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NONATOMIC STATES

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ABSTRACT. We introduce the definition of nonatomicity for a state defined on an orthomodular poset and we prove that the characterization of Boolean algebras which admit nonatomic states does not hold in the case of concrete orthomodular posets. Finally, we show an hypothesis under which this characterization is true for concrete orthomodular posets as well.

1. Introduction

There are satisfactory necessary and sufficient conditions for a Boolean algebra to have nonatomic states defined on it ([2; Theorem 5.3.2]). Now, let P be an orthomodular poset, a problem recently stated was as follows: if $s\colon P \to [0,1]$ is a nonatomic state then has P all those properties that it would have if it was a Boolean algebra on which a nonatomic charge ([2; Definition 2.1.1]) was defined or, equivalently, does the characterization of Boolean algebras for which the set of nonatomic states defined on them is non-empty continue to hold in orthomodular posets? Here, investigating this problem in concrete orthomodular posets, we prove that the answer to this question turns out to be negative and this essentially happens since an orthomodular poset could be made of an initial system of blocks (an almost disjoint system of Boolean algebras) ([6; Definition 2.4.2]) two by two having trivial intersection and, therefore, completely independent. Nevertheless, some implications of the characterization showed by K. P. S. Bhaskara Rao and by M. Bhaskara Rao continue to hold and, under a particular hypothesis on the Stone space ([4; p. 78]) of the Boolean algebra into which a concrete OMP can be embedded ([7; Proposition 1.3]), it is possible to characterize concrete orthomodular posets having a nonatomic state defined on them using techniques similar to those utilized for Boolean algebras.

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2. Preliminaries

A pair $(P, \leq)$ where P is a non empty set and $\leq$ is a partial ordering is called *partially ordered set* or *poset*. If the supremum (resp. the infimum) of $\{x, y\} \subseteq P$ exists in P we shall denote it by $x \vee y$ (resp. $x \wedge y$).

An *orthoposet* is a quintuple $(P, \leq, 0, 1, ')$ that fulfils the following requirements:

1. $(P, \leq)$ is a poset having a least and a greatest element, 0, 1,
2. $' \colon P \to P$ is an orthocomplementation, i.e.
   for any x, y in P the following conditions are satisfied:
   (i) $x'' = x$,
   (ii) $x \leq y \implies x' \geq y'$,
   (iii) $x \vee x' = 1$.

Two elements x, y in P are said to be *orthogonal* if $x \leq y'$ and, in this case, we shall write $x \perp y$.

An orthoposet P is an *orthomodular poset* if

(i) $x \vee y$ exists for any pair (x, y) of orthogonal elements in P,
(ii) $y = x \vee (y \wedge x')$, for any x, y in P such that $x \leq y$ (orthomodular law).

Let P be an OMP (orthomodular poset), for x, y in P, we shall say that x *commutes* with y and we shall write $x \mathrm{C} y$ if $x \wedge y$ and $x \wedge y'$ both exist in P and $x = (x \wedge y) \vee (x \wedge y')$.

Let $C = \{x \in P : x \mathrm{C} y \text{ for all } y \in P\}$, C is called the *center* of P. The set C is a Boolean algebra ([1], [5]).

A subset P_1 of an OMP P closed under orthocomplementation to which 0 and 1 belong is an *orthomodular subposet* of P if and only if $x \perp y$, $x, y \in P_1$, implies $x \vee y \in P_1$.

An OMP is said to be *concrete* if it consists of a family of subsets of a set Ω with the following properties:

(i) $\emptyset \in P$,
(ii) if $A \in P$ $(A \subseteq \Omega)$, then $\Omega \setminus A \in P$,
(iii) if $A, B \in P$ $(A, B \subseteq \Omega)$ and $A \cap B = \emptyset$ then $A \cup B \in P$ ([6; p. 2]).

Let P and Q be two orthomodular posets. An *homomorphism* $f \colon P \to Q$ is a mapping that satisfies the following conditions:

(i) $f(0) = 0$,
(ii) $f(x') = f(x)'$, for any x in P,
(iii) $f(x \vee y) = f(x) \vee f(y)$, for any x, y in P with $x \perp y$ ([6; Definition 1.2.7]).

When f is bijective and f^{-1} is a homomorphism as well, we shall say that f is an *isomorphism*.

If $f \colon P \to f(P) \subseteq Q$ is an isomorphism, we shall call f an *embedding* and we shall say that P is an orthomodular subposet of Q.

Observe that if f is a bijective homomorphism such that $f(x) \leq f(y')$ implies $x \leq y'$ for any x, y in P, then f is an isomorphism.

If an OMP P is isomorphic to a concrete OMP, we shall call it *concrete*.

Let P be an OMP. We shall call a *state* on P a mapping $s\colon P \to [0,1]$ with the following properties:

1. $s(1) = 1$,
2. for any $x, y \in P$ such that $x \perp y$ $\quad s(x \vee y) = s(x) + s(y)$.

3. Nonatomic states

DEFINITION 3.1. Let P be an OMP. An element x in P is said to be an *atom* if it satisfies:

(i) $x \neq 0$,
(ii) if $y \in P$ and $y \leq x$, then either $y = x$ or $y = 0$.

DEFINITION 3.2. Let P be an OMP and let $s \in S(P)$. An element x in P is said to be an s*-atom* if it fulfils:

(i) $s(x) \neq 0$,
(ii) if $y \in P$ and $y \leq x$, then either $s(y) = s(x)$ or $s(y) = 0$.

DEFINITION 3.3. Let P be an OMP and s a state on P. The state s is said to be *nonatomic* on P if there are no s-atoms in P or, equivalently, if, for any x in P satisfying $s(x) > 0$, there exists y in P such that $y < x$ and $0 < s(y) < s(x)$.

DEFINITION 3.4. Let P be an OMP and let s be a state on P, s is said to be *strongly continuous* on P if for every $\varepsilon > 0$ there exists a partition $\{x_1, \dots, x_n\}$ of 1 in P such that $s(x_i) < \varepsilon$, for every $i \leq n$.

DEFINITION 3.5. Let P be an OMP. A collection of non-zero elements $\{x_{i_1,\dots,i_k} :\ i_1, \dots, i_k$ is any finite sequence of 0's and 1's, $k \geq 1\}$ in P is said to be a *tree* in P if the following conditions are satisfied:

(i) $x_1 = (x_0)'$,
(ii) $(x_{i_1,\dots,i_{k-1},1}) \leq (x_{i_1,\dots,i_{k-1},0})'$,
(iii) $x_{i_1,\dots,i_{k-1},0} \vee x_{i_1,\dots,i_{k-1},1} = x_{i_1,\dots,i_{k-1}}$.

To prove that some implications of the characterization of Boolean algebras for which the set of nonatomic states defined on them is non-empty ([2; Theorem 5.3.2]) continue to hold in concrete orthomodular posets, we first recall a definition and two theorems already known.

DEFINITION 3.6. ([6; Definition 1.3.18]) Let P be an OMP. A subset X of P is called *compatible* if for any finite subset $\{x_1, \dots, x_n\}$ of X there exists a finite subset $G = \{y_1, \dots, y_m\}$ of P such that

(i) the set G consists of mutually orthogonal elements,

(ii) for any i $(i \le n)$ there exists a subset H_i of the set $\{1, \dots, m\}$ such that

$$x_i = \bigvee_{j \in H_i} y_j .$$

THEOREM 3.7. *Let P be an OMP and let X be a compatible subset of P. Then there exists a Boolean subalgebra P_0 of P such that $X \subseteq P_0$* [6; Theorem 1.3.23]).

THEOREM 3.8. *Let A be a Boolean algebra and let s be a charge on A. If s is strongly continuous then s is nonatomic as well* ([2; Theorem 5.1.6]).

THEOREM 3.9. *Let P be a concrete OMP. Consider the following statements:*

(i) *there is a nonatomic state on P,*

(ii) *P contains a tree,*

(iii) *P has a countable atomless subalgebra,*

(iv) *there is a strongly continuous state on P.*

Then the following implications hold: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). *Also,* (ii) *and* (iii) *are equivalent,* (ii) *does not imply* (i) *and* (iv) *does not imply* (iii).

Proof.

(i) $\Longrightarrow$ (ii): Since $s(1) = 1$, there exists x in P such that $0 < s(x) < 1$ and $0 < s(x') < 1$. Define $x = x_0$ and $x' = x_1$. Applying the same technique to x_0 and x_1 separately, we obtain x_{00}, x_{01}, x_{10} and x_{11}. ($\exists\, x_{00} < x_0$ s.t. $0 < s(x_{00}) < s(x_0)$ and, since $x_{00} \,\mathrm{C}\, x_0$, $\exists\, (x_{00})' \wedge x_0$. Since P is orthomodular and $s \in S(P)$, it happens that

$$s(x_0) = s(x_{00}) + s\big((x_{00})' \wedge x_0\big),$$

i.e.

$$0 < s\big((x_{00})' \wedge (x_0)\big) = s(x_0) - s(x_{00}) < s(x_0),$$

define $(x_{00})' \wedge x_0 = x_{01}$ and continue in this way). Proceeding in this manner, we clearly construct a tree.

(ii) $\Longrightarrow$ (iii): Let $T = \{x_{i_1,\dots,i_k} : i_1, \dots, i_k$ is any finite sequence of 0's and 1's, $k \ge 1\}$ be a tree in P. Then T is a compatible subset of P, because two elements of T either are orthogonal or are one contained into the other, therefore, by Theorem 3.7, there exists a Boolean subalgebra B of P that contains T. Moreover $B = \Big\{ \bigvee_{i \le n} x_i : n \in \mathbb{N}$ and $\forall i \in \{1, \dots, n\}$ $x_i \in T \Big\}$ is countable.

(iii) $\Longrightarrow$ (iv): Let B be a countable atomless subalgebra of P. Starting from B it is easy to obtain a tree. Let $T = \{x_{i_1,\ldots,i_k} : i_1, \ldots, i_k$ is any finite sequence of 0's and 1's, $k \geq 1\}$ be such tree. Define a state on the subalgebra A generated by T in P in the following way:

$$s(x_{i_1,\ldots,i_{k-1}}) = \frac{1}{2^{k-1}},$$

s is strongly continuous and, by [3; Theorem 4.3], has at least an extension to P which is a state and, obviously, still is strongly continuous.

(iii) $\Longrightarrow$ (ii) is straightforward.

To prove that (ii) does not imply (i), consider the following example:

EXAMPLE 3.10. Let $\mathbb{N} = \{1, \ldots, n, \ldots\}$ and let $T = \{X_{i_1,\ldots,i_k} : i_1, \ldots, i_k$ is any finite sequence of 0's and 1's, $k \geq 1\}$ be a tree in $P(\mathbb{N})$ constructed seeing that every $X_{i_1,\ldots,i_k}$ contains infinitely many even numbers and infinitely many odd numbers. It is clear that the Boolean algebra B generated by T in $P(\mathbb{N})$ is countable and hence $T \subset P(\mathbb{N})$. Let $X \in P(\mathbb{N})$ be the set so obtained:

$$X = \{x_{i_1,\ldots,i_k} : i_1, \ldots, i_k \text{ is any finite sequence of 0's and 1's, } k \geq 1\},$$

where we choose $x_{i_1,\ldots,i_k}$ in $2\mathbb{N} \cap X_{i_1,\ldots,i_k}$.

Let P be the concrete OMP generated by B and X in $P(\mathbb{N})$. It is straightforward to verify that

$$P = \{Y : Y \in B\} \cup \{X, \mathbb{N} \setminus X\}.$$

Then P is not a Boolean algebra, because $X \wedge X_0 = \emptyset$, but $X \subseteq (X_0)' = X_1$ is false. For any state s on P at least one of the two, X and $\mathbb{N} \setminus X$, is an atom. Further, let s be the state on P defined in this way:

$$\forall X_{i_1,\ldots,i_k} \in T \qquad s(X_{i_1,\ldots,i_k}) = \frac{1}{2^k},$$

and

$$s(X) = 1.$$

It is obvious that s is strongly continuous, therefore this example shows how, unlike the case of Boolean algebras, in orthomodular posets, in general, strong continuity does not imply nonatomicity.

To show that the strong continuity of a state on a concrete OMP P does not imply the existence of a Boolean atomless countable subalgebra of P, consider the following example:

EXAMPLE 3.11. Let $\mathbb{N} = \{1, \dots, n, \dots\}$ and let $P_1 = \{X_0, X_1\}$ be a partition of $\mathbb{N}$, where X_0 and X_1 are both infinite. Let $X_j = \bigcup_{i=1}^{4} Y_{j,i}$, $j = 0, 1$, with any $Y_{j\,i}$ infinite subset of $\mathbb{N}$, and consider the partition of $\mathbb{N}$, $P_2 = \{X_{00}, X_{01}, X_{10}, X_{11}\}$, where $X_{00} = Y_{0,1} \cup Y_{1,1}$, $X_{01} = Y_{0,2} \cup Y_{1,2}$, $X_{10} = Y_{0,3} \cup Y_{1,3}$ and $X_{11} = Y_{0,4} \cup Y_{1,4}$. Then P_2 has the property that each one of its elements has an infinite intersection with every element of P_1. Let

$$Y_{j,i} = \bigcup_{k=1}^{8} Y_{j,i,k}$$

and let

$$X_{000} = \left(\bigcup_{i=1}^{4} Y_{0,i,1}\right) \cup \left(\bigcup_{i=1}^{4} Y_{1,i,1}\right),$$

$$X_{001} = \left(\bigcup_{i=1}^{4} Y_{0,i,2}\right) \cup \left(\bigcup_{i=1}^{4} Y_{1,i,2}\right),$$

$$X_{010} = \left(\bigcup_{i=1}^{4} Y_{0,i,3}\right) \cup \left(\bigcup_{i=1}^{4} Y_{1,i,3}\right),$$

$$X_{011} = \left(\bigcup_{i=1}^{4} Y_{0,i,4}\right) \cup \left(\bigcup_{i=1}^{4} Y_{1,i,4}\right),$$

$$\vdots$$

$$X_{111} = \left(\bigcup_{i=1}^{4} Y_{0,i,8}\right) \cup \left(\bigcup_{i=1}^{4} Y_{1,i,8}\right).$$

Continuing this way, we obtain for any $k \geq 1$ a partition of $\mathbb{N}$, $P_k = \{X_{i_1,\dots,i_k} :$ $i_1, \dots, i_k$ is any sequence of 0's and 1's of length $k\}$ consisting of 2^k subsets of $\mathbb{N}$ such that

$$X_{i_1,\dots,i_k} \cap X_{i_1,\dots,i_j} \neq \emptyset$$

for any pair of sequences $\big((i_1, \dots, i_k), (i_1, \dots, i_j)\big)$ of 0's and 1's of length k and j respectively, with $j \leq k$.

For any $k \geq 1$, let B_k be the Boolean algebra generated by the elements of P_k in $P(\mathbb{N})$. Then B_k obviously consists of $\emptyset$, $\mathbb{N}$ and unions of elements of P_k. Let K be the disjoint union of all P_k's. Let P be the horizontal sum of B_k's ([5], [6]), it is clear that

$$P = \{\emptyset, \mathbb{N}\} \cup \{X : X \text{ is union of elements of } K \text{ which are in the same partition of } \mathbb{N}\}.$$

Using the technique utilized previously, it is possible to define on P a strongly continuous state. Every element of K is an atom of P, hence, if B is a Boolean subalgebra of P, either it contains an element of K and, in this case, it has an atom, or it does not contain elements of K, but unions of elements of K which, clearly, either are its atoms or contain its atoms.

$\square$

THEOREM 3.12. *Let P be a concrete OMP which can be embedded into a Boolean algebra whose Stone space has countably many points. Then* (iii) $\Longrightarrow$ (i) *in the previous theorem.*

Proof. By the previous theorem, it is sufficient to prove (ii) $\Longrightarrow$ (i). Let $T = \{X_{i_1,\dots,i_k} : i_1,\dots,i_k$ is any finite sequence of 0's and 1's, $k \geq 1\}$ be a tree in P and $f\colon P \to B$ the embedding of P into B. We can suppose that B is the field of all clopen sets of a compact Hausdorff totally disconnected space X having countably many points ([4; Theorem 6]). Then $f(T) = \{f(x_{i_1,\dots,i_k}) : x_{i_1,\dots,i_k} \in T\}$ is a tree in B. Let A be the subalgebra of B generated by $f(T)$. Consider the state s_A on A defined below

$$\forall x_{i_1,\dots,i_k} \in T \qquad s_A\big(f(x_{i_1,\dots,i_k})\big) = \frac{1}{2^k}\,.$$

The state s_A is strongly continuous on A and hence, nonatomic by Theorem 3.8. Let s_B be an extension of s_A on B ([2; Theorem 3.3.4]). Define a state s on P in the following way:

$$\forall x \in P \qquad s(x) = s_B\big(f(x)\big)\,.$$

The state s is nonatomic, in fact, let x be an element of P such that $s(x) \neq 0$. If $x = 1$, then any z in T satisfies

$$0 < s(z) = s_B(f(z)) < s(1) = 1\,.$$

If $x \neq 1$, we shall prove that there exists z in T such that $0 < s(z) < s(x)$. The set $f(x)$ is a clopen subset of B and so is $\big(f(x)\big)'$ as well and moreover $\big(f(x)\big)'$ is countable. Thus there exists $\{a_n\}_{n\in\mathbb{N}}$ in X such that $\big(f(x)\big)' = \{a_1,\dots,a_n,\dots\}$. We shall obtain a cover of $\big(f(x)\big)'$ consisting of clopen subsets of X in $f(T)$ in the following way:

let V_{a_1} be the one of the two, $f(x_0)$ and $f(x_1)$, to which a_1 belongs,
let V_{a_2} be the one of the four, $f(x_{00})$, $f(x_{01})$, $f(x_{10})$ and $f(x_{11})$, to which a_2 belongs,

$$\vdots$$

let V_{a_n} be the one of the nth power of 2 elements of $f(T)$ in $\{f(x_{i_1,\dots,i_n}) : x_{i_1,\dots,i_n} \in T\}$ to which a_n belongs,

$$\vdots$$

Proceeding in this way, we obtain a sequence of clopen sets

$$\{V_{a_n}\}_{n\in\mathbb{N}}$$

with the property:

$$(f(x))' \subseteq \bigcup_{n\in\mathbb{N}} V_{a_n} .$$

As $\big(f(x)\big)'$ is a closed subset of a compact space, there exists a finite subset I of $\mathbb{N}$ such that

$$(f(x))' \subseteq \bigcup_{i\in I} V_{a_i} .$$

But then

$$f(x) \supseteq \bigcap_{i\in I} (V_{a_i})' .$$

Since the complement of an element of $f(T)$ is a union of elements of $f(T)$, there exists Y in $f(T)$ that fulfils $Y \subseteq f(x)$ and then there exists y in T satisfying $f(y) = Y \subseteq f(x)$. Hence, chosen k in $\mathbb{N}$ such that $\frac{1}{2^k} < s(x)$, there exists z in T such that $z < y$ and $s(z) \leq \frac{1}{2^k}$, so x is not an atom. $\square$

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