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CONTROL AND SEPARATING POINTS OF MODULAR FUNCTIONS

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ABSTRACT. Let L be a complemented or sectionally complemented lattice and let X be a locally convex linear space. We prove the existence of an X-valued control for uniformly exhaustive sequences of X-valued modular functions on L. Moreover, if L is σ-complete, we prove that every countable family of σ-o.c. nonatomic group-valued modular functions on L admits separating points.

Introduction

If μ is an X-valued modular function on a lattice L and $\mathcal{U}(\mu)$ is the weakest lattice uniformity on L which makes μ uniformly continuous, the *control problem* for μ is to find a real-valued modular function ν such that $\mathcal{U}(\mu) = \mathcal{U}(\nu)$. A related question is to investigate, for a family M of X-valued modular functions, under which conditions there exists an X-valued modular function ν on L such that $\mathcal{U}(\nu) = \sup\{\mathcal{U}(\mu) : \mu \in M\}$.

In [W5], H. Weber studied the existence of a real-valued control, proving Bartle-Dunford-Schwartz and Rybakov type theorems for modular functions on complemented lattices, and in [F-T1] I. Fleischer and T. Traynor, correcting a statement of [K2], proved the Rybakov theorem for modular functions on distributive lattices.

In this paper, we study the control problem for families of modular functions on complemented or sectionally complemented lattices (Section 2): We prove that, if X is a complete Hausdorff locally convex linear space, every uniformly exhaustive sequence of X-valued modular functions has an X-valued control. This result has been proved by L. Drewnowski in [D1] and A. Basile in [B1] for measures on Boolean rings. We also study some consequences of the Bartle-Dunford-Schwartz theorem.

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In the second part (Section 3), we study the question of when a countable family M of group-valued modular functions on L admits a *separating point*, i.e. a point $a \in L$ such that $\mu(a) \neq \nu(a)$ for any pair (μ, ν) of different members of M. In particular, we prove that, if L is σ-complete, every countable family of σ-o.c. nonatomic group-valued modular functions on L admits a dense G_δ-set of separating points with respect to a suitable lattice uniformity. This result has been proved in [B-W2] for countable families of measures on Boolean rings.

Essential tools are the theory of lattice uniformities developed in [W1] and, in Section 2, the main result of [A-W], which allows us, as in [W5], to reduce the search for a control for modular functions to the search for a control for suitable measures on Boolean algebras.

The study of modular functions on complemented lattices includes the study of measures on Boolean algebras and, more generally, of additive functions (with respect to disjoint elements) on orthomodular lattices (see the Introduction to [W4]), which are connected with non-commutative measure theory. Moreover, linear operators on Riesz spaces and, more generally, measures on minimal clans (see [S] and [C]) or on Δ-ℓ-semigroups (see [B-W1]) are examples of modular functions on lattices. Other examples are in [K1], [K-W], [P1] and [P2]. For an investigation of modular functions, we refer, for example, to [B2], [F-T1], [F-T2], [T], [W2], [W3], [W4] and [W5] (see also [A1], [A2] and [A-L]).

1. Preliminaries

Let L be a lattice. If L has a smallest or a greatest element, we denote these elements, respectively, by 0 and 1. We set $\Delta = \{(a,b) \in L \times L : a = b\}$. If $\{a_\alpha\}$ is an increasing net and $a = \sup a_\alpha$ in L (respectively, if $\{a_\alpha\}$ is decreasing and $a = \inf a_\alpha$ in L), we write $a_\alpha \uparrow a$ (respectively, $a_\alpha \downarrow a$). If L has 0, a sequence $\{a_n\}$ in L is called *disjoint* if $a_i \wedge a_j = 0$ for each $i \neq j$ and *strongly disjoint* if, for every $i \geq 2$, $a_i \wedge \bigvee_{j<i} a_j = 0$. If $a \leq b$, we set $[a,b] = \{c \in L : a \leq c \leq b\}$. If N_0 is a *congruence relation*, i.e., an equivalence relation which is closed with respect to finite supremum and infimum, we denote by $\hat{a}$ the equivalence class of an element $a \in L$. If L has 0 and 1, the set $C(L)$ of all complemented neutral elements of L is called the *centre* of L. By [G2; Chapter III, Section 4], $C(L)$ is a Boolean algebra. If L is a complemented lattice, we denote by a' a complement of an element $a \in L$. We refer to [G2] or [B2] for the standard definitions and notions for lattices.

A *lattice uniformity* $\mathcal{U}$ on L is a uniformity on L which makes the lattice operations of L uniformly continuous (for an investigation, see [W1]). The set of all lattice uniformities on L is a complete lattice with the discrete uniformity as greatest element and the trivial uniformity as smallest element. We set $N(\mathcal{U}) = \bigcap\{U : U \in \mathcal{U}\} = \overline{\Delta}^{\mathcal{U}}$. $N(\mathcal{U})$ is a congruence relation. $\mathcal{U}$ is called *exhaustive* if every monotone sequence in L is Cauchy in $(L,\mathcal{U})$, *σ-order continuous* (σ-o.c.) if $a_n \uparrow a$ or $a_n \downarrow a$ imply $a_n \to a$ in $(L,\mathcal{U})$ and *order continuous* (o.c.) if the same condition holds for nets. If $\mathcal{U}$ is a lattice uniformity, we denote by $\mathcal{LU}(L,\mathcal{U})$ the lattice of all the lattice uniformities on L weaker than $\mathcal{U}$.

We shall use the following results.

THEOREM (1.1). ([W1; 1.2.4] and [A-W; 5.2]) *Let $\mathcal{W}$ be a lattice uniformity, let $\hat{L}$ be the quotient of L with respect to $N(\mathcal{W})$ and, for $\mathcal{U} \in \mathcal{LU}(L,\mathcal{W})$, let $\hat{\mathcal{U}}$ be the quotient uniformity on $\hat{L}$ generated by $\mathcal{U}$. Then:*

(1) *$\hat{\mathcal{W}}$ is a Hausdorff lattice uniformity on $\hat{L}$ and $\mathcal{U}$ is exhaustive if and only if $\hat{\mathcal{U}}$ is exhaustive.*
(2) *The map $\mathcal{U} \in \mathcal{LU}(L,\mathcal{W}) \to \hat{\mathcal{U}} \in \mathcal{LU}(\hat{L},\hat{\mathcal{W}})$ is a lattice isomorphism.*

THEOREM (1.2). ([W1; 1.3.1, 6.3], [A-W; 5.3], [W5; 3.1.8]) *Let $\mathcal{W}$ be a Hausdorff lattice uniformity and $(\tilde{L},\tilde{\mathcal{W}})$ the uniform completion of $(L,\mathcal{W})$. Then:*

(1) *$\tilde{L}$ becomes a lattice if one extends the lattice operations from L to $\tilde{L}$ by continuity, $\tilde{\mathcal{W}}$ is a lattice uniformity and $\tilde{\mathcal{U}} \in \mathcal{LU}(\tilde{L},\tilde{\mathcal{W}})$ is exhaustive if and only if $\tilde{\mathcal{U}}|_L$ is exhaustive.*
(2) *The map $\tilde{\mathcal{U}} \in \mathcal{LU}(\tilde{L},\tilde{\mathcal{W}}) \to \tilde{\mathcal{U}}|_L \in \mathcal{LU}(L,\mathcal{W})$ is a lattice isomorphism.*
(3) *If $\mathcal{W}$ is exhaustive, then $\tilde{\mathcal{W}}$ is o.c. and $(\tilde{L},\leq)$ is complete.*
(4) *If $\mathcal{W}$ is exhaustive and L is modular and complemented or sectionally complemented, then $\tilde{L}$ is complemented and modular.*

If $(G,+)$ is an Abelian group, a function $\mu\colon L \to G$ is called *modular* if, for every $a,b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$. We set

$$N(\mu) = \{(a,b) \in L \times L : \mu \text{ is constant on } [a \wedge b, a \vee b]\}.$$

By [W4; 2.5], $N(\mu)$ is a congruence.

If G is a topological group and $\mu\colon L \to G$ is a modular function, by [W4; 3.1], there exists the weakest lattice uniformity which makes μ uniformly continuous and a basis of $\mathcal{U}(\mu)$ is the family consisting of the sets

$$\{(a,b) \in L \times L : \mu(c) - \mu(d) \in W \text{ for all } c,d \in [a \wedge b, a \vee b]\},$$

where W is a 0-neighbourhood in G. Moreover, if G is Hausdorff, $N(\mathcal{U}(\mu)) = N(\mu)$. We denote by $\tau(\mu)$ the topology generated by $\mathcal{U}(\mu)$. If L is a Boolean algebra and μ is a measure, then $\tau(\mu)$ is the Fréchet-Nikodym-topology (FN-topology) generated by μ. A modular function μ is called, respectively, *exhaustive*, *σ-o.c.* or *o.c.* if $\mathcal{U}(\mu)$ is, respectively, exhaustive, σ-o.c. or o.c.. By [W4; 3.5 and 3.6], μ is exhaustive if and only if, for every monotone sequence $\{a_n\}$

in L, $\{\mu(a_n)\}$ is Cauchy in G and μ is σ-o.c. (respectively, o.c.) if and only if $\mu(a_\alpha) \to \mu(a)$ for every monotone sequence (respectively, net) $\{a_\alpha\}$ order-convergent to a in L.

THEOREM (1.3). ([W4; 3.7 and 3.8]) *Let $\mathcal{W}$ be an exhaustive Hausdorff lattice uniformity on L, $(\tilde{L}, \tilde{\mathcal{W}})$ the uniform completion of $(L, \mathcal{W})$ and G a complete Hausdorff topological group. Then an arbitrary uniformly continuous modular function $\mu\colon (L, \mathcal{W}) \to G$ extends uniquely to a uniformly continuous o.c. modular function $\tilde{\mu}\colon (\tilde{L}, \tilde{\mathcal{W}}) \to G$. Moreover, $\mathcal{U}(\tilde{\mu})|_L = \mathcal{U}(\mu)$ and $\tau(\tilde{\mu})|_L = \tau(\mu)$.*

THEOREM (1.4). *Let G be a topological group and let $\mu\colon L \to G$ be a modular function. Then:*

(1) *If $N \subseteq N(\mu)$ is a congruence and $\hat{L} = L/N$, then the function $\hat{\mu}\colon \hat{L} \to G$ defined by $\hat{\mu}(\hat{a}) = \mu(a)$ for $a \in \hat{a} \in \hat{L}$ is well defined and $\mathcal{U}(\hat{\mu})$ is the quotient uniformity generated by $\mathcal{U}(\mu)$ on $\hat{L}$.*

(2) *If $N = N(\mu)$, then $\hat{L} = L/N$ is modular. Moreover, if L is complemented or sectionally complemented, then $\hat{L}$ is relatively complemented.*

(3) *Let G be Hausdorff, $\mathcal{W}$ a lattice uniformity on L, $\hat{L} = L/N(\mathcal{W})$ and $\hat{\mathcal{W}}$ the quotient uniformity generated by $\mathcal{W}$ on $\hat{L}$. Then, if $\mu\colon (L, \mathcal{W}) \to G$ is uniformly continuous, $\hat{\mu}(\hat{a}) = \mu(a)$ for $a \in \hat{a} \in \hat{L}$ defines a uniformly continuous modular function $\hat{\mu}\colon (\hat{L}, \hat{\mathcal{W}}) \to G$.*

Proof.

(1) is trivial.

(2) has been proved in [W4; 2.5].

(3) Since $\mathcal{U}(\mu)$ is the weakest lattice uniformity which makes μ uniformly continuous, we get $\mathcal{U}(\mu) \leq \mathcal{W}$ and therefore $N(\mathcal{W}) \subseteq N(\mathcal{U}(\mu)) = N(\mu)$. Hence $\hat{\mu}$ is well defined by (1). Moreover, by (1.1), $\mathcal{U}(\hat{\mu}) \leq \hat{\mathcal{W}}$ and therefore $\hat{\mu}$ is uniformly continuous with respect to $\hat{\mathcal{W}}$. □

COROLLARY (1.5). *For each $\alpha \in A$, let G_α be a topological Abelian group and $M = \{\mu_\alpha : \alpha \in A\}$ a family of G_α-valued modular functions on L. Then:*

(1) *The quotient $L' = L/\bigcap_{\alpha \in A} N(\mu_\alpha)$ is modular and the functions $\hat{\mu}_\alpha\colon L' \to G_\alpha$ defined by $\hat{\mu}_\alpha(\hat{a}) = \mu_\alpha(a)$ for $a \in \hat{a} \in L'$ are well defined modular functions. Moreover, if L is complemented or sectionally complemented, L' is relatively complemented.*

(2) *If $\mathcal{U} = \sup\{\mathcal{U}(\mu_\alpha) : \alpha \in A\}$ and G_α are Hausdorff, then $L' = L/N(\mathcal{U})$.*

Proof.

(1) follows by (1.4), since, if we set $\lambda = (\mu_\alpha)_{\alpha \in A}$, then $\lambda\colon L \to \prod_{\alpha \in A} G_\alpha$ is a modular function such that $N(\lambda) = \bigcap_{\alpha \in A} N(\mu_\alpha)$.

(2) Since G_α are Hausdorff, $N(\mathcal{U}) = \bigcap_{\alpha\in A} N(\mathcal{U}(\mu_\alpha)) = \bigcap_{\alpha\in A} N(\mu_\alpha)$. □

Remark (1.6). If $\mu: L \to G$ is a modular function, the function $\mu': L \to G$ defined by $\mu'(a) = \mu(a) - \mu(0)$ for $a \in L$ is a modular function with $\mu'(0) = 0$ and $\mathcal{U}(\mu') = \mathcal{U}(\mu)$.

We shall use the following notation.

NOTATION (1.7). If $\mathcal{W}$ is a lattice uniformity on L, we set $\hat{L} = L/N(\mathcal{W})$, $\hat{\mathcal{W}}$ is the quotient uniformity on $\hat{L}$ and $(\tilde{L}, \tilde{\mathcal{W}})$ is the completion of $(\hat{L}, \hat{\mathcal{W}})$.

If $\mathcal{U} \in \mathcal{LU}(L, \mathcal{W})$, then $\tilde{\mathcal{U}}$ is the unique element of $\mathcal{LU}(\tilde{L}, \tilde{\mathcal{W}})$ such that $\tilde{\mathcal{U}}|_{\hat{L}} = \hat{\mathcal{U}}$ and $\bar{\mathcal{U}}$ is the restriction of $\tilde{\mathcal{U}}$ to the centre $C(\tilde{L})$ of $\tilde{L}$.

If $\mathcal{W}$ is exhaustive, G is a Hausdorff topological group and $\mu: (L, \mathcal{W}) \to G$ is a uniformly continuous modular function, then $\hat{\mu}: \hat{L} \to G$ is the function defined by $\hat{\mu}(\hat{a}) = \mu(a)$ for $a \in \hat{a} \in \hat{L}$. Moreover, if G is also complete, $\tilde{\mu}: (\tilde{L}, \tilde{\mathcal{W}}) \to G$ is the unique uniformly continuous o.c. extension of $\hat{\mu}$ to $\tilde{L}$ and $\bar{\mu}$ is the restriction of $\tilde{\mu}$ to $C(\tilde{L})$.

THEOREM (1.8). *Let X be a complete Hausdorff locally convex linear space, let L be complemented or sectionally complemented and let $\mathcal{W}$ be an exhaustive lattice uniformity on L. Then:*

(1) *If $M_{\mathcal{W}}(L)$ denotes the linear space of all uniformly continuous modular functions $\mu: (L, \mathcal{W}) \to X$ with $\mu(0) = 0$, and L is modular or orthomodular, then the map $\mu \in M_{\mathcal{W}}(L) \to \bar{\mu} \in M_{\bar{\mathcal{W}}}(C(\tilde{L}))$ is a monomorphism and the restriction of $\mathcal{U}(\tilde{\mu})$ to $C(\tilde{L})$ is $\mathcal{U}(\bar{\mu})$.*

(2) *If $X = \mathbb{R}$ and $\mathcal{W}$ is the supremum of the lattice uniformities generated by all the bounded real-valued modular functions, then the map $\mu \to \bar{\mu}$ of (1) is an isometric isomorphism.*

Proof.

(2) has been proved in [W5; 3.2.3].

(1) is a modification of (3.2.4) of [W5]. For the proof, we first observe that, by (1.3) and (1.4), the map $\mu \in M_{\mathcal{W}}(L) \to \tilde{\mu} \in M_{\tilde{\mathcal{W}}}(\tilde{L})$ is an isomorphism. Then we can apply 4.1 of [W2] to $\tilde{L}$, in which we replace the assumption $\bigcap\{N(\mu) : \mu: L \to \mathbb{R}$ o.c. modular function$\} = \Delta$ with the assumption that L is modular or orthomodular, and observe that, as in 3.2.4 of [W5], we can replace real-valued modular functions by X-valued modular functions via the Hahn-Banach theorem. □

By $\mathbb{N}$ and $\mathbb{R}$ we denote the sets of all integer and of all real numbers, respectively.

2. Control of modular functions

In this section, L is a *complemented* or a *sectionally complemented lattice* and X, Y are *complete Hausdorff locally convex linear spaces.*

DEFINITION (2.1). Let $\mu\colon L \to X$, $\nu\colon L \to Y$ be modular functions. We say that μ is *ν-continuous* (and we write $\mu \ll \nu$) if $\mathcal{U}(\mu) \leq \mathcal{U}(\nu)$ or, equivalently, if μ is uniformly continuous with respect to $\mathcal{U}(\nu)$.

DEFINITION (2.2). If $\mathcal{U}$ is a lattice uniformity on L, a *control* for $\mathcal{U}$ in Y is a Y-valued modular function ν such that $\mathcal{U} = \mathcal{U}(\nu)$.

If M is a family of X-valued modular functions on L, a control for M in Y is a control for $\mathcal{U}(M) = \sup\{\mathcal{U}(\mu):\ \mu \in M\}$.

In particular, a modular function $\nu\colon L \to Y$ is a control for a modular function $\mu\colon L \to X$ if and only if $\mu \ll \nu \ll \mu$.

PROPOSITION (2.3). *Let μ and ν be as in* (2.1), *with* $\mu(0) = \nu(0) = 0$. *Then the following conditions are equivalent:*

(1) $\mu \ll \nu$.
(2) *μ is continuous with respect to $\tau(\nu)$.*
(3) *For every 0-neighbourhood W in X, there exists a 0-neighbourhood W_0 in Y such that $a \in L$ and $\nu([0,a]) \subseteq W_0$ imply $\mu(a) \in W$.*

Moreover, if μ is exhaustive, another equivalent condition is:

(4) *For every continuous and linear functional x' on X, $x' \circ \mu \ll \nu$.*

Proof. By [A-L; 3.3.1], μ is continuous in 0 if and only if μ is uniformly continuous with respect to $\mathcal{U}(\nu)$. Then the equivalence of (1), (2) and (3) follows, since (3), by [A1; 1.5], means that μ is continuous in 0 with respect to $\tau(\nu)$.

(1) $\Longrightarrow$ (4) is trivial. (4) $\Longrightarrow$ (2) follows by the fact that, by [W4; 6.3],

$$\tau(\mu) = \sup\{\tau(x' \circ \mu):\ x' \in X'\},$$

where X' is the dual of X. □

PROPOSITION (2.4). *Let μ, ν be as in* (2.1), *with* $\mu(0) = \nu(0) = 0$. *Suppose that one of the following conditions is satisfied:*

(1) *L is complete and μ, ν are o.c.*
(2) *L is σ-complete, μ, ν are σ-o.c. and $\mathcal{U}(\mu)$, $\mathcal{U}(\nu)$ have a countable base.*

Then $\mu \ll \nu$ if and only if, for every $a \in L$, $\nu(b) = 0$ for every $b \leq a$ implies $\mu(b) = 0$ for every $b \leq a$.

Proof.

(1) $\Longrightarrow$ is trivial.

$\Longleftarrow$ Denote by $\mathcal{W}$ the supremum of all o.c. lattice uniformities on L. By [W1; 6.3], $(L, \mathcal{W})$ is exhaustive and complete. By assumption, $\overline{\{0\}}^{\mathcal{U}(\mu)} \supseteq \overline{\{0\}}^{\mathcal{U}(\nu)}$, since, for an X-valued modular function μ, with $\mu(0) = 0$, $\overline{\{0\}}^{\mathcal{U}(\mu)} = \{a \in L : \mu([0,a]) = \{0\}\}$. Since $\overline{\{0\}}^{\mathcal{U}(\mu)}$ and $\overline{\{0\}}^{\mathcal{U}(\nu)}$ are the kernel ideals, respectively, of $N(\mathcal{U}(\mu))$ and $N(\mathcal{U}(\nu))$, by [G2; Theorem III.3.10] we obtain $N(\mathcal{U}(\mu)) \supseteq N(\mathcal{U}(\nu))$. Since $\mathcal{U}(\mu), \mathcal{U}(\nu) \leq \mathcal{W}$, by [W1; 6.7] we get $\tau(\mu) \leq \tau(\nu)$, from which, by (2.3), $\mu \ll \nu$.

(2) Set $\mathcal{U} = \mathcal{U}(\mu) \vee \mathcal{U}(\nu)$ and $\hat{L} = L/N(\mathcal{U})$. By [W1; 7.1.9], $\hat{L}$ is σ-complete and $\hat{\mathcal{U}}$ is σ-o.c. Moreover $\hat{\mathcal{U}}$ is metrizable and, by [W1; 8.1.2], is exhaustive. Then, by [W1; 8.1.4], $\hat{L}$ is complete and $\hat{\mathcal{U}}$ is o.c. Therefore $\hat{\mu}$ and $\hat{\nu}$ are o.c., too. Since L can be replaced by $\hat{L}$ and μ, ν by $\hat{\mu}$, $\hat{\nu}$, we may suppose L complete and μ, ν o.c. Then (2) follows by (1). □

Definition (2.5). Let M be a family of X-valued modular functions on L.

We say that M is *uniformly exhaustive* if, for every monotone sequence $\{a_n\}$ in L, $\{\mu(a_n)\}$ is Cauchy in X uniformly for $\mu \in M$.

If $\mathcal{U}$ is a lattice uniformity on L, we say that M is *$\mathcal{U}$-equicontinuous* if M is uniformly equicontinuous with respect to $\mathcal{U}$.

Remark (2.6). Let M be as in (2.5). Set $\lambda = (\mu)_{\mu \in M}$, and denote by $\mathcal{U}_\infty$ the uniformity of the uniform convergence in X^M. Then, by (2.5), we get:

M is uniformly exhaustive if and only if $\lambda \colon L \to (X^M, \mathcal{U}_\infty)$ is exhaustive;

If $\mathcal{U}$ is a lattice uniformity on L, then M is $\mathcal{U}$-equicontinuous if and only if $\lambda \colon (L, \mathcal{U}) \to (X^M, \mathcal{U}_\infty)$ is uniformly continuous.

By (2.3), (2.6) and by [A-L; 3.2.2], we obtain the following result.

Proposition (2.7). *Let M be a family of X-valued modular functions on L such that $\mu(0) = 0$ for each $\mu \in M$. Then:*

1. *M is uniformly exhaustive if and only if, for every strongly disjoint sequence $\{a_n\}$ in L, $\{\mu(a_n)\}$ converges to 0 in X uniformly with respect to $\mu \in M$.*
2. *If $\mathcal{U}$ is a lattice uniformity on L, then M is $\mathcal{U}$-equicontinuous if and only if, for every 0-neighbourhood W in X, there exists $U \in \mathcal{U}$ such that $a \in L$ and $(a, 0) \in U$ imply $\mu(a) \in W$ for every $\mu \in M$.*

An essential tool for the proof of the main result is the following.

Theorem (2.8). ([A-W; 2.1, 5.6, 5.7 and 5.10], [W5; 3.2.1]) *Let L be a modular or orthomodular lattice and $\mathcal{W}$ an exhaustive lattice uniformity on L.*

With the notation (1.7)*, the map* $\phi\colon \mathcal{U} \in \mathcal{LU}(L, \mathcal{W}) \to \bar{\mathcal{U}} \in \mathcal{LU}(C(\tilde{L}), \bar{\mathcal{W}}) \to \left(\sup \overline{\{0\}}^{\bar{\mathcal{U}}}\right)' \in C(\tilde{L})$ *is a lattice isomorphism.*

As noted in Section 1 (1.8), in [W5; 3.2.3 and 3.2.4], H. Weber proved that the map $\mu \to \bar{\mu}$ defines a monomorphism from the linear space of all exhaustive X-valued modular functions on L with $\mu(0) = 0$ into the linear space of all order-continuous X-valued measures on $C(\tilde{L})$. Moreover, if $X = \mathbb{R}$ and $\mathcal{W}$ is the supremum of the lattice uniformities generated by all the bounded real-valued modular functions, the map $\mu \to \bar{\mu}$ is an isometric isomorphism.

A consequence of this result and of (2.8) is that an exhaustive X-valued modular function μ has a real-valued control if and only if $\bar{\mu}$ has a real-valued control. Instead, if M is a family of X-valued modular functions, the existence of an X-valued control for $\{\bar{\mu} : \mu \in M\}$ is not sufficient, since, in the locally convex case, it is not known if the map $\mu \to \bar{\mu}$ is surjective. To be precise, the following result holds.

LEMMA (2.9). *Let* $\mathcal{W}$ *be an exhaustive lattice uniformity on* L *and* M *a family of exhaustive* X*-valued modular functions on* L*. With the notation* (1.7)*, we have:*

1. *Suppose* L *is modular or orthomodular and that every element of* M *is uniformly continuous with respect to* $\mathcal{W}$*. Then, if* $\nu\colon (L, \mathcal{W}) \to Y$ *is a uniformly continuous modular function,* ν *is a control for* M *if and only if* $\bar{\nu}$ *is a control for the family* $\bar{M} = \{\bar{\mu} : \mu \in M\}$*.*
2. *If* $\mathcal{W}$ *is the supremum of the lattice uniformity generated by all the bounded real-valued modular functions,* M *has an exhaustive real-valued control if and only if* $\bar{M}$ *has an exhaustive real-valued control.*

Proof.

(1) Set $\mathcal{U} = \sup\{\mathcal{U}(\mu) : \mu \in M\}$ and $\mathcal{V} = \sup\{\mathcal{U}(\bar{\mu}) : \mu \in M\}$. By (2.8), the map $\phi\colon \mathcal{U} \in \mathcal{LU}(L, \mathcal{W}) \to \bar{\mathcal{U}} \in \mathcal{LU}(C(\tilde{L}), \bar{\mathcal{W}})$ is a lattice isomorphism. Then $\mathcal{U}(\nu) = \mathcal{U}$ if and only if $\mathcal{U}(\tilde{\nu})|_{C(\tilde{L})} = \tilde{\mathcal{U}}|_{C(\tilde{L})}$ and, by (1.8), we get

$$\tilde{\mathcal{U}}|_{C(\tilde{L})} = \sup\{\mathcal{U}(\tilde{\mu})|_{C(\tilde{L})} : \mu \in M\} = \sup\{\mathcal{U}(\bar{\mu}) : \mu \in M\} = \mathcal{V}$$

and

$$\mathcal{U}(\tilde{\nu})|_{C(\tilde{L})} = \mathcal{U}(\bar{\nu}) .$$

Therefore $\mathcal{U}(\nu) = \mathcal{U}$ if and only if $\mathcal{U}(\bar{\nu}) = \mathcal{V}$.

(2) By (1.5), we may suppose L modular. By [W4; 6.3], for each $\mu \in M$, $\tau(\mu) = \sup\{\tau(x' \circ \mu) : x' \in X'\}$, where X' is the dual of X. By [A-W; 2.1], we get $\mathcal{U}(\mu) = \sup\{\mathcal{U}(x' \circ \mu) : x' \in X'\}$. By (2.7), every $x' \circ \mu$ is exhaustive and

therefore bounded. Hence every $\mu \in M$ is uniformly continuous with respect to $\mathcal{W}$. Then (2) follows by (1) and by the fact that, by (2.8), $\nu \to \bar{\nu}$ is an isomorphism. $\square$

For the proof of the control theorem, we use the following result of [B1] (see proof of Lemma 1 of [B1]).

LEMMA (2.10). *Let $\mathcal{A}$ be a complete Boolean algebra, τ_0 an o.c. Hausdorff FN-topology on $\mathcal{A}$ and M a family of τ_0-continuous X-valued measures on $\mathcal{A}$. Set $\tau = \sup\{\tau(\mu) : \mu \in M\}$ (where $\tau(\mu)$ is the FN-topology generated by μ), $a_M = \left(\sup \overline{\{0\}}^{\tau}\right)'$ and $a_\mu = \left(\sup \overline{\{0\}}^{\tau(\mu)}\right)'$ for $\mu \in M$. Suppose that, for each $n \in \mathbb{N}$, there exist $a_n \in \mathcal{A}$, $\mu_n \in M$ and a τ_0-continuous measure ν_n^* on $\mathcal{A}$ such that the following conditions hold:*

(1) *$\{a_n\}$ is disjoint.*
(2) *$a_n \le a_{\mu_n}$ for each $n \in \mathbb{N}$.*
(3) *$a_M = \sup\limits_n a_n$.*
(4) *$\tau(\nu_n^*) = \tau(\nu_n)$, where $\nu_n(a) = \mu_n(a \wedge a_n)$ for $a \in \mathcal{A}$.*
(5) *The series $\sum\limits_{n=1}^{\infty} \nu_n^*$ is uniformly convergent on $\mathcal{A}$.*

Then the measure $\gamma = \sum\limits_{n=1}^{\infty} \nu_n^*$ is a control for M.

Now we can prove the main result of this section, which has been proved in [B1; Theorem 2] for measures on Boolean rings.

THEOREM (2.11). *Every uniformly exhaustive sequence $\{\mu_n : n \in \mathbb{N}\}$ of X-valued modular functions on L has a control in X.*

Proof. By (1.6), we can suppose $\mu_n(0) = 0$ for each $n \in \mathbb{N}$. We use the notation (1.7) with $\mathcal{W} = \sup\{\mathcal{U}(\mu_n) : n \in \mathbb{N}\}$. By (1.5), $\hat{L}$ is sectionally complemented and modular and, by (1.1), $\hat{\mathcal{W}}$ is exhaustive and $\mathcal{LU}(L, \mathcal{W})$, $\mathcal{LU}(\hat{L}, \hat{\mathcal{W}})$ are isomorphic. Then, by (2.8), the map

$$\phi : \mathcal{U} \in \mathcal{LU}(L, \mathcal{W}) \to \bar{\mathcal{U}} \in \mathcal{LU}(C(\tilde{L}), \bar{\mathcal{W}}) \to \left(\sup \overline{\{0\}}^{\bar{\mathcal{U}}}\right)' \in C(\tilde{L})$$

is a lattice isomorphism.

The first step is to prove that the assumptions of (2.10) are satisfied for the family $\bar{M} = \{\bar{\mu}_n : n \in \mathbb{N}\}$.

Denote by τ_0 and τ_n the topologies generated, respectively, by $\bar{\mathcal{W}}$ and by $\mathcal{U}(\tilde{\mu}_n)|_{C(\tilde{L})}$. By [W1; 6.10], τ_0 is an o.c. Hausdorff FN-topology on $C(\tilde{L})$. Moreover, since ϕ is a lattice isomorphism, we have $\tau_0 = \sup\{\tau_n : n \in \mathbb{N}\}$ and, using again [W1; 6.10] and the equality $\mathcal{U}(\tilde{\mu}_n)|_{C(\tilde{L})} = \mathcal{U}(\bar{\mu}_n)$ of (1.8), τ_n is the

FN-topology generated by $\bar{\mu}_n$. Then $\bar{M}$ is a family of τ_0-continuous measures on the complete Boolean algebra $C(\tilde{L})$.

Now, for each $n \in \mathbb{N}$, set

$$a_{\mu_n} = \phi(\mathcal{U}(\mu_n)), \quad a_1 = a_{\mu_1}, \quad a_n = a_{\mu_n} \setminus \Big(\bigvee_{i \leq n-1} a_{\mu_i} \Big) \qquad \text{for} \quad n \geq 2.$$

Hence $\{a_n\}$ is a disjoint sequence in $C(\tilde{L})$, with $a_n \leq a_{\mu_n} = \big(\sup \overline{\{0\}}^{\tau_n}\big)'$ and, since ϕ is a lattice isomorphism,

$$\sup_n a_n = \sup_n a_{\mu_n} = \sup\{\phi(\mathcal{U}(\mu_n)) : n \in \mathbb{N}\} = \phi(\mathcal{W}) = \big(\sup \overline{\{0\}}^{\tau_0}\big)'.$$

Set, for $a \in \tilde{L}$,

$$\tilde{\nu}_n(a) = \tilde{\mu}_n(a \wedge a_n), \quad \tilde{\nu}_n^* = 2^{-n}\tilde{\nu}_n \qquad \text{for} \quad n \in \mathbb{N}.$$

Since $\{a_n\} \subseteq C(\tilde{L})$, $\tilde{\nu}_n$ is a $\tilde{\mathcal{W}}$-continuous modular function. Moreover, since $\mathcal{U}(\tilde{\nu}_n^*) = \mathcal{U}(\tilde{\nu}_n)$, by (2.3), $\mathcal{U}(\tilde{\nu}_n^*) \leq \tilde{\mathcal{W}}$.

Now we prove that the series $\sum_{n=1}^{\infty} \tilde{\nu}_n^*$ is uniformly convergent on $\tilde{L}$.

Observe that, since $\{\mu_n : n \in \mathbb{N}\}$ is uniformly exhaustive, by (1.3), (1.4) and (2.6), $\{\tilde{\mu}_n : n \in \mathbb{N}\}$ is uniformly exhaustive, too. Then, by (2.7), we get $\lim_n \tilde{\nu}_n(a) = 0$ for each $a \in \tilde{L}$, since $\{a \wedge a_n\}$ is a strongly disjoint sequence. Therefore, if we set

$$\tilde{h} = (\tilde{\nu}_n)_{n\in\mathbb{N}} \qquad \text{and} \qquad H = \Big\{ f \in X^N : f(N) \text{ is bounded} \Big\},$$

$\tilde{h}$ is a H-valued modular function and, by the uniform exhaustivity of $\{\tilde{\mu}_n\}$, $\tilde{h}$ is exhaustive in $(H, \mathcal{U}_\infty)$, where $\mathcal{U}_\infty$ is the uniformity of the uniform convergence. Then, by [W4; 2.3], $\tilde{h}$ is bounded in $(H, \mathcal{U}_\infty)$ (we use that $(H, \mathcal{U}_\infty)$ is locally convex and the notion of boundedness given in [W4] coincides with the classical notion in locally convex linear spaces). This is equivalent to the boundedness of $\bigcup_{n\in\mathbb{N}} \tilde{\nu}_n(\tilde{L})$ in X. Hence, if W is an absolutely convex 0-neighbourhood in X, we may choose $\varepsilon > 0$ such that $\varepsilon\Big(\bigcup_{n\in\mathbb{N}} \tilde{\nu}_n(\tilde{L}) \Big) \subseteq W$. Now let $r \in \mathbb{N}$ be such that $\sum_{i>r} 2^{-i} < \varepsilon$. Then, if $p > q \geq r$ and $a \in \tilde{L}$, we get $\sum_{i=q}^{p} \tilde{\nu}_i^*(a) = \sum_{i=q}^{p} \frac{1}{2^i\varepsilon}(\varepsilon\tilde{\nu}_i(a))$ $\in W$, namely, the series $\sum_{n=1}^{\infty} \tilde{\nu}_n^*$ is uniformly convergent on $\tilde{L}$.

Then all the assumptions of (2.10) are satisfied.

Therefore, if we set

$$\tilde{\gamma}(a) = \sum_{n=1}^{\infty} \tilde{\nu}_n^*(a) \qquad \text{for} \quad a \in \tilde{L},$$

by (2.10), $\tilde{\gamma}|_{C(\tilde{L})}$ is a control for $\tilde{M}$. Moreover, since $\mathcal{U}(\tilde{\nu}_n^*) \le \tilde{\mathcal{W}}$ and the series $\sum_{n=1}^{\infty} \tilde{\nu}_n^*$ is uniformly convergent on $\tilde{L}$, we get $\mathcal{U}(\tilde{\gamma}) \le \tilde{\mathcal{W}}$. Then we can apply (2.9) to the family $\tilde{M} = \{\tilde{\mu}_n : n \in \mathbb{N}\}$ and we obtain that $\tilde{\gamma}$ is a control for $\tilde{M}$. By (1.1), (1.2), (1.3) and (1.4), $\{\mu_n : n \in \mathbb{N}\}$ has an X-valued control. □

As proved in $[B_1]$, in (2.11) the uniform exhaustivity cannot be omitted. This is possible if $\sup\{\mathcal{U}(\mu_n) : n \in \mathbb{N}\}$ has a countable base, since an immediate consequence of the Bartle-Dunford-Schwartz theorem for modular functions (see [W5; 1.3.9]) is the following result.

PROPOSITION (2.12). *If $\{X_\alpha : \alpha \in A\}$ is a family of locally convex Hausdorff linear spaces and, for each $\alpha \in A$, μ_α is an exhaustive X_α-valued modular function on L, then $M = \{\mu_\alpha : \alpha \in A\}$ has a real-valued control if and only if $\mathcal{U}(M) = \sup\{\mathcal{U}(\mu_\alpha) : \alpha \in A\}$ has a countable base (and, in this case, for each $\alpha \in A$, M has an X_α-valued control, too).*

P r o o f. We apply (1.3.9) of [W5] to the function $\lambda \colon L \to \prod_{\alpha \in A} X_\alpha$ defined by $\lambda(a) = \big(\mu(a)\big)_{\mu \in M}$ for $a \in L$, since $\mathcal{U}(\lambda) = \mathcal{U}(M)$. □

The next result, which has been proved in [D1; 10.7] and [D2; 3.2] for measures on Boolean rings, gives a representation of the control in the case that X is a Banach space.

THEOREM (2.13). *Let X be a Banach space and M a family of exhaustive X-valued modular functions on L such that $\mathcal{U}(M) = \sup\{\mathcal{U}(\mu) : \mu \in M\}$ has a countable base. Then there exist a sequence $\{c_n\}$ in $\mathbb{R}$ and a sequence $\{\mu_n\}$ in M such that $\sum_{n=1}^{\infty} |c_n| < +\infty$, the series $\sum_{n=1}^{\infty} \|c_n \mu_n(a)\|$ is uniformly convergent for $a \in L$ and $\mu_0 = \sum_{n=1}^{\infty} c_n \mu_n$ is a control for M.*

P r o o f. We use the notation (1.7) with $\mathcal{W} = \mathcal{U}(M)$. We may suppose that $\mathcal{W}$ is Hausdorff, since we can replace L by $L/N(\mathcal{W})$. Since $\mathcal{U}(M)$ has a countable base, we can find $\{\mu_n\}$ in M such that $\mathcal{U}(M) = \sup\{\mathcal{U}(\mu_n) : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, set $\mu_n'(a) = \mu_n(a) - \mu_n(0)$ for $a \in L$ and denote by $\tilde{\lambda}_n$ the uniformly continuous extension of μ_n' to $(\tilde{L}, \tilde{\mathcal{W}})$. Then, by [W4; 2.3], every $\tilde{\lambda}_n$ is bounded, since, by (1.2), it is exhaustive. Set $\alpha_n = \sup\{\|\tilde{\lambda}_n(a)\| : a \in \tilde{L}\}$. We may suppose $\alpha_n > 0$ for each $n \in \mathbb{N}$. Observe that, if we set $\nu_n = \frac{\mu_n'}{n\alpha_n}$ for each $n \in \mathbb{N}$, then

$$\mathcal{U}(M) = \sup\{\mathcal{U}(\nu_n) : n \in \mathbb{N}\}.$$

Since $\{\bar{\nu}_n : n \in \mathbb{N}\}$ is a sequence of exhaustive measures on the complete Boolean algebra $C(\tilde{L})$, which is pointwise convergent on $C(\tilde{L})$, by the theorem of Brooks-Jewett $\{\bar{\nu}_n : n \in \mathbb{N}\}$ is uniformly exhaustive and therefore $\bar{\nu} = (\bar{\nu}_n)_{n\in\mathbb{N}}$ is exhaustive with respect to the uniformity $\mathcal{U}_\infty$ of the uniform convergence in $X^{\mathbb{N}}$. Then, if we consider the measure which corresponds to $\bar{\nu}$ on the Stone algebra of $C(\tilde{L})$ and the extension with respect to $\mathcal{U}_\infty$ to the σ-algebra generated, we obtain a uniformly σ-additive sequence of measures on a σ-algebra. Applying the control theorem of Drewnowski for measures (see [D1; 10.7]) and the Stone argument, we obtain a sequence $\{d_n\}$ in $\mathbb{R}$ such that the series $\sum_{n=1}^{\infty} \|d_n\bar{\nu}_n\|$ is uniformly convergent on $C(\tilde{L})$ and the function

$$\bar{\nu}_0 = \sum_{n=1}^{\infty} d_n\bar{\nu}_n$$

is a control for $\{\bar{\nu}_n : n \in \mathbb{N}\}$. Moreover, by the proof of 10.7 of [D1], we can see that it is possible to choose d_n such that, setting $c_n = \frac{d_n}{n\alpha_n}$, the series

$$\sum_{n=1}^{\infty} |c_n|, \qquad \sum_{n=1}^{\infty} |d_n| \qquad \text{and} \qquad \sum_{n=1}^{\infty} \|c_n\mu_n(0)\|$$

are convergent. For $a \in \tilde{L}$, set

$$\tilde{\nu}_0(a) = \sum_{n=1}^{\infty} d_n\tilde{\nu}_n(a)\,.$$

Since, for $a \in \tilde{L}$,

$$\sum_{k=n+1}^{n+h} \|d_k\tilde{\nu}_k(a)\| \le \sum_{k=n+1}^{n+h} |d_k|\frac{\|\tilde{\lambda}_k(a)\|}{k\alpha_k} \le \sum_{k=n+1}^{n+h} |d_k|,$$

the series $\sum_{n=1}^{\infty} \|d_n\tilde{\nu}_n\|$ is uniformly convergent on $\tilde{L}$. Since $\mathcal{U}(d_n\tilde{\nu}_n) = \mathcal{U}(\tilde{\nu}_n) \le \tilde{\mathcal{W}}$, we get $\mathcal{U}(\tilde{\nu}_0) \le \tilde{\mathcal{W}}$. Then we can apply (2.9) to $\tilde{M} = \{\tilde{\nu}_n : n \in \mathbb{N}\}$ and we obtain that $\tilde{\nu}_0$ is a control for $\tilde{M}$. By (1.1), (1.2), (1.3) and (1.4), the function

$$\nu_0 = \sum_{n=1}^{\infty} d_n\nu_n$$

is a control for $\{\nu_n : n \in \mathbb{N}\}$ and therefore for M.

Now we prove that $\mu_0 = \sum_{n=1}^{\infty} c_n\mu_n$ is a control for M. Since

$$\nu_0(a) = \sum_{n=1}^{\infty} \frac{d_n}{n\alpha_n}\mu_n'(a) = \mu_0(a) - \sum_{n=1}^{\infty} c_n\mu_n(0)$$

and the series $\sum_{n=1}^{\infty} \|c_n\mu_n(0)\|$ is convergent, we get $\mathcal{U}(\mu_0) = \mathcal{U}(\nu_0) = \mathcal{U}(M)$. Moreover, since $\|c_n\mu_n(a)\| \le \|c_n\mu'_n(a)\| + \|c_n\mu_n(0)\| \le |d_n| + \|c_n\mu_n(0)\|$ for $a \in L$, the series $\sum_{n=1}^{\infty} \|c_n\mu_n\|$ is uniformly convergent on L. □

Other immediate consequences of (2.9) are the following results, which have been proved in [L] and [D2] for measures on Boolean rings.

COROLLARY (2.14). *Let $\mu\colon L \to X$ be an exhaustive modular function such that $\mathcal{U}(\mu)$ has a countable base. Then there exist a metrizable locally convex linear space Z and a continuous linear map $h\colon X \to Z$ such that the function $h \circ \mu$ is a control for μ in Z.*

Proof. We may suppose $\mu(0) = 0$. We use the notation (1.7) with $\mathcal{W} = \mathcal{U}(\mu)$. By (1.8), $\mathcal{U}(\bar\mu) = \mathcal{U}(\tilde\mu)|_{C(\tilde L)}$ and then $\mathcal{U}(\bar\mu)$ has a countable base, too. Hence, if we apply the Stone argument and the result of Lipecki for measures (see Theorem 1 and Note before Theorem 1 in [L]), we obtain a metrizable locally convex linear space Z and a continuous linear map $h\colon X \to Z$ such that $h \circ \bar\mu$ is a control for $\bar\mu$. Set $\lambda = h \circ \mu$. Since $\tau(\tilde\lambda) \le \tau(\tilde\mu)$, by (2.3) and (2.9) $\tilde\lambda$ is a control for $\tilde\mu$ and therefore λ is a control for μ. □

We say that X has the *countable summability condition* (CSC) if every family $\{x_\alpha : \alpha \in A\}$ in $X \setminus \{0\}$, such that every countable subfamily is summable, is at most countable.

Examples of spaces with CSC are in [D2].

COROLLARY (2.15). *If X verifies CSC, then, for every complemented lattice L_0, every exhaustive X-valued modular function on L_0 has a real-valued control.*

Proof. We use the notation (1.7) with $\mathcal{W}$ as in (2.9)(2). By [D2; 4.4] and the Stone argument, for every complemented lattice L_0 and for every exhaustive modular function $\mu\colon L_0 \to X$, $\bar\mu$ has a real-valued control. Then we apply (2.9)(2). □

We recall that a Banach space X is called a *Gould space* (*G-space*) if it has the following property: For every sequence $\{x_n\}$ in X, such that $\{\|x_n\|\}$ has a positive lower bound, for every $c > 0$, there exists a finite subsequence $\{x_{n_r}\}$ such that $\|\sum_r x_{n_r}\| > c$.

Gould has shown that every weakly complete Banach space is a G-space (see [G1]).

The following result is a consequence of the Rybakov theorem for modular functions ([W5; 3.2.5]) and it has been proved in the Boolean case in [O].

COROLLARY (2.16). *Let X be a Banach space and $\mu: L \to X$ a modular function. Then:*

(1) *If μ is exhaustive, μ has a control $\nu: L \to [0, +\infty[$ such that $\nu(a) \leq \sup\{\|\mu(b)\| : b \leq a\}$ for every $a \in L$.*

(2) *If X is a G-space, μ is exhaustive if and only if it is bounded (and therefore in (1) we can replace "exhaustive" with "bounded").*

P r o o f.

(1) By the Rybakov theorem for modular functions, there exists $x' \in X'$ such that

$$\mathcal{U}(\mu) = \mathcal{U}(x' \circ \mu) \tag{$*$}$$

and we may suppose that $\|x'\| \leq 1/2$. Denote by ν the total variation of $x' \circ \mu$. By [W4; 2.7 and 2.8], ν is bounded and therefore, by [W5; 1.3.11], $\tau(x' \circ \mu) = \tau(\nu)$. Hence, by [A-W; 2.1] and by $(*)$, we get that ν is a $[0, +\infty[$-valued control for μ. Moreover, by [A1; 1.8], for $a \in L$, we get

$$\nu(a) \leq 2 \sup\{|x'(\mu(b))| : b \leq a\} \leq \sup\{\|\mu(b)\| : b \leq a\}.$$

(2) By [W4; 2.3], every exhaustive modular function is bounded. Now suppose μ bounded and not exhaustive. We may suppose $\mu(0) = 0$. By (2.7), we can find a strongly disjoint sequence $\{a_n\}$ in L and $\varepsilon > 0$ such that $\|\mu(a_n)\| \geq \varepsilon$ for each $n \in \mathbb{N}$. Since X is a G-space, for every $c > 0$ we can find a finite subsequence $\{a_{n_r}\}$ such that $\|\sum_r \mu(a_{n_r})\| > c$. Since $\{a_n\}$ is strongly disjoint and $\mu(0) = 0$, we obtain, by induction, $\mu\left(\bigvee_r a_{n_r}\right) = \sum_r \mu(a_{n_r})$. Then, for every $c > 0$, we can find $a \in L$ such that $\|\mu(a)\| > c$, a contradiction. $\square$

Now, to study some consequences of (2.12), we need some facts concerning the countable property, which has been introduced in [W1] and plays the role of the countable chain condition in Boolean algebras.

DEFINITION (2.17). We say that L has the *countable property* (CP) if every subset A of L contains an at most countable subset which has the same upper bounds and the same lower bounds as A.

PROPOSITION (2.18). *Suppose L with CP and let $D \subseteq L$ be dense, i.e. for every $a \in L$, with $a \neq 0$, there exists $d \in D$ such that $d \neq 0$ and $d \leq a$. Then, for every $a \in L$, with $a \neq 0$, there exists an at most countable subset D_0 of D such that $a = \sup D_0$.*

P r o o f. Let $a \in L$, with $a \neq 0$, and set $D_1 = \{d \in D : 0 \neq d \leq a\}$. By assumption, $D_1 \neq \emptyset$. We prove that $a = \sup D_1$. Let c be an upper bound of D_1 and suppose that $c \geq a$ is not true. Let d_1 be a relative complement of

$a \wedge c$ in $[0,a]$. Hence $d_1 \neq 0$ and, by assumption, we can find $d_0 \in D$ such that $0 \neq d_0 \leq d_1$. Therefore $d_0 \in D_1$ and $c \geq d_0$. Hence $d_0 \leq d_1 \wedge (c \wedge a) = 0$, a contradiction. By (CP), we can find $D_0 \subseteq D_1 \subseteq D$ at most countable such that $a = \sup D_0$. □

We use the following result of [W6].

PROPOSITION (2.19). *Suppose L is modular or orthomodular. If $\mathcal{U}$ is an exhaustive lattice uniformity on L and $\tilde{L}$ is the completion of $L/N(\mathcal{U})$, then the following conditions are equivalent:*

(1) *$\mathcal{U}$ has a countable base.*
(2) *$\mathcal{LU}(L,\mathcal{U})$ has the Countable Chain Condition.*
(3) *$\mathcal{C}(\tilde{L})$ has the Countable Chain Condition.*
(4) *$\tilde{L}$ has (CP).*

Moreover, if L is σ-complete and $\mathcal{U}$ is σ-o.c., another equivalent condition is:

(5) *$L/N(\mathcal{U})$ has (CP).*

COROLLARY (2.20). *Let $\mu\colon L \to X$ be an exhaustive modular function. Then the following conditions are equivalent:*

(1) *μ has a real-valued control.*
(2) *$\mathcal{U}(\mu)$ has a countable base.*
(3) *The completion of $L/N(\mathcal{U}(\mu))$ has (CP).*
(4) *$\mathcal{LU}(L,\mathcal{U}(\mu))$ has the Countable Chain Condition.*

Moreover, if L is σ-complete and μ is σ-o.c., another equivalent condition is:

(4) *$L/N(\mathcal{U}(\mu))$ has (CP).*

Proof.
(1) $\Longrightarrow$ (2) is trivial.
(2) $\Longrightarrow$ (1) follows by [W5; 1.3.9].
The other equivalences follow by (2.19), since we may suppose L to be modular. □

COROLLARY (2.21). *If X is a Banach space and M is a uniformly exhaustive family of X-valued modular functions on L, then $\mathcal{U}(M) = \sup\{\mathcal{U}(\mu) : \mu \in M\}$ has a countable base.*

Proof. By (1.5), we may suppose L to be modular. Set $\nu = (\mu)_{\mu \in M}$. By assumption, $\nu\colon L \to G^M$ is an exhaustive modular function with respect to the uniformity generated by $\|f\| = \sup\{\|f(\mu)\| : \mu \in M\}$ in G^M and then $\mathcal{U}(\nu)$ has a countable base. Since $\mathcal{U}(M) \leq \mathcal{U}(\nu)$, by (2.19) we get that $\mathcal{LU}(L,\mathcal{U}(M)) \subseteq \mathcal{LU}(L,\mathcal{U}(\nu))$ has the Countable Chain Condition and therefore $\mathcal{U}(M)$ has a countable base, too. □

COROLLARY (2.22). *Suppose L to be modular or orthomodular and σ-complete. Then, if L has (CP), every σ-o.c. lattice uniformity on L has a countable base.*

Proof. By [W1; 8.1.4], $\mathcal{U}$ is o.c. Hence, by [W1; 6.2 and 4.4] the quotient map $\pi\colon L \to L/N(\mathcal{U})$ has the following property: If $A \subseteq L$, $a = \sup A$ and $b = \inf A$, then $\pi(a) = \sup \pi(A)$ and $\pi(b) = \inf \pi(A)$. Then $L/N(\mathcal{U})$ has (CP) and we can apply (2.19). □

Now we can study some consequences of (2.12).

COROLLARY (2.23). *Let M be a family of X-valued modular functions on L. Then the following conditions are equivalent:*

(1) *M is uniformly exhaustive and $\mathcal{U}(M) = \sup\{\mathcal{U}(\mu) : \mu \in M\}$ has a countable base.*
(2) *There exists an exhaustive modular function $\nu\colon L \to \mathbb{R}$ such that M is ν-equicontinuous.*

Moreover, if X is normed, the condition that $\mathcal{U}(M)$ has a countable base is superfluous.

Proof.
(1) $\Longrightarrow$ (2) By (2.12), M has a control $\nu\colon L \to \mathbb{R}$. Then every $\mu \in M$ is continuous with respect to $\mathcal{U}(\nu)$. By [W4; 6.2], M is equicontinuous with respect to $\mathcal{U}(\nu)$.

(2) $\Longrightarrow$ (1) We may suppose L to be modular by (1.4), since we can replace L by $L/N(\mathcal{U}(\nu))$ and M by $\hat{M} = \{\hat{\mu} : \mu \in M\}$. Since $\mathcal{U}(M) \leq \mathcal{U}(\nu)$, by (2.19) we get, as in (2.21), that $\mathcal{U}(M)$ has a countable base. Moreover the function $\lambda = (\mu)_{\mu\in M}\colon (L,\mathcal{U}(\nu)) \to (X^M,\mathcal{U}_\infty)$ (where $\mathcal{U}_\infty$ is the uniformity of the uniform convergence) is uniformly continuous and therefore exhaustive. Hence, by (2.6), M is uniformly exhaustive.

If X is normed and (2) holds, then $\mathcal{U}(M)$ has a countable base by (2.21). □

COROLLARY (2.24). *Suppose L to be modular or orthomodular. Then:*

(1) *If $\mathcal{U}$ is an exhaustive lattice uniformity on L with a countable base, for every locally convex linear space Y there exists a modular function $\nu\colon L \to \mathbb{R}$ with the following properties:*
   (a) *ν is uniformly continuous with respect to $\mathcal{U}$.*
   (b) *For every uniformly continuous modular function $\mu\colon (L,\mathcal{U}) \to Y$, $\mu \ll \nu$.*
(2) *If L is σ-complete with (CP) and X is a Banach space, there exists an o.c. modular function $\lambda\colon L \to X$ such that, for every σ-o.c. modular function $\mu\colon L \to X$, $\mu \ll \lambda$.*

P r o o f . Set $M = \{\mu\colon L \to Y : \mathcal{U}(\mu) \le \mathcal{U}\}$ and $\mathcal{V} = \sup\{\mathcal{U}(\mu) : \mu \in M\}$. Since $\mathcal{V} \le \mathcal{U}$, by (2.19), as in (2.21), we get that $\mathcal{V}$ has a countable base. Now the result follows applying (2.12) to M.

(2) Let $\mathcal{U} = \sup\{\mathcal{U}(\mu) : \mu\colon L \to X \ \sigma\text{-o.c. modular function}\}$. By (2.22), $\mathcal{U}$ has a countable base. By (2.12), $\mathcal{U}$ has a control $\lambda\colon L \to X$. By [W1; (8.1.4)], λ is o.c. □

COROLLARY (2.25). *Suppose L to be modular or orthomodular. Let X be a Banach space and $\mathcal{U}$ an exhaustive lattice uniformity on L with the following properties:*

(1) *$\mathcal{U}$ has a countable base.*
(2) *For every $\mathcal{V} \in \mathcal{LU}(L,\mathcal{U})$, there exists a non-constant modular function $\mu\colon L \to X$ such that $\mathcal{U}(\mu) \le \mathcal{V}$.*

Then $\mathcal{U}$ has a control in X.

P r o o f . By (2.19) and (2.8), $\mathcal{LU}(L,\mathcal{U})$ is a Boolean algebra with the Countable Chain Condition. Hence we can apply (2.18) to

$$D = \{\mathcal{V} \in \mathcal{LU}(L,\mathcal{U}) : \text{ exists a modular function } \mu\colon L \to X : \mathcal{U}(\mu) = \mathcal{V}\}$$

and we can find a sequence $\{\mu_n\}$ of X-valued modular functions on L such that $\mathcal{U} = \sup\{\mathcal{U}(\mu_n) : n \in \mathbb{N}\}$. Now we apply (2.12). □

Remark. The converse of (2.25) is also true, since a consequence of the general decomposition theorem of [W6] is that, if $\mathcal{U}$ is an exhaustive lattice uniformity which has a control, then every element of $\mathcal{LU}(L,\mathcal{U})$ has a control.

COROLLARY (2.26). *Suppose L to be σ-complete, modular or orthomodular with (CP). Suppose that, for every $a \in L \setminus \{0\}$, there exists a σ-o.c. modular function $\mu_a\colon L \to X$ such that $\mu_a(a) \ne 0$. Then there exists an o.c. modular function $\mu\colon L \to [0,+\infty[$, with $\mu(0) = 0$, such that, for every $a,b \in L$, with $a \ne b$, μ is not constant on $[a \wedge b, a \vee b]$ (in particular, since μ is monotone, $\mu(a) > 0$ for each $a \ne 0$).*

P r o o f . Let $\mathcal{U} = \sup\{\mathcal{U}(\mu_a) : a \in L \setminus \{0\}\}$. By (2.22), $\mathcal{U}$ has a countable base and, by [W1; 8.1.2], μ_a is exhaustive for each $a \in L \setminus \{0\}$. Then, by (2.12), $\mathcal{U}$ has a real-valued control μ, which we may suppose takes values in $[0,+\infty[$ and satisfies $\mu(0) = 0$ (see [W5; 1.3.9]). Therefore we have to prove that $\mathcal{U}$ is o.c. and Hausdorff. $\mathcal{U}$ and $\mathcal{U}(\mu_a)$ are o.c. by [W1; 8.1.4]. Then, if we set $b = \sup \overline{\{0\}}^{\mathcal{U}}$ and $b_a = \sup \overline{\{0\}}^{\mathcal{U}(\mu_a)}$, we get $\overline{\{0\}}^{\mathcal{U}} = [0,b]$ and $\overline{\{0\}}^{\mathcal{U}(\mu_a)} = [0,b_a]$, since L is complete by (CP). Since L is sectionally complemented, by [G1; Theorem III.3.10], $N(\mathcal{U}) = \theta_b$ and $N(\mathcal{U}(\mu_a)) = \theta_{b_a}$, where, for $c \in L$, $\theta_c = \{(a,b) \in L \times L : a \vee c = b \vee c\}$. Since $\mathcal{U} \ge \mathcal{U}(\mu_a)$, we get $b \le b_a$ and therefore $\mu_a(b) = 0$ for each $a \in L \setminus \{0\}$. The assumption on $\{\mu_a : a \in L \setminus \{0\}\}$ implies $b = 0$, from which $N(\mathcal{U}) = \Delta$. □

3. Separating points of sequences of modular functions

In this section (see 3.11), we extend to modular functions a result of [B-W2] concerning with the set of separating points of a sequence of nonatomic or exhaustive measures on Boolean rings.

We denote by G a *topological Abelian group.*

First we give characterizations of nonatomic lattice uniformities. We introduce $\perp$-lattices to unify the proofs of the results which involve strongly disjoint sequences in sectionally complemented lattices and orthogonal sequences in orthomodular lattices.

A $\perp$*-lattice* is a lattice with 0 and a binary relation $\perp$ which satisfies the following conditions:

$a \perp 0$ for each $a \in L$;
$a \perp a \implies a = 0$;
$a \perp b \implies b \perp a$;
$a \perp b$ and $c \leq a \implies c \perp b$.

Then $a \perp b$ implies $a \wedge b = 0$.

$(L, \perp)$ is called *sectionally semi-orthocomplemented* (s.s.c.) if, for every $a, b \in L$, with $a \leq b$, there exists $c \in L$, called *relative semi-orthocomplement* of a in $[0, b]$, such that $c \perp a$ and $c \vee a = b$. A sequence $\{a_n\}$ in $(L, \perp)$is called *semi-independent* if, for every $n \geq 2$, $a_n \perp \bigvee_{j<n} a_j$.

Every lattice is a $\perp$-lattice if we define $a \perp b$ if and only if $a \wedge b = 0$. Every orthomodular lattice is a s.s.c. $\perp$-lattice if we define $a \perp b$ if and only if $a \leq b^{\perp}$, where $b^{\perp}$ is the orthocomplement of b, and a sequence is semi-independent if and only if is orthogonal.

DEFINITION (3.1). If L has 0, we say that a lattice uniformity $\mathcal{U}$ on L is *nonatomic* if, for every $a \in L$ and every 0-neighbourhood U_0 in $\mathcal{U}$, there exists a finite subset A of U_0 such that $a = \sup A$.

If μ is a G-valued modular function on L, we say that μ is nonatomic if $\mathcal{U}(\mu)$ is nonatomic.

PROPOSITION (3.2). *Let L be a sectionally complemented lattice and $\mathcal{U}$ a lattice uniformity on L. Then the following conditions are equivalent:*

(1) *$\mathcal{U}$ is nonatomic.*
(2) *For every $a, b \in L$, with $a \leq b$, and every $U \in \mathcal{U}$, there exist $a_1, \ldots, a_{n-1}$ in L such that $a = a_0 \leq a_1 \leq \cdots \leq a_n = b$ and $(a_{i-1}, a_i) \in U$ for each $i \leq n$.*

Moreover, if L has 1, another equivalent condition is:

(3) *For every 0-neighbourhood U_0 in $\mathcal{U}$, there exists a finite subset A of U_0 such that $1 = \sup A$.*

If L is a s.s.c. $\perp$-lattice, another equivalent condition is:

(4) *For every $a \in L$ and every 0-neighbourhood U_0 in $\mathcal{U}$, there exists a semi-independent finite set $A \subseteq U_0$ such that $a = \sup A$.*

If L is a s.s.c. $\perp$-lattice with 1, another equivalent condition is:

(5) *For every 0-neighbourhood U_0 in $\mathcal{U}$, there exists a semi-independent finite set $A \subseteq U_0$ such that $1 = \sup A$.*

Proof. For the equivalence of (1) and (2), see proof of 1.3 of [A1].

(1) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (2):

(i) We first prove (2) with $a = 0$ and $b = 1$. Let $U, V \in \mathcal{U}$ such that $\Delta \vee V \subseteq U$ and choose, by (3), $a_1, \dots, a_n$ in L such that $\bigvee_{i\le n} a_i = 1$ and $(0, a_i) \in V$ for each $i \le n$. Set $b_0 = 0$ and $b_i = \bigvee_{j \le i} a_j$ for $i \le n$. Then $0 = b_0 \le b_1 \le \cdots \le b_n = 1$ and

$$(b_{i-1}, b_i) = \Big(\bigvee_{j\le i-1} a_i, \bigvee_{j\le i-1} a_i\Big) \vee (0, a_i) \in \Delta \vee V \subseteq U .$$

(ii) Now let a, b in L with $a \le b$ and $U \in \mathcal{U}$. Choose V, V' in $\mathcal{U}$ such that $V' \wedge \Delta \subseteq V$ and $V \vee \Delta \subseteq U$ and, by (i), $b_1, \dots, b_{n-1}$ in L such that $0 = b_0 \le b_1 \le \cdots \le b_n = 1$ and $(b_i, b_{i-1}) \in V'$ for each $i \le n$. Set, for $i \le n$, $c_i = (b_i \wedge b) \vee a$. Then $a = c_0 \le c_1 \le \cdots \le c_n = b$ and

$$(c_{i-1}, c_i) = \big((b_{i-1}, b_i) \wedge (b, b)\big) \vee (a, a) \in (V' \wedge \Delta) \vee \Delta \subseteq V \vee \Delta \subseteq U .$$

(1) $\Longrightarrow$ (4) Let $a \in L$ and $U \in \mathcal{U}$. Choose symmetric V, V', V'' in $\mathcal{U}$ such that $V' \circ V' \subseteq V$, $\Delta \wedge V \subseteq U$ and the following condition holds:

$$(a_i, b_i) \in V'' \text{ for all } i \le n \Longrightarrow \bigvee_{j\le i} (a_j, b_j) \in V' \text{ for all } i \le n .$$

By (1), we can choose $a_1, \dots, a_n$ in L such that $a = \sup_{i\le n} a_i$ and $(0, a_i) \in V''$ for each $i \le n$. Set $b_i = \bigvee_{j\le i} a_i$ for $i \le n$, $c_1 = b_1$ and, for $i \ge 2$, let c_i be such that $c_i \perp b_{i-1}$ and $c_i \vee b_{i-1} = b_i$. Then $\{c_1, \dots, c_n\}$ is a semi-independent set, since, for each $i \ge 2$, $c_i \perp b_{i-1}$ and $\bigvee_{j\le i-1} c_j \le b_{i-1}$. Moreover, by induction, $\bigvee_{i\le n-1} c_i = b_{n-1}$ and therefore $\bigvee_{i\le n} c_i = c_n \vee b_{n-1} = b_n = \bigvee_{i\le n} a_i = a$. Finally, since $\Big(0, \bigvee_{j\le i} a_j\Big) \in V'$ for $i \le n$, we get

$$\begin{aligned}(0, c_i) &= (c_i \wedge b_{i-1}, c_i \wedge b_i) \\ &= (c_i, c_i) \wedge \Big(\bigvee_{j\le i-1} a_j, \bigvee_{j\le i} a_j\Big) \in \Delta \wedge (V' \circ V') \subseteq \Delta \wedge V \subseteq U .\end{aligned}$$

(4) $\implies$ (5) $\implies$ (3) are trivial. □

If μ is a G-valued modular function on a sectionally complemented lattice, with $\mu(0) = 0$, by [A1; 1.5], a base of 0-neighbourhoods in $\mathcal{U}(\mu)$ is the family consisting of the sets $\{a \in L : \mu([0,a]) \subseteq W\}$, where W is a 0-neighbourhood in G. Using this fact, an immediate consequence of (3.2) is the following result, which has been proved in a different way in [A2] in the sectionally complemented case.

COROLLARY (3.3). *Let L be sectionally complemented and $\mu: L \to G$ a modular function with $\mu(0) = 0$. Then the following conditions are equivalent:*

(1) *μ is nonatomic.*
(2) *For every $a \in L$ and every 0-neighbourhood W in G, there exists a finite $A \subseteq L$ such that $a = \sup A$ and $\mu([0,b]) \subseteq W$ for each $b \in A$.*
(3) *For every $a \in L$ and every 0-neighbourhood W in G, there exist $a_1, \dots, a_n$ in L such that $a = \sup_{i \le n} a_i$, $a_i \wedge \bigvee_{j<i} a_j = 0$ for each $i \ge 2$ and $\mu([0,a_i]) \subseteq W$ for each $i \le n$.*
(4) *For every a, b in L, with $a \le b$, and every 0-neighbourhood W in G, there exist $a_1, \dots, a_{n-1}$ in L such that $a = a_0 \le a_1 \le \dots \le a_n = b$ and $\mu(c) - \mu(d) \in W$ for every $c, d \in [a_{i-1}, a_i]$ with $c \ge d$ and every $i \le n$.*

If L has 1, another equivalent condition is:

(5) *For every 0-neighbourhood W in G, there exist $a_1, \dots, a_n$ in L such that $1 = \sup_{i \le n} a_i$, $a_i \wedge \bigvee_{j<i} a_j = 0$ for each $i \ge 2$ and $\mu([0,a_i]) \subseteq W$ for each $i \le n$.*

Moreover, if L is orthomodular, in (3) and in (5) we may replace the condition $a_i \wedge \bigvee_{j<i} a_j = 0$ with the condition that $\{a_1, \dots, a_n\}$ is orthogonal.

We need the following results in the proofs of (3.7), (3.10) and (3.11).

LEMMA (3.4). *If $\{\mathcal{U}_\alpha : \alpha \in A\}$ is a family of nonatomic lattice uniformities on L, then $\mathcal{U} = \sup\{\mathcal{U}_\alpha : \alpha \in A\}$ is nonatomic, too.*

Proof. Let $a \in L$ and $U \in \mathcal{U}$. Choose $\alpha_1, \dots, \alpha_n \in A$ and $U_i \in \mathcal{U}_{\alpha_i}$ such that $\bigcap_{i \le n} U_i \subseteq U$. Moreover, by [W1; 1.1.3], choose, for each $i \le n$, $V_i \in \mathcal{U}_{\alpha_i}$ such that

$$V_i \subseteq U_i \text{ and } [a \wedge b, a \vee b]^2 \subseteq V_i \text{ whenever } (a,b) \in V_i. \qquad (*)$$

We inductively prove that we can find $a_1, \dots, a_r \in \bigcap_{i \le n} V_i(0)$ such that $a = \bigvee_{i \le r} a_i$. First suppose $n = 2$. Choose $b_1, \dots, b_s \in V_1(0)$ such that $a = \bigvee_{i \le s} b_i$.

Moreover, for each $i \le s$, choose $c_{1i}, \dots, c_{ri} \in V_2(0)$ such that $\bigvee_{j\le r} c_{ji} = b_i$. Since $c_{ji} \in [0, b_i]$, by $(*)$ we get $c_{ji} \in V_1(0) \cap V_2(0)$ for each $j \le r$ and $i \le s$. Moreover, if we denote by $a_1, \dots, a_t$ the elements $\{c_{ji} : j \le r,\ i \le s\}$, then $\bigvee_{i\le t} a_i = a$.

Now, by induction, choose $a_1, \dots, a_s$ in $\bigcap_{i\le n-1} V_i(0)$ such that $a = \bigvee_{i\le s} a_i$ and, for each $i \le s$, $b_{1i}, \dots, b_{ri}$ in $V_n(0)$ such that $\bigvee_{j\le r} b_{ji} = a_i$. As before, since $b_{ji} \in [0, a_i]$, we get $b_{ji} \in \bigcap_{j\le n} V_j(0)$ and $\bigvee_{i\le s} \bigvee_{j\le r} b_{ji} = a$. □

LEMMA (3.5). *Let L be complemented or sectionally complemented, N_0 a congruence relation and $\hat{L} = L/N_0$. Then, for every disjoint (respectively, strongly disjoint) sequence $\{\hat{a}_n\}$ in $\hat{L}$, there exists a disjoint (respectively, strongly disjoint) sequence $\{b_n\}$ in L such that $\hat{b}_n = \hat{a}_n$.*

Proof. Suppose L to be sectionally complemented and let $\{\hat{a}_n\}$ be disjoint in $\hat{L}$. For each $n \in \mathbb{N}$, let b_n be a relative complement of $\sup_{i<n}(a_n \wedge b_i)$ in $[0, a_n]$, where $b_0 = 0$. Then, since $\hat{a}_n \wedge \hat{b}_i \le \hat{a}_n \wedge \hat{a}_i = \hat{0}$ for each $i < n$, we get

$$\hat{b}_n = \hat{b}_n \vee \hat{0} = \hat{b}_n \vee \sup_{i<n}(\hat{a}_n \wedge \hat{b}_i) = \hat{a}_n .$$

Moreover, if $n < m$,

$$b_n \wedge b_m = b_n \wedge a_m \wedge b_m \le \sup_{i<m}(b_i \wedge a_m) \wedge b_m = 0 .$$

For disjoint sequences in complemented lattices, the result has been proved in [W4] (see proof of 2.6). For strongly disjoint sequences, see [A-L; 3.2.1]. □

DEFINITION (3.6). If L has a 0, we say that a lattice uniformity $\mathcal{U}$ on L is *strongly exhaustive* if every disjoint sequence converges to 0 in $\mathcal{U}$.

A modular function $\mu\colon L \to G$ is called *strongly exhaustive* if $\mathcal{U}(\mu)$ is strongly exhaustive.

PROPOSITION (3.7). *Let L be complemented or sectionally complemented and $\mu\colon L \to G$ a modular function with $\mu(0) = 0$. Then μ is strongly exhaustive if and only if $\mu(a_n) \to 0$ for every disjoint sequence $\{a_n\}$ in L.*

Proof.

$\Longrightarrow$ is trivial, since μ is uniformly continuous with respect to $\mathcal{U}(\mu)$.

$\Longleftarrow$ We may suppose L to be sectionally complemented, since, by (3.5), we can replace L by $\hat{L} = L/N(\mu)$ and μ by $\hat{\mu}$. Suppose that $\{a_n\}$ is a disjoint sequence in L which is not convergent to 0 in $\mathcal{U}(\mu)$. By [A1; 1.5] (see remarks before (3.3)), we can find a 0-neighbourhood W in G such that, for each $s \in \mathbb{N}$, there are $n > s$ and $c \leq a_n$ such that $\mu(c) \notin W$. Therefore we can inductively obtain a strictly increasing sequence $\{n_k\}$ in $\mathbb{N}$ and a sequence $\{c_{n_k}\}$ in L such that $\mu(\{c_{n_k}\}) \notin W$ for each $k \in \mathbb{N}$. Since $\{c_{n_k}\}$ is disjoint, we obtain a contradiction. □

Remark. By (1.6) and (3.7) and by [A-L; 3.2.2], we obtain that, if L is complemented or sectionally complemented, every strongly exhaustive modular function is exhaustive. The converse is not true: If L is the orthomodular lattice of all subspaces of $\mathbb{R}^2$ and $\mu(H) = \dim H$ for each subspace H of $\mathbb{R}^2$, then μ is exhaustive, but not strongly exhaustive.

In the proof of (3.10), we need the following result.

PROPOSITION (3.8).

(1) *If $(L, \perp)$ is a s.s.c. $\perp$-lattice, then a semi-independent sequence of non-zero elements exists in L if and only if a strictly increasing sequence exists in L.*

(2) *If L is an infinite sectionally complemented lattice, then there exists a disjoint sequence $\{a_n\}$ in L, with $a_n \neq 0$ for each $n \in \mathbb{N}$.*

Proof.

(1) Let $\{a_n\}$ be a semi-independent sequence in L, with $a_n \neq 0$ for each $n \in \mathbb{N}$, and set $b_n = \bigvee_{i \leq n-1} a_i$, with $b_1 = 0$. Suppose that there exist $n, m \in \mathbb{N}$, with $n < m$, such that $b_n = b_m$. Then, since $\{a_n\}$ is semi-independent, we get $a_n = a_n \wedge b_m = a_n \wedge b_n = 0$, a contradiction. Hence $\{b_n\}$ is a strictly increasing sequence.

Now let $\{a_n\}$ be a strictly increasing sequence in L and, for each $n \in \mathbb{N}$, let b_n be a relative semi-orthocomplement of a_n in $[0, a_{n+1}]$. By [A-L; 2.4], $\{b_n\}$ is a semi-independent sequence and $b_n \neq 0$ for each $n \in \mathbb{N}$, since $a_n \neq a_{n+1}$ for each $n \in \mathbb{N}$.

(2) It is easy to find two non-zero elements a, b in L with $a \wedge b = 0$. By induction, let $a_1, \ldots, a_n$ be such that $a_i \neq 0$ for each $i \leq n$ and $a_i \wedge a_j = 0$ for $i \neq j$. Choose $b \in L$ such that

$$b \neq 0 \qquad \text{and} \qquad b \neq \bigvee_{j \in J} a_j \quad \text{for each } J \subseteq \{1, \ldots, n\},$$

and suppose that $c = b \wedge \bigvee_{j \leq n} a_j \neq 0$. If $c \neq b$, we choose by a_{n+1} a relative complement of c in $[0, b]$, since $a_{n+1} \neq 0$ because of $b \neq c$ and, for each $i \leq n$,

$$a_{n+1} \wedge a_i \leq a_{n+1} \wedge b \wedge \bigvee_{j \leq n} a_j = a_{n+1} \wedge c = 0 .$$

Now suppose that $c = b$. Set $A = \{j \leq n : a_j \leq b\}$ and let d be a relative complement of $\bigvee_{j \in A} a_j$ in $[0, b]$. If there is $h \leq n$ such that $a_h \leq d$, we get $a_h = a_h \wedge d \leq \bigvee_{j \in A} a_j \wedge d = 0$, a contradiction. Hence, for each $j \leq n$, $d \wedge a_j \neq a_j$, and $d \neq 0$ since $b \neq \bigvee_{j \in A} a_j$.

Now let $j \leq n$. If $a_j \wedge d = 0$, we set $b_j = a_j$. If $a_j \wedge d \neq 0$, we choose by b_j a relative complement of $d \wedge a_j$ in $[0, a_j]$. Then $b_j \neq 0$ since $d \wedge a_j \neq a_j$. Moreover, for $i, j \leq n$ and $i \neq j$, we have $b_i \wedge b_j \leq a_i \wedge a_j = 0$ and, for each $i \leq n$, $d \wedge b_i = d \wedge b_i \wedge a_i = 0$. Then, if we set $b_{n+1} = d$, $\{b_1, \ldots, b_{n+1}\}$ is a disjoint family of non-zero elements of L. □

In the next result, we use that, if L is sectionally complemented and $\mathcal{U}$ is a lattice uniformity on L, by [A-W; 2.1] a base of $\mathcal{U}$ is the family consisting of the sets

$$\{(a, b) \in L \times L : \exists c \in F : (a \wedge b) \vee c = a \vee b\},$$

where F is a 0-neighbourhood in $\mathcal{U}$. In particular, every lattice uniformity on L is uniquely determinated by the topology.

Moreover we use the fact that, for a G-valued modular function μ, with $\mu(0) = 0$,

$$\overline{\{0\}}^{\mathcal{U}(\mu)} = \{a \in L : \mu([0, a]) = \{0\}\}.$$

The following result is an essential tool for the proof of the main result.

THEOREM (3.9). *Suppose G Hausdorff and L sectionally complemented. Let $\mathcal{U}$ be a lattice uniformity on L and M a family of G-valued uniformly continuous modular functions on $(L, \mathcal{U})$, with $\mu(0) = 0$ for each $\mu \in M$. Set $A_M = \{a \in L : \mu(a) \neq 0$ for each $\mu \in M\}$ and $I(\mu) = \{a \in L : \mu([0, a]) = \{0\}\}$ for $\mu \in M$. Then:*

1. *If M is finite, A_M is a dense open set in $(L, \mathcal{U})$ if and only if, for each $\mu \in M$, $I(\mu)$ is not open in $(L, \mathcal{U})$.*
2. *If M is countable and $(L, \mathcal{U})$ is a Baire space, A_M is a dense G_δ-set in $(L, \mathcal{U})$ if and only if, for each $\mu \in M$, $I(\mu)$ is not open in $(L, \mathcal{U})$.*

P r o o f. Set $A_\mu = \{a \in L : \mu(a) = 0\}$ for $\mu \in M$. First suppose A_M dense in $(L, \mathcal{U})$. Then, for each $\mu \in M$, A_μ has no inner point in $(L, \mathcal{U})$. Let $\mu \in M$ and suppose $I(\mu)$ is open in $(L, \mathcal{U})$. Set, for $a \in A_\mu$,

$$U_a = \{b \in L : \exists c \in I(\mu) : a \vee b = (a \wedge b) \vee c\}.$$

Then U_a is a neighbourhood of a in $\mathcal{U}$. We prove that $U_a \subseteq A_\mu$, a contradiction.

Let $b \in U_a$ and choose $U, V \in \mathcal{U}(\mu)$ such that $\Delta \vee V \subseteq U$. Choose $c \in I(\mu)$ such that $a \vee b = (a \wedge b) \vee c$. Then

$$(a \wedge b, a \vee b) = (a \wedge b, a \wedge b) \vee (0, c) \in \Delta \vee V \subseteq U,$$

from which $(a\wedge b, a\vee b) \in N(\mu)$. Then μ is constant on $[a\wedge b, a\vee b]$, in particular $\mu(b) = \mu(a) = 0$, from which $b \in A_\mu$.

Now suppose $I(\mu)$ is not open for each $\mu \in M$. We prove that, for each $\mu \in M$, $L \setminus A_\mu$ is dense in $(L,\mathcal{U})$.

Let $\mu \in M$, $a \in A_\mu$ and $U \in \mathcal{U}$. Choose symmetric V, $V' \in \mathcal{U}$ such that $V' \vee \Delta \subseteq U$, $V \subseteq V'$ and $[a \wedge b, a \vee b]^2 \subseteq V$ whenever $(a,b) \in V$ (see [W1; 1.1.3]). We prove that there exists $d \in V(0)$ such that $\mu(d) \neq 0$.

Suppose $\mu(d) = 0$ for each $d \in V(0)$. Then, since $[0,d] \subseteq V(0)$ for each $d \in V(0)$, we get $V(0) \subseteq I(\mu)$. Let $a \in I(\mu)$ and

$$U_a = \{b \in L : \ \exists c \in V(0) : \ a \vee b = (a \wedge b) \vee c\}.$$

Then U_a is a neighbourhood of a in $\mathcal{U}$. Let $b \in U_a$ and $c \in V(0) \subseteq I(\mu)$ such that $a \vee b = (a \wedge b) \vee c$. Since $I(\mu)$ is an ideal, we have c, $a \wedge b \in I(\mu)$, from which $a \vee b \in I(\mu)$ and therefore $b \in I(\mu)$. Hence, for each $a \in I(\mu)$, $U_a \subseteq I(\mu)$, a contradiction.

Now choose $d \in V(0)$ such that $\mu(d) \neq 0$ and let b, c be relative complements of $a \wedge d$ in $[0,d]$ and in $[0,a]$, respectively. Since $d = b \vee (a \wedge d)$, $b \wedge a \wedge d = 0$ and $\mu(d) \neq 0$, we get

$$\mu(b) \neq 0 \qquad \text{or} \qquad \mu(a \wedge d) \neq 0. \tag{$*$}$$

Moreover, since $b \wedge a = b \wedge a \wedge d = 0$ and $b \vee a = a \vee (a \wedge d) \vee b = a \vee d$, we get $\mu(a \vee d) = \mu(b) + \mu(a) = \mu(b)$ and $\mu(c) = \mu(a) - \mu(a \wedge d) = -\mu(a \wedge d)$, since $a \in A_\mu$. By $(*)$, we get $a \vee d \notin A_\mu$ or $c \notin A_\mu$. Since $[0,d] \subseteq V(0) \subseteq V'(0)$, we get

$$(a, a \vee d) = (a,a) \vee (0,d) \in \Delta \vee V' \subseteq U$$

and

$$(a,c) = \big(c \vee (a \wedge d), c\big) = (c,c) \vee (a \wedge d, 0) \in \Delta \vee V' \subseteq U.$$

Then, in any case, we obtain an element $e \in L \setminus A_\mu$ such that $(a,e) \in U$.

Now, since each $\mu \in M$ is uniformly continuous in $(L,\mathcal{U})$, $L \setminus A_\mu$ is open in $(L,\mathcal{U})$ for each $\mu \in M$. Then, if M is a finite, $A_M = \bigcap\limits_{\mu \in M} (L \setminus A_\mu)$ is a finite intersection of open dense sets. Instead, if M is countable, A_M is a G_δ-set, which is dense in $(L,\mathcal{U})$, since $(L,\mathcal{U})$ is a Baire space. □

COROLLARY (3.10). *Suppose G to be Hausdorff and L to be sectionally complemented. Let $\mathcal{U}$ be a lattice uniformity on L, $A \subseteq N$ and $\{\mu_n : \ n \in A\}$ a family of uniformly continuous G-valued modular functions on $(L,\mathcal{U})$, with $\mu_n(0) = 0$ for each $n \in \mathbb{N}$. Suppose that one of the following conditions is satisfied:*

(1) *$\mathcal{U}$ is nonatomic and, for each $n \in A$, μ_n is non-zero.*

(2) *$\mathcal{U}$ is exhaustive and, for each $n \in A$, $\hat{L}_n = L/N(\mu_n)$ contains an infinite chain.*

(3) *$\mathcal{U}$ is strongly exhaustive and, for each $n \in A$, $\hat{L}_n = L/N(\mu_n)$ is infinite.*

Moreover suppose that one of the following conditions is satisfied:

(i) *A is finite.*

(ii) *A is countable, L is σ-complete and $\mathcal{U}$ is σ-o.c.*

Then the set $\{a \in L : \ \mu_n(a) \neq 0 \text{ for each } n \in A\}$ is a dense G_δ-set with respect to $\mathcal{U}$ and therefore non-empty.

Proof. We prove that, in all the cases (1), (2) and (3), for each $n \in A$, $I(\mu_n) = \{a \in L : \ \mu_n([0,a]) = \{0\}\}$ is not open in $(L,\mathcal{U})$.

Suppose that there exists $n \in \mathbb{N}$ such that $I(\mu_n)$ is open in $(L,\mathcal{U})$.

(1): Since $\mathcal{U}$ is nonatomic, for each $a \in L$, we can find $a_1,\dots,a_r$ in $I(\mu_n)$ such that $a = \bigvee_{i\leq r} a_i$ and therefore, since $I(\mu_n)$ is an ideal, we get $I(\mu_n) = L$, from which $\mu_n = 0$.

(2) and (3): By (3.8), we can find in $\hat{L}_n$ a strongly disjoint sequence $\{\hat{a}_k\}$ in the case (2) (disjoint in the case (3)), with $\hat{a}_k \neq \hat{0}$ for each $k \in \mathbb{N}$. By (3.5), we obtain in L a strongly disjoint sequence $\{b_k\}$ in the case (2) (disjoint in the case (3)) such that $\hat{b}_k = \hat{a}_k$ for each $k \in \mathbb{N}$.

In the case (2), since $\mathcal{U}$ is exhaustive, $b_k \to 0$ in $\mathcal{U}$ by [A-L; 3.1.4].

In the case (3), $b_k \to 0$ in $\mathcal{U}$, since $\mathcal{U}$ is strongly exhaustive.

Then, in both the cases (2) and (3), we can find $\nu \in \mathbb{N}$ such that $b_k \in I(\mu_n)$ for each $k > \nu$. Therefore we get $\mu_n([0,b_k]) = 0$ for each $k > \nu$, from which $\hat{a}_k = \hat{b}_k = \hat{0}$ for each $k > \nu$, a contradiction.

Now the result follows by (3.9), since, if (ii) holds, $(L,\mathcal{U})$ is a Baire space by [W1; 3.15]. □

COROLLARY (3.11). *Let L be complemented or sectionally complemented, G as in (3.10), $A \subseteq \mathbb{N}$ and $\{\mu_n : \ n \in A\}$ a family of G-valued modular functions on L, with $\mu_n(0) = 0$ for each $n \in A$ and $\mu_n \neq \mu_m$ for each n, $m \in A$, with $n \neq m$. Suppose that one of the following conditions is satisfied:*

(1) *For each $n \in A$, μ_n is nonatomic.*

(2) *For each $n \in A$, μ_n is exhaustive and, for each $n,m \in A$, with $n \neq m$, $L/N(\mu_n - \mu_m)$ contains an infinite chain.*

(3) *For each $n \in A$, μ_n is strongly exhaustive and, for each $n,m \in A$, with $n \neq m$, $L/N(\mu_n - \mu_m)$ is infinite.*

Moreover suppose that one of the following conditions is satisfied:

(i) *A is finite.*

(ii) *A is countable, L is σ-complete and, for each $n \in \mathbb{N}$, μ_n is σ-o.c.*

Then the set $\{a \in L:\ \mu_n(a) \neq \mu_m(a)$ *for each* $n \neq m\}$ *of separating points of the family* $\{\mu_n:\ n \in A\}$ *is a dense* G_δ*-set with respect to* $\mathcal{U} = \sup\{\mathcal{U}(\mu_n):\ n \in A\}$ *and therefore non-empty.*

Proof. Set $M_0 = \{\mu_n - \mu_m:\ n, m \in A,\ n \neq m\}$ and observe that every element of M_0 is uniformly continuous with respect to $\mathcal{U}$, since, for each $n, m \in A$, we have $\mathcal{U}(\mu_n - \mu_m) \leq \mathcal{U}(\mu_n) \vee \mathcal{U}(\mu_m) \leq \mathcal{U}$. Moreover, in the case (1), $\mathcal{U}$ is nonatomic by (3.4) and, in the cases (2) and (3), $\mathcal{U}$ is exhaustive or strongly exhaustive, respectively. Then, if L is sectionally complemented, we can apply (3.10) to M_0. The complemented case follows by the sectionally complemented case, since we can replace L by $\hat{L} = L/N(\mathcal{U})$ and μ_n by $\hat{\mu}_n$ defined by $\hat{\mu}_n(\pi(a)) = \mu_n(a)$ for $a \in L$, where $\pi\colon L \to \hat{L}$ is the quotient map. This is trivial in the case (1). In the cases (2) and (3), it is sufficient to observe that, for each $n \in \mathbb{N}$, the quotients $L/N(\mu_n - \mu_m)$ and $\hat{L}/N(\hat{\mu}_n - \hat{\mu}_m)$ are isomorphic, since, for $a, b \in L$,

$$(a, b) \in N(\mu_n - \mu_m) \iff (\pi(a), \pi(b)) \in N(\hat{\mu}_n - \hat{\mu}_m)\,.$$

□

In [B-W2; 2.2 and 2.3], the authors proved results similar to (3.10) and (3.11) for exhaustive measures with infinite quotients on Boolean rings. The following simple example proves that (3.10) and (3.11) fail for exhaustive modular functions on sectionally complemented lattices if the quotients are infinite, but do not contain infinite chains.

EXAMPLE. Let L be the orthomodular lattice of all subspaces of $\mathbb{R}^2$ and $\mu\colon L \to \mathbb{R}$ the modular function defined by $\mu(H) = \dim H$ for a subspace H of $\mathbb{R}^2$. In this case, $\mathcal{U}(\mu)$ is the discrete uniformity and therefore the set $\{a \in L:\ \mu(a) \neq 0\}$ is not dense with respect to $\mathcal{U}(\mu)$.

In this example, μ is exhaustive, but not strongly exhaustive, and $L/N(\mu) = L$ does not contain infinite chains.

Remark. As proved in [B-W2], it is not possible in (3.11) to remove the assumptions that A is countable, μ_n σ-o.c. and L σ-complete.

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