}^2 \mid x^2+y^2=n\} \qquad (n\in\mathbb{N}).$$

COROLLARY 1. *For arbitrary fixed* $k\in\mathbb{N}$,

$$\sum_{1\le n\le t} n^k r(n) \sim \frac{\pi}{1+k}t^{1+k} \qquad (t\to\infty).$$

[1] See Fricker [2] for an enlightening survey and also Recknagel [6] for interesting results concerning mean value theorems of this arithmetic function.

More precisely, for arbitrary fixed $k \in \mathbb{N}$,

$$\sum_{1 \le n \le t} n^k r(n) = \frac{\pi}{1+k} t^{1+k} + O\big(t^{k+\frac{1}{3}}\big)\,.$$

Proof. We put $a = b = 0$, $r = \sqrt{t}$, and consider the polynomial $p(X,Y) = (X^2+Y^2)^k$. Then

$$\sum_{1 \le n \le t} n^k r(n) = R\big(p; 0, 0; \sqrt{t}\big)\,.$$

We compute

$$\iint\limits_{x^2+y^2 \le r^2} (x^2+y^2)^k \, \mathrm{d}(x,y) = 2\pi \int_0^r \rho^{2k} \rho \, \mathrm{d}\rho = \frac{\pi}{k+1}\big(r^{2k+2}\big)\,.$$

Now we apply the Theorem and this concludes the proof of Corollary 1. $\square$

Remark. Of course, the results of Corollary 1 remain valid if $k = 0$. Furthermore, the O-estimate is not best possible. In fact, for $k = 0$ we have $O\big(t^{23/73+\varepsilon}\big)$ (cf. Proposition 1 in Section 4) and thus, by Abelian summation, one can derive $O\big(t^{k+23/73+\varepsilon}\big)$ in the general case.

The expansion of $\sum\limits_{1 \le n \le t} n^k r(n)$ yields an insight into the asymptotic behaviour of $r(n)$. It is well known (cf. [2]) that $r(n) = O(n^{\varepsilon})$ and $r(n) = \Omega\big((\log n)^{\theta}\big)$ for every $\varepsilon > 0$ and every θ. Furthermore, for every $\alpha > -1$,

$$\sum_{1 \le n \le t} n^{\alpha} = \frac{1}{1+\alpha} t^{1+\alpha} + O(t^{\alpha}) + O(1)\,.$$

Thus, the main terms of the asymptotic expansions of the summatory function of $n^k r(n)$ on one hand, and n^k on the other hand, differ by the factor π. This yields the following nice formula.

COROLLARY 2. *For every* $k \in \mathbb{N}_0$,

$$\lim_{t \to \infty} \frac{\sum\limits_{1 \le n \le t} n^k r(n)}{\sum\limits_{1 \le n \le t} n^k} = \pi\,.$$

Remark. By applying Abelian summation, it is straightforward to prove that the above formula remains true if the function n^k is replaced by an arbitrary monotonic sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers satisfying the condition $\Big|\sum\limits_{n=1}^{N} a_n\Big| \gg (1 + |a_N|) N^{1/3}$ $(N \to \infty)$.

3. Preparation of the proof. Some lemmata

Let the rounding error functions ψ and ψ_1 be defined by

$$\psi(z) = z - [z] - \frac{1}{2} \quad (z \in \mathbb{R}) \qquad \text{and} \qquad \psi_1(z) = \begin{cases} \psi(z), & z \notin \mathbb{Z}, \\ 1/2, & z \in \mathbb{Z} \end{cases} \quad (z \in \mathbb{R})$$

throughout the paper. ([ ] are Gauss brackets.)

Lemma 1. (Abelian summation, cf. [4]) *For arbitrary $P, Q \in \mathbb{Z}$, $P \le Q$, let g, h be real valued functions defined on $[P, Q]$, g being continuous on $[P, Q]$ and continuously differentiable on $]P, Q[$. Then*

$$\sum_{k=P}^{Q} g(k)h(k) = g(Q) \sum_{k=P}^{Q} h(k) - \int_P^Q g'(t) \Big(\sum_{P \le k \le t} h(k) \Big) \, \mathrm{d}t \, .$$

Lemma 2. (Euler summation formula, cf. [2], [4]) *For every real valued function f continuous on $[\alpha, \beta]$ and continuously differentiable on $]\alpha, \beta[$,*

$$\sum_{\alpha \le k \le \beta} f(k) = \int_\alpha^\beta f(t) \, \mathrm{d}t + \psi_1(\alpha) f(\alpha) - \psi(\beta) f(\beta) + \int_\alpha^\beta \psi(t) f'(t) \, \mathrm{d}t \, .$$

Lemma 3. (van der Corput, cf. [1], [4]) *Let f be a real valued function which is twice continuously differentiable on $[\alpha, \beta] \subset \mathbb{R}$. Furthermore, let f'' be monotonic and nonzero on $[\alpha, \beta]$. Then for $\varphi \in \{\psi, \psi_1\}$,*

$$\sum_{\alpha \le k \le \beta} \varphi(f(k)) \ll \int_\alpha^\beta |f''(t)|^{\frac{1}{3}} \, \mathrm{d}t + |f''(\alpha)|^{-\frac{1}{2}} + |f''(\beta)|^{-\frac{1}{2}} \, ,$$

where the $\ll$-constant is absolute.

The following lemma is an immediate consequence of the second mean value theorem.

Lemma 4. *Let f be a real valued function which is continuous and piecewise monotonic on $[\alpha, \beta] \subset \mathbb{R}$. Then for $\varphi \in \{\psi, \psi_1\}$,*

$$\left| \int_\alpha^\beta \varphi(t) f(t) \, \mathrm{d}t \right| \le \frac{c}{4} \max_{\alpha \le t \le \beta} |f(t)| \, ,$$

where c is the number of monotonic pieces of f.

4. Preparation of the proof. Some propositions

For $k, l \in \mathbb{N}_0$ and $a, b, r \in \mathbb{R}$, $r \geq 1$, let

$$R_{k,l}(a,b;r) = \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x-a)^2+(y-b)^2\leq r^2}} (x-a)^k (y-b)^l .$$

For the special case $k = l = 0$ we are dealing with the circle problem itself. Indeed, $R_{0,0}$ is identical to the number of lattice points that lie within a circle of radius r.

Proposition 1. *We have*

$$R_{0,0}(a,b;r) = r^2\pi + O\big(r^{\frac{46}{73}}(\log r)^{\frac{315}{146}}\big)$$

uniformly in a, b.

This deep result was proved by Huxley in 1993. A proof can be found in [3; Theorem 18.3.2].

Proposition 2. *If k and l are even, then*

$$\iint\limits_{(x-a)^2+(y-b)^2\leq r^2} (x-a)^k (y-b)^l \,\mathrm{d}(x,y) = r^{k+l+2}\pi \, \frac{\prod\limits_{i=1}^{k/2-1}(1+2i) \prod\limits_{j=1}^{l/2-1}(1+2j)}{2^{(k+l)/2}\big(\frac{k+l}{2}+1\big)!} .$$

The double integral is always zero if either k or l is odd.

A proof of Proposition 2 is straightforward, if we use a well-known formula involving the beta-function.

Proposition 3. *For arbitrary* $k, l \in \mathbb{N}_0$,

$$R_{k,l}(a,b;r) = \iint\limits_{(x-a)^2+(y-b)^2\leq r^2} (x-a)^k (y-b)^l \,\mathrm{d}(x,y) + O\big(r^{k+l+2/3}\big) + O\big((k+l)r^{k+l}\big) .$$

The O-constants are absolute. In particular, they do not depend on k or l.

Proposition 3 is the main result of the present paper. A proof is given in Section 6.

The next proposition deals with the special case that the center coordinates of the circle are integers or half odd integers, respectively. Clearly, because of symmetry, we have.

Proposition 4.

(i) *If $2a \in \mathbb{Z}$ and k is odd, then $R_{k,l} = 0$ for every l.*
(ii) *If $2b \in \mathbb{Z}$ and l is odd, then $R_{k,l} = 0$ for every k.*

5. Proof of Theorem

Via

$$X^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k}(X-a)^k , \qquad Y^n = \sum_{l=0}^{n} \binom{n}{l} b^{n-l}(Y-b)^l$$

we can write

$$R(p;a,b;r) = \sum_{(m,n)\in I} \sum_{k=0}^{m}\sum_{l=0}^{n} \binom{m}{k}\binom{n}{l} \alpha_{m,n} a^{m-k} b^{n-l} R_{k,l}(a,b;r) ,$$

where $R_{k,l}(a,b;r)$ is defined as in Section 4. Applying Proposition 1 and Proposition 3 we obtain

$$R(p;a,b;r) = T(p;a,b;r) + \Delta(p;a,b;r) ,$$

where $T(p;a,b;r)$ is exactly the same as in the Theorem, and

$$\Delta(p;a,b;r) = \sum_{(m,n)\in I} \sum_{k=0}^{m}\sum_{l=0}^{n} \binom{m}{k}\binom{n}{l} \alpha_{m,n} a^{m-k} b^{n-l} \Delta_{k,l} ,$$

with (uniformly in a,b,r,k,l)

$$\Delta_{0,0} = O\big(r^{\frac{46}{73}} (\log r)^{\frac{315}{146}}\big) \qquad \text{and} \qquad \Delta_{k,l} = O\big(r^{k+l+2/3}\big) + O\big((k+l) r^{k+l}\big) .$$

(Proposition 2 yields the formula for the double integral in the Theorem.)

Furthermore, by Proposition 4, $\Delta_{k,l} = 0$ if either $2a \in \mathbb{Z}$ and k is odd or $2b \in \mathbb{Z}$ and l is odd.

Now let d be the degree and h be the height of the polynomial p. We distinguish the cases $k = l = 0$ and $(k,l) \neq (0,0)$. We have

$$|\Delta(p;a,b;r)| \le h(\Delta_1 + \Delta_2)$$

with

$$\Delta_1 = \sum_{(m,n)\in I} |a|^m |b|^n |\Delta_{0,0}|$$

and

$$\Delta_2 = \sum_{(m,n)\in I} \sum_{\substack{(k,l)\neq(0,0) \\ k\le m;\, l\le n}} \binom{m}{k}\binom{n}{l} |a|^{m-k} |b|^{n-l} |\Delta_{k,l}| .$$

Now, for the sake of simplicity and w.l.o.g. we assume $|a| \ge |b|$. Then

$$\Delta_1 \ll d^2 (1 + |a|^d) r^{\frac{46}{73}} (\log r)^{\frac{315}{146}}$$

and

$$\Delta_2 \ll r^{2/3} \sum_{(m,n)\in I} \sum_{\substack{(k,l)\neq(0,0)\\ k\le m;\, l\le n}} (k+l)\binom{m}{k}\binom{n}{l}\left(\max_{\substack{(k,l)\neq(0,0)\\ k\le m;\, l\le n}} |a|^{m+n-(k+l)} r^{k+l}\right)$$

$$\le r^{2/3} \max_{(m,n)\in I}\left(\max\{|a|^{m+n-1}r, r^{m+n}\}\right) \sum_{(m,n)\in I} \sum_{k=0}^{m}\sum_{l=0}^{n}(k+l)\binom{m}{k}\binom{n}{l}$$

$$= d^2 2^d\left(O\left(r^{5/3}|a|^{d-1}\right) + O\left(r^{d+2/3}\right)\right).$$

Clearly, if $r \ll |a|^{\frac{219}{227}}$ then $2^d\Delta_1 \gg \Delta_2$.

Finally, we consider the case that $2a, 2b \in \mathbb{Z}$. In that case

$$\Delta_{1,0} = \Delta_{0,1} = \Delta_{1,1} = \Delta_{2,1} = \Delta_{1,2} = 0$$

and (k,l) in the inner sum in the estimate of Δ_2 runs through $\{(0,2),(2,0),(2,2),\dots\}$ and for $d \ge 2$ we obtain

$$\Delta_2 \ll d^2 2^d r^{2/3} \max_{(m,n)\in I}\left(\max\{|a|^{m+n-2}r^2, r^{m+n}\}\right)$$

$$= d^2 2^d\left(O\left(r^{8/3}|a|^{d-2}\right) + O\left(r^{d+2/3}\right)\right),$$

which is dominated by $2^d\Delta_1$ if $r \ll |a|^{\frac{219}{223}}$.

Clearly, if $d < 2$ then the inner sum is empty and thus $\Delta_2 = 0$.

This concludes the proof of the Theorem.

6. Proof of Proposition 3

We have

$$R_{k,l} = R_{k,l}(a,b;r) = \sum_{b-r\le y\le b+r} (y-b)^l \sum_{\alpha(y)\le x\le \beta(y)} (x-a)^k,$$

where

$$\alpha(y) = a - \sqrt{r^2-(y-b)^2} \qquad \text{and} \qquad \beta(y) = a + \sqrt{r^2-(y-b)^2}.$$

We apply Lemma 2 to the inner sum and obtain

$$R_{k,l} = \sum_{b-r\le y\le b+r} (y-b)^l \left(\int_{\alpha(y)}^{\beta(y)} (x-a)^k \, \mathrm{d}x + k\int_{\alpha(y)}^{\beta(y)} (x-a)^{k-1}\psi(x)\, \mathrm{d}x\right)$$
$$+ (-1)^k S_1 - S_2,$$

where

$$S_1 = \sum_{b-r\le y\le b+r} (y-b)^l \big(r^2-(y-b)^2\big)^{k/2}\psi_1(\alpha(y))$$

and

$$S_2 = \sum_{b-r\le y\le b+r} (y-b)^l \big(r^2-(y-b)^2\big)^{k/2}\psi(\beta(y))\,.$$

First we concentrate on the integrals. By Lemma 4 we have

$$\sum_{b-r\le y\le b+r} (y-b)^l k \int\limits_{\alpha(y)}^{\beta(y)} (x-a)^{k-1}\psi(x)\,\mathrm{d}x \ll \sum_{b-r\le y\le b+r} (y-b)^l k r^{k-1} \ll k r^{l+k}\,.$$

Since $\alpha(b\pm r) = a = \beta(b\pm r)$, by Lemma 2,

$$\sum_{b-r\le y\le b+r} (y-b)^l \int\limits_{\alpha(y)}^{\beta(y)} (x-a)^k\,\mathrm{d}x$$

$$= \iint\limits_{(x-a)^2+(y-b)^2\le r^2} (x-a)^k(y-b)^l\,\mathrm{d}(x,y) + \int\limits_{b-r}^{b+r} F(y)\psi(y)\,\mathrm{d}y\,,$$

where

$$F(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left((y-b)^l \int\limits_{\alpha(y)}^{\beta(y)} (x-a)^k\,\mathrm{d}x\right).$$

Now for $b-r<y<b+r$,

$$\begin{aligned}
&F(y) = \\
&= \frac{1+(-1)^k}{k+1}\frac{\mathrm{d}}{\mathrm{d}y}\Big((y-b)^l\big(r^2-(y-b)^2\big)^{\frac{k+1}{2}}\Big)\\
&= \frac{1+(-1)^k}{k+1}(y-b)^{l-1}\big(r^2-(y-b)^2\big)^{\frac{k-1}{2}}\Big(l\big(r^2-(y-b)^2\big)-(k+1)(y-b)^2\Big)\\
&\ll \left(\frac{l}{k+1}+1\right) r^{k+l}\,,
\end{aligned}$$

and thus, by Lemma 4,

$$R_{k,l} = \iint\limits_{(x-a)^2+(y-b)^2\le r^2} (x-a)^k(y-b)^l\,\mathrm{d}(x,y) + (-1)^k S_1 - S_2 + O\big((k+l)r^{k+l}\big)\,.$$

Now we estimate the sums S_1 and S_2. For $t \in [b-r, b+r]$, let

$$\Psi_1(t) = \sum_{b-r \le y \le t} \psi_1\left(a - \sqrt{r^2-(y-b)^2}\right).$$

Then, by Lemma 1, with $g(y) = (y-b)^l\left(r^2-(y-b)^2\right)^{k/2}$,

$$\begin{aligned} S_1 &= g([b+r])\Psi_1([b+r]) - \int_{-[r-b]}^{[b+r]} g'(t)\Psi_1(t)\,\mathrm{d}t \\ &\ll \Big(\max_{b-r\le t\le b+r} |\Psi_1(t)|\Big)\Big(r^{l+\frac{k}{2}} + \max_{b-r\le t\le b+r} |g(t)|\Big) \\ &\ll r^{k+l} \max_{b-r\le t\le b+r} |\Psi_1(t)|, \end{aligned}$$

all $\ll$-constants being absolute.

Now we apply van der Corput's method (Lemma 3) on $\Psi_1(t)$ for every t and obtain $\Psi_1(t) \ll r^{2/3}$ uniformly in t and this yields $S_1 \ll r^{k+l+2/3}$. Clearly, the sum S_2 can be treated analogously and we obtain $S_2 \ll r^{k+l+2/3}$, too.

This concludes the proof of Proposition 3.

Remark. One might expect that with the help of the Discrete Hardy-Littlewood Method[2] the estimates of S_1, S_2 might be sharpened by improving the upper bound $r^{2/3}$ of $\Psi_1(t)$ to $r^{\frac{46}{73}}(\log r)^{\frac{315}{146}}$.

Unfortunately, this is not possible as explained in [5], where we had similar ψ-sums to estimate.

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[2] See Huxley [3] for a profound presentation of the method and its various applications to important problems of geometry and analytic number theory.

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Institut für Mathematik u.a.St.
Universität für Bodenkultur
Gregor Mendel-Straße 33
A-1180 Wien
AUSTRIA