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THE THIRD BOUNDARY VALUE PROBLEM FOR A NONLINEAR SYSTEM OF SECOND ORDER HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. The paper concerns a boundary value problem for a system of nonlinear second order integro-differential equations whose leading parts contain the operator $\frac{\partial^2 u}{\partial x \partial y}$, with the linear boundary conditions containing both the unknown function u and its normal derivatives. The problem is reduced to an auxiliary problem (Σ) and hence the local existence of its solution is proved by using the Schauder fixed point theorem.

Introduction

Boundary value problems with the third of Neumann boundary conditions have been examined intensively for second order hyperbolic partial differential equations whose leading parts correspond to the second canonical form $\square u := \frac{\partial^2 u}{\partial \xi_1^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial \xi_2^2}$ (cf. [2], [3], [4] – [12], [14] and the references therein). To the best of our knowledge, analogous problems for the equations with the leading parts corresponding to the first canonical form $Lu := \frac{\partial^2 u}{\partial x \partial y}$ have not been taken up so far (cf. Remark 3 in the sequel).

In this paper we deal with the third boundary value problem for a system of nonlinear integro-differential equations of the form $Lu = F$. We reduce the problem to an auxiliary problem (Σ) and hence prove the existence of a solution by using Schauder's fixed point theorem.

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1.

Let $\mathcal{D} = [0,A] \times [0,B]$, where $0 < A, B < \infty$, and consider the curves Γ and $\tilde{\Gamma}$ of equations $y = f(x)$ and $x = g(y)$, respectively, where $f\colon [0,A] \to [0,B]$ and $g\colon [0,B] \to [0,A]$ are functions of class C^1. We introduce the class $\mathcal{K}$ of all functions $u = (u^k)\colon \mathcal{D} \to \mathbb{R}^n$ (n being arbitrarily fixed in $\mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers) such that the derivatives $v := \frac{\partial}{\partial x} u$, $w := \frac{\partial}{\partial y} u$ and Lu exist and are continuous.

We deal with the system of integro-differential equations

$$Lu(x,y) = F\big[x, y, u(x,y), \Phi(x,y), \Omega(x,y)\big] \tag{1}$$

where $\Phi = (v, w)$;

$$\Omega(x,y) = \int_0^x \int_0^y \mathcal{E}\big[x, y; t, \tau, u(t,\tau), \Phi(t,\tau)\big]\, \mathrm{d}\tau\, \mathrm{d}t\,, \tag{2}$$

and F, $\mathcal{E}$ are given functions.

By a solution of system (1) in $\mathcal{D}$ we mean a function $u \in \mathcal{K}$ satisfying (1) at each point $(x,y) \in \mathcal{D}$.

Denote by $\mathbf{n}$ and $\tilde{\mathbf{n}}$ the unit normal vectors to Γ and $\tilde{\Gamma}$, respectively.

We examine the following boundary value problem (P):

(P) Find a solution of system (1) in $\mathcal{D}$ satisfying the boundary conditions

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\mathbf{n}} u\big[x, f(x)\big] + \gamma(x) u\big[x, f(x)\big] = M(x)\,, \\ &\frac{\mathrm{d}}{\mathrm{d}\tilde{\mathbf{n}}} u\big[g(y), y\big] + \tilde{\gamma}(y) u\big[g(y), y\big] = N(y) \end{aligned} \tag{3}$$

($(x,y) \in \mathcal{D}$), where γ, $\tilde{\gamma}$, M and N are given functions.

We make the following assumptions.

I. The function $F\colon \mathcal{D} \times \mathbb{R}^{4n} \to \mathbb{R}^n$ satisfies the conditions

$$|F(0,0,\mathbf{0},(\mathbf{0}),\mathbf{0})| = 0\,;$$

$$\begin{aligned} &|F(x,y,\xi,\eta,\zeta) - F(\bar{x},\bar{y},\bar{\xi},\bar{\eta},\bar{\zeta})| \\ &\le K_1\big((\bar{x}-x)^{\alpha_1} + (\bar{y}-y)^{\alpha_1}\big) + K_2\big(|\xi - \bar{\xi}| + |\zeta - \bar{\zeta}|\big) + K_3 \sum_{\nu=1}^{2} |\eta_\nu - \bar{\eta}_\nu| \end{aligned} \tag{4}$$

($(\mathbf{0}) = \mathbf{0}, \mathbf{0}$; $\mathbf{0}$ is the system of n zeros; $\eta = (\eta_1, \eta_2)$, $0 \le x \le \bar{x} \le A$; $0 \le y \le \bar{y} \le B$), where $\alpha_1 \in (0,1]$, and K_i ($i = 1,2,3$) are positive constants.

II. The function $\mathcal{E}\colon \mathcal{D}^2 \times \mathbb{R}^{3n} \to \mathbb{R}^n$ is continuous and fulfils the condition

$$|\mathcal{E}(x,y;t,\tau;\xi,\eta)| \le K_4 + K_5\Big(|\xi| + \sum_{\nu=1}^{2} |\eta_\nu|\Big) \tag{5}$$

where K_4 and K_5 are positive constants. Moreover, it satisfies Hölder's conditions with respect to x and y, with a positive coefficient and an exponent $h_1 \in (0,1]$.

III. Let c, $\tilde{c}$, s, $\tilde{s}$, m, $\tilde{m}$ and a, $\tilde{a}$ be positive constants such that

$$s_* := \max(s, \tilde{s}) \leq \alpha_1 ; \tag{6}$$

$$\max_{[0,A]} |f'(x)| \leq a ; \qquad \max_{[0,B]} |g'(y)| \leq \tilde{a} ; \tag{7}$$

$$m\tilde{m} < \min\bigl(1, (a\tilde{a})^{-\beta_0}\bigr) \tag{8}$$

for a certain number $\beta_0 \in (0,1)$. All the mentioned constants except c and $\tilde{c}$ are required to be independent of A and B, while c and $\tilde{c}$ are demanded to satisfy the condition $\bigl[\min(c,\tilde{c})\bigr]^{-1} \leq b$ where b is a constant independent of A and B.

We assume that the functions f and g fulfil the inequalities

$$f'(x) \geq \max\left(\frac{f(x)}{mx}, cx^s\right); \qquad g'(y) \geq \max\left(\frac{g(y)}{\tilde{m}y}, \tilde{c}y^{\tilde{s}}\right) \tag{9}$$

($x \in (0,A]$; $y \in (0,B]$). Moreover, the derivatives f' and g' satisfy Hölder's condition with a positive coefficient and an exponent $h_2 \in (0,1]$.

IV. The functions $\gamma\colon [0,A] \to \mathbb{R}$ and $\tilde{\gamma}\colon [0,B] \to \mathbb{R}$ satisfy Hölder's condition with a positive coefficient K_6 and an exponent $\alpha_2 \in (0,1]$, and fulfil the equalities

$$\gamma(0) = \tilde{\gamma}(0) = 0 . \tag{10}$$

V. The functions $M\colon [0,A] \to \mathbb{R}^n$ and $N\colon [0,B] \to \mathbb{R}^n$ satisfy the conditions

$$\begin{gathered} M(0) = N(0) = 0 , \\ |M(x) - M(\bar{x})| \leq K_7 \bar{x}^{\alpha_3 - h_3} (\bar{x} - x)^{h_3} , \qquad |N(y) - N(\bar{y})| \leq K_7 \bar{y}^{\alpha_3 - h_3} (\bar{y} - y)^{h_3} \end{gathered} \tag{11}$$

where K_7 is a positive constant, $h_3 \in (0,1]$ and α_3 is a number such that

$$s_* + 1 < \alpha_3 . \tag{11'}$$

Moreover, at the common points of Γ and $\tilde{\Gamma}$ the functions M and N satisfy suitable compatibility conditions.

COROLLARY 1. *It follows from Assumptions* I, IV *and* V *that the inequalities*

$$\begin{gathered} |F(x,y,\xi,\eta,\zeta)| \le K_1\left(x^{\alpha_1}+y^{\alpha_1}\right)+K_2\left(|\xi|+|\zeta|\right)+K_3\sum_{\nu=1}^{2}|\eta_\nu|\,; \\ |\gamma(x)| \ \le K_6 x^{\alpha_2}\,; \qquad |\tilde\gamma(y)| \ \le K_6 y^{\alpha_2}\,; \\ |M(x)| \le K_7 x^{\alpha_3}\,; \qquad |N(y)| \le K_7 y^{\alpha_3} \end{gathered} \tag{12}$$

hold good.

Remark 1. Assume that $f(A)=B$; $g(B)=A$, and that the curves Γ and $\tilde\Gamma$ have no common point apart from $(0,0)$ and (A,B). Setting $\Gamma_0=\Gamma\cup\tilde\Gamma$ and denoting by $\mathcal{D}_0$ the domain bounded by Γ_0, we can assert that (P) is in the considered case the third boundary value problem for the domain $\mathcal{D}_0$ with the conditions (3) given on its boundary Γ_0. The compatibility conditions for M and N are in this case $M(0)=N(0)$; $M(A)=N(B)$.

Remark 2. If $\gamma(x)=\tilde\gamma(x)=0$ then (P) is the Neumann problem for system (1). Let us point out that, due to the assumptions and the method of treating the problem, the present result is not contained in that of paper [1] devoted to a Neumann-type problem for a system of high order integro-differential equations.

Remark 3. Let us observe that the present problem cannot be, in general, obtained from those concerning the equation

(i) $\Box u = F(\xi_1,\xi_2)$ (for simplicity we discuss the linear case).

Indeed, every of the problems for equation (i) dealt with in the papers mentioned in the Introduction contains the initial conditions

(ii) $u(\xi_1,0)=u_0(\xi_1)$; $\dfrac{\partial}{\partial\xi_2}u(\xi_1,0)=u_1(\xi_1)$; $0\le\xi_1\le a$,
where $a>0$ and $u_\nu(\xi_1)$ $(\nu=0,1)$ are given functions.

The linear map τ: $x=\xi_1-c\xi_2$; $y=\xi_1+c\xi_2$ transforms equation (i) to

(iii) $L\tilde u=\tilde F(x,y)\equiv\frac14 F\left(\frac{x+y}{2},\frac{y-x}{2c}\right)$,

and conditions (ii) to

(iv) $\tilde u(x,x)=\tilde u_0(x)\equiv u_0(x)$; $\dfrac{\mathrm{d}}{\mathrm{d}\mathbf{n}}\tilde u(x,x)=\tilde u_1(x)\equiv\dfrac{1}{c\sqrt2}u_1(x)$.

Problem (P) does not contain conditions (iv) and hence it cannot be obtained from the said problems by using transformation τ.

EXAMPLE. We give an example of the curves satisfying Assumption III. Let $A = B$ and $\nu \in (1, \alpha_1 + 1)$. Set $f(x) = A^{1-\nu}x^\nu$ and $g(y) = A^{1-\nu}y^\nu$. Assumption III is satisfied with $m = \tilde{m} = 1/\nu$; $c = \tilde{c} = \nu A^{1-\nu}$, $s = \tilde{s} = \nu - 1$, $a = \tilde{a} = \nu$ and any $\beta_0 \in (0,1)$.

2.

Assume that the normal vectors $\mathbf{n}$ and $\tilde{\mathbf{n}}$ are directed so that

$$\begin{aligned} \cos(x,\mathbf{n}) &= -\frac{f'(x)}{e(x)}; & \cos(y,\mathbf{n}) &= \frac{1}{e(x)}; \\ \cos(x,\tilde{\mathbf{n}}) &= \frac{1}{\tilde{e}(y)}; & \cos(y,\tilde{\mathbf{n}}) &= -\frac{g'(y)}{\tilde{e}(y)}, \end{aligned} \tag{13}$$

where

$$e(x) = \sqrt{1 + f'^2(x)}; \qquad \tilde{e}(y) = \sqrt{1 + g'^2(y)}, \tag{14}$$

and denote by $\mathcal{K}_1$ the class of all functions $u \in \mathcal{K}$ such that

$$u(0,0) = v(0,0) = w(0,0) = 0. \tag{15}$$

It is easily observed that, in the class $\mathcal{K}_1$, problem (P) is equivalent to the following problem (Σ) (cp. with that in [13]):

(Σ) Find a solution $u \in \mathcal{K}_1$ of system (1) in $\mathcal{D}$ satisfying the boundary conditions

$$v[x, f(x)] = G_\Phi(x); \qquad w[g(y), y] = H_\Phi(y) \tag{16}$$

$((x,y) \in \mathcal{D})$, where

$$\begin{aligned} &G_\Phi(0) = H_\Phi(0) = 0; \\ &G_\Phi(x) = \check{G}_\Phi(x) + \hat{G}_\Phi(x) \qquad \text{for} \quad x \in (0, A], \\ &H_\Phi(y) = \check{H}_\Phi(y) + \hat{H}_\Phi(y) \qquad \text{for} \quad y \in (0, B] \end{aligned} \tag{17}$$

with

$$\check{G}_\Phi(x) = \frac{1}{f'(x)} w[x, f(x)]; \qquad \check{H}_\Phi(y) = \frac{1}{g'(y)} v[g(y), y], \tag{18}$$

$$\begin{aligned} \hat{G}_\Phi(x) &= \frac{e(x)}{f'(x)} \Big(\gamma(x) u[x, f(x)] - M(x) \Big), \\ \hat{H}_\Phi(y) &= \frac{\tilde{e}(y)}{g'(y)} \Big(\tilde{\gamma}(y) u[g(y), y] - N(y) \Big). \end{aligned} \tag{19}$$

It follows from Taylor's formula with the integral remainder that if $u \in \mathcal{K}_1$, then

$$\begin{aligned} u(x,y) = \Lambda^1_\Phi(x,y) &:= \int_0^x v(\xi,y)\,\mathrm{d}\xi + \int_0^y w(0,\eta)\,\mathrm{d}\eta \\ &= \int_0^y w(x,\eta)\,\mathrm{d}\eta + \int_0^x v(\xi,0)\,\mathrm{d}\xi \end{aligned} \tag{20}$$

$((x,y) \in \mathcal{D})$.

If, moreover, u is a solution of system (1) in $\mathcal{D}$, then

$$\begin{aligned} v(x,y) &= v(x,0) + \int_0^y F\big[x,\eta,u(x,\eta),\Phi(x,\eta),\Omega(x,\eta)\big]\,\mathrm{d}\eta\,, \\ w(x,y) &= w(0,y) + \int_0^x F\big[\xi,y,u(\xi,y),\Phi(\xi,y),\Omega(\xi,y)\big]\,\mathrm{d}\xi \end{aligned} \tag{21}$$

$((x,y) \in \mathcal{D})$.

In the sequel, Λ^*_Φ denotes the expression Ω (cp. (2)) with $u = \Lambda^1_\Phi$, while Λ^2_Φ and Λ^3_Φ stand for v and w given by (21), respectively, with $u = \Lambda^1_\Phi$, $\Omega = \Lambda^*_\Phi$.

Now, let us consider the following system of integro-functional equations

$$v(x,y) = T_\Phi(x,y)\,; \qquad w(x,y) = \hat{T}_\Phi(x,y) \tag{22}$$

$((x,y) \in \mathcal{D})$ with the unknown vector Φ (cp. (1)), where

$$T_\Phi(x,y) = \mathcal{G}_\Phi(x) + \int_{f(x)}^y \vartheta_\Phi(x,\eta)\,\mathrm{d}\eta\,; \qquad \hat{T}_\Phi(x,y) = \mathcal{H}_\Phi(y) + \int_{g(y)}^x \vartheta_\Phi(\xi,y)\,\mathrm{d}\xi\,. \tag{23}$$

Above, $\mathcal{G}_\Phi(0) = \mathcal{H}_\Phi(0) = 0$; $\mathcal{G}_\Phi(x) = \check{\mathcal{G}}_\Phi(x) + \hat{\mathcal{G}}_\Phi(x)$ for $x \in (0,A]$, $\mathcal{H}_\Phi(y) = \check{\mathcal{H}}_\Phi(y) + \hat{\mathcal{H}}_\Phi(y)$ for $y \in (0,B]$, where $\check{\mathcal{G}}_\Phi$, $\check{\mathcal{H}}_\Phi$ denote the expressions (18), respectively, with $v = \Lambda^2_\Phi$, $w = \Lambda^3_\Phi$, and $\hat{\mathcal{G}}_\Phi$, $\hat{\mathcal{H}}_\Phi$ the expressions (19) respectively, with $u = \Lambda^1_\Phi$. Moreover, ϑ_Φ is given by

$$\vartheta_\Phi(x,y) = F\big[x,y,\Lambda^1_\Phi(x,y),\Phi(x,y),\Lambda^*_\Phi(x,y)\big]\,. \tag{24}$$

One can prove the following lemma:

LEMMA 1. *If u is a solution of problem (Σ), then Φ is a continuous solution of system* (22). *Conversely, if Φ is a continuous solution of system* (22), *then the function $u = \Lambda^1_\Phi$ is a solution of problem (Σ).*

3.

We shall prove the existence of a solution to system (22) (and hence of problem (P)) by using Schauder's fixed point theorem.

Let $\mathcal{S}$ be the set of all systems $\Phi = (v, w)$, where the components $v\colon \mathcal{D}_* \to \mathbb{R}^n$ and $w\colon \mathcal{D}_* \to \mathbb{R}^n$ (with $\mathcal{D}_* = \mathcal{D} \setminus \{0,0\}$) are continuous functions such that

$$B_\Phi := \max\Big(\sup_{\mathcal{D}_*}\big[(x+y)^{-1}|v(x,y)|\big],\ \sup_{\mathcal{D}_*}\big[(x+y)^{-1}|w(x,y)|\big]\Big) < \infty. \tag{25}$$

We define the addition of points and the multiplication of a point by a number in the ordinary way, and introduce the norm

$$\|\Phi\| = B_\Phi \tag{26}$$

($\Phi \in \mathcal{S}$). It is easily observed that $\mathcal{S}$ is a Banach space.

We consider the set $\mathcal{Z}$ of all points $\Phi \in \mathcal{S}$ satisfying the conditions

$$|v(x,y)| \le \varrho_1 (x+y)^{1+h}; \qquad |w(x,y)| \le \varrho_2 (x+y)^{1+h} \tag{27}$$

and

$$\begin{aligned} |v(x,y) - v(\bar{x},\bar{y})| &\le \kappa_1(\bar{x}+\bar{y})\big[(\bar{x}-x)^h + (\bar{y}-y)^h\big], \\ |w(x,y) - w(\bar{x},\bar{y})| &\le \kappa_2(\bar{x}+\bar{y})\big[(\bar{x}-x)^h + (\bar{y}-y)^h\big] \end{aligned} \tag{28}$$

($0 < x \le \bar{x} \le A$; $0 < y \le \bar{y} \le A$), where $\varrho_1, \varrho_2, \kappa_1, \kappa_2 > 0$ are parameters to be chosen later, and

$$0 < h < h_* := \min\Big(\min_{1\le\nu\le3} h_\nu,\ \beta_0,\ \alpha_1 - s_*,\ \alpha_2,\ \alpha_3 - s_* - 1\Big). \tag{29}$$

Evidently, $\mathcal{Z}$ is a closed and convex set. In order to prove its compactness, let us consider an arbitrary sequence $\{\Phi_m\}$, where $\Phi_m = (v_m, w_m) \in \mathcal{Z}$, and introduce the sequence $\{\overset{*}{\Phi}_m\}$, where $\overset{*}{\Phi}_m = (\overset{*}{v}_m, \overset{*}{w}_m)$ with $\overset{*}{v}_m(x,y) = (x+y)^{-1} v_m(x,y)$ for $(x,y) \in \mathcal{D}_*$; $\overset{*}{v}_m(0,0) = 0$ and $\overset{*}{w}_m(x,y) = (x+y)^{-1} w_m(x,y)$ for $(x,y) \in \mathcal{D}_*$; $\overset{*}{w}_m(0,0) = 0$. By (27), the functions $\overset{*}{v}_m$ and $\overset{*}{w}_m$, defined on the closed and bounded set $\mathcal{D}$, are uniformly bounded:

$$|\overset{*}{v}_m(x,y)| \le (x+y)^h \varrho_1 \le (2\mathcal{A})^h \varrho_1; \qquad |\overset{*}{w}_m(x,y)| \le (x+y)^h \varrho_2 \le (2\mathcal{A})^h \varrho_2$$

($(x,y) \in \mathcal{D}$), where $\mathcal{A} = \max(A, B)$.

We shall prove the equicontinuity of $\overset{*}{v}_m$ and $\overset{*}{w}_m$.

Let us observe that if $(x,y) = (0,0)$, then

$$\begin{aligned} r_1(x,y,\bar{x},\bar{y}) &:= |\overset{*}{v}_m(x,y) - \overset{*}{v}_m(\bar{x},\bar{y})| \le 2\mathcal{A}\varrho_1\big[(\bar{x}-x)^h + (\bar{y}-y)^h\big]; \\ r_2(x,y,\bar{x},\bar{y}) &:= |\overset{*}{w}_m(x,y) - \overset{*}{w}_m(\bar{x},\bar{y})| \le 2\mathcal{A}\varrho_2\big[(\bar{x}-x)^h + (\bar{y}-y)^h\big]. \end{aligned}$$

For $(x,y) \in \mathcal{D}_*$; $(\bar{x},\bar{y}) \in \mathcal{D}_*$, we have

$$r_1(x,y,\bar{x},\bar{y}) \le \kappa_1\big[(\bar{x}-x)^h + (\bar{y}-y)^h\big] + \varrho_1 \frac{(x+y)^h}{\bar{x}+\bar{y}}\big[(\bar{x}-x)^h\bar{x}^{1-h} + (\bar{y}-y)^h\bar{y}^{1-h}\big] \le (\varrho_1+\kappa_1)\big[(\bar{x}-x)^h + (\bar{y}-y)^h\big]$$

and similarly

$$r_2(x,y,\bar{x},\bar{y}) \le (\varrho_2+\kappa_2)\big[(\bar{x}-x)^h + (\bar{y}-y)^h\big].$$

Thus, for all $(x,y) \in \mathcal{D}$; $(\bar{x},\bar{y}) \in \mathcal{D}$, we get

$$r_1(x,y,\bar{x},\bar{y}) \le \big[\max(2\mathcal{A},1)\varrho_1 + \kappa_1\big]\big[(\bar{x}-x)^h + (\bar{y}-y)^h\big];$$
$$r_2(x,y,\bar{x},\bar{y}) \le \big[\max(2\mathcal{A},1)\varrho_2 + \kappa_2\big]\big[(\bar{x}-x)^h + (\bar{y}-y)^h\big]$$

which shows that $\overset{*}{v}_m$ and $\overset{*}{w}_m$ are equicontinuous.

By the Arzelà theorem there is a subsequence $\{\overset{*}{\Phi}_{m_k}\}$, where $\overset{*}{\Phi}_{m_k} = (\overset{*}{v}_{m_k}, \overset{*}{w}_{m_k})$, uniformly convergent in $\mathcal{D}$ and hence the relations

$$\sup_{\mathcal{D}} |\overset{*}{v}_{m_k}(x,y) - \overset{*}{v}_{m_l}(x,y)| = \sup_{\mathcal{D}_*}\big[(x+y)^{-1}|v_{m_k}(x,y) - v_{m_l}(x,y)|\big] < \varepsilon;$$
$$\sup_{\mathcal{D}} |\overset{*}{w}_{m_k}(x,y) - \overset{*}{w}_{m_l}(x,y)| = \sup_{\mathcal{D}_*}\big[(x+y)^{-1}|w_{m_k}(x,y) - w_{m_l}(x,y)|\big] < \varepsilon$$

$(\varepsilon > 0;\ k,l > N(\varepsilon))$.

It follows from the said relations that the subsequence $\{\Phi_{m_k}\}$, where $\Phi_{m_k} = (v_{m_k}, w_{m_k})$ of $\{\Phi_m\}$ satisfies the Cauchy condition in the norm (26) and hence, by the completeness of $\mathcal{S}$ and the closedness of $\mathcal{Z}$, its limit exists and belongs to $\mathcal{Z}$. Thus, $\mathcal{Z}$ is compact, as required.

In view of system (22), we map $\mathcal{Z}$ by the transformation T defined by (cp. (23))

$$\tilde{v}(x,y) = T_\Phi(x,y); \qquad \tilde{w}(x,y) = \hat{T}_\Phi(x,y) \tag{30}$$

$((x,y) \in \mathcal{D}_*)$.

We shall find sufficient conditions for the inclusion $T(\mathcal{Z}) \subset \mathcal{Z}$.

In order to estimate the functions $\tilde{v}$ and $\tilde{w}$, let us first observe that by Assumption II, Corollary 1 and relations (20), (27) we have

$$|\Lambda^1_\Phi(x,y)| \le \mathrm{const}(\varrho_1+\varrho_2)(x+y)^{2+h}; \tag{31}$$

$$|\Lambda^*_\Phi(x,y)| \le \mathrm{const}\, e(\varrho)(x+y)^{2+h}; \tag{32}$$

(above and in the sequel, const denotes a positive constant independent of ϱ_1, ϱ_2, κ_1, κ_2), where

$$e(\varrho) = 1 + \varrho_1 + \varrho_2 \tag{33}$$

$(\varrho = (\varrho_1, \varrho_2))$.

Using Assumption III, Corollary 1 and relations (18), (21), (31), (32), we get the estimate

$$|\check{\mathcal{G}}_\Phi(x)| \le \{m^{1+h} a^h \varrho_2 + \mathrm{const}\, e(\varrho)(A^{\omega_1} + K_3 A^{1-s})\} x^{1+h} \tag{34}$$

with $\omega_1 = \alpha_1 - (s_* + h)$.

Moreover, by Assumptions III – V and formulae (14), (19) and (31), we obtain

$$|\hat{\mathcal{G}}_\Phi(x)| \le \mathrm{const}\, e(\varrho) A^{\omega_2} x^{1+h}, \tag{35}$$

where $\omega_2 = \min[\alpha_2 - (s_* - 1),\, \alpha_3 - (s_* + h + 1)]$.

Finally, we have (cp. Corollary 1 and formulae (24), (27), (31), (32))

$$\left| \int\limits_{f(x)}^{y} \vartheta_\Phi(x, \eta)\, \mathrm{d}\eta \right| \le \mathrm{const}\, e(\varrho) \mathcal{A}^{\omega_3} (x + y)^{1+h} \tag{36}$$

where $\omega_3 = \min(\alpha_1 - h, 1)$.

Relations (17), (23) and (34) – (36) yield

$$|T_\Phi(x, y)| \le \{m^{1+h} a^h \varrho_2 + \mathrm{const}\, e(\varrho)(\mathcal{A}^\omega + K_3 A^{1-s_*})\}(x + y)^{1+h}, \tag{37}$$

with $\omega = \min\limits_{1 \le \nu \le 3} \omega_\nu$.

In a similar way we get the estimate (cp. (23)).

$$|\hat{T}_\Phi(x, y)| \le \{\tilde{m}^{1+h} \tilde{a}^h \varrho_1 + \mathrm{const}\, e(\varrho)(\mathcal{A}^\omega + K_3 B^{1-s_*})\}(x + y)^{1+h}. \tag{38}$$

Thus, by (30), (37) and (38), the functions $\tilde{v}$ and $\tilde{w}$ satisfy conditions (27) if the following system of inequalities

$$\begin{aligned} m^{1+h} a^h \varrho_2 + C e(\varrho)(\mathcal{A}^\omega + K_3 \mathcal{A}^{1-s_*}) &\le \varrho_1, \\ \tilde{m}^{1+h} \tilde{a}^h \varrho_1 + C e(\varrho)(\mathcal{A}^\omega + K_3 \mathcal{A}^{1-s_*}) &\le \varrho_2 \end{aligned} \tag{39}$$

holds good, where C is a positive constant independent of ϱ, κ_1, κ_2.

Let us observe that by (8)[1], there is a number $\theta \in (0, 1)$ fulfilling the condition

$$(m\tilde{m})^{1+h}(a\tilde{a})^h = \theta^2.$$

We choose ϱ_1 and ϱ_2 in (27) so that

$$\frac{\varrho_2}{\varrho_1} = \frac{\theta}{a^h m^{1+h}} = \frac{\tilde{a}^h \tilde{m}^{1+h}}{\theta}, \tag{40}$$

[1] Due to the condition $h < \beta_0$ (cp. (29)), inequality (8) implies $(m\tilde{m})^{1+h}(a\tilde{a})^h < 1$ and $m\tilde{m}(a\tilde{a})^h < 1$.

whence (39) reduces to

$$Ce(\varrho)(\mathcal{A}^{\omega}+K_3\mathcal{A}^{1-s_*})\leq(1-\theta)\min(\varrho_1,\varrho_2). \tag{41}$$

It is easily seen that inequality (41) is satisfied, provided that

$$\mathcal{A}\leq\left\{\frac{(1-\theta)\min(\varrho_1,\varrho_2)}{\tilde{C}[1+2\max(\varrho_1,\varrho_2)]}\right\}^{1/\omega'} \tag{42}$$

where $\omega'=\min(\omega,1-s_*)$ and $\tilde{C}$ is a positive constant of the same type as C above.

We proceed to the examination of conditions (28).

Basing on Assumptions I, II and relations (20), (27), (28), we get

$$|\Lambda^1_\Phi(x,y)-\Lambda^1_\Phi(\bar{x},\bar{y})|\leq\operatorname{const}(\varrho_1+\varrho_2+\kappa_1)(\bar{x}+\bar{y})^2\big[(\bar{x}-x)^h+(\bar{y}-y)^h\big]\,; \tag{43}$$

$$|\Lambda^*_\Phi(x,y)-\Lambda^*_\Phi(\bar{x},\bar{y})|\leq\operatorname{const}(\varrho_1+\varrho_2+\kappa_1)(\bar{x}+\bar{y})^2\big[(\bar{x}-x)^h+(\bar{y}-y)^h\big]\,, \tag{44}$$

whence, and from Assumptions I, III, Corollary 1 and relations (18), (21), (27), (28), we obtain

$$|\check{\mathcal{G}}_\Phi(x)-\check{\mathcal{G}}_\Phi(\bar{x})|\leq\big\{ma^h\kappa_2+\operatorname{const}\big[\varrho_2+\tilde{e}(\varrho,\kappa)(A^{\check{\omega}}+K_3A^{1-s})\big]\big\}\,, \tag{45}$$

where

$$\tilde{e}(\varrho,\kappa)=1+\sum_{\nu=1}^{2}(\varrho_\nu+\kappa_\nu) \tag{46}$$

(ϱ is as in (33) and $\kappa=(\kappa_1,\kappa_2)$), and $\check{\omega}=\min(1-h,\ 1-h+\alpha_1-2s_*)$.

Furthermore, by Assumptions III – V and formulae (14), (31) and (34), we get

$$\left|\frac{e(x)}{f'(x)}\gamma(x)\Lambda^1_\Phi[x,f(x)]-\frac{e(\bar{x})}{f'(\bar{x})}\gamma(\bar{x})\Lambda^1_\Phi[\bar{x},f(\bar{x})]\right|\leq\operatorname{const}\tilde{e}(\varrho,\kappa)A^{\hat{\omega}_1}\bar{x}(\bar{x}-x)^h$$

with $\hat{\omega}_1=\min\big(\alpha_2-(s-1),\ h_2-(s_*-1)\big)$, and

$$\left|\frac{e(x)}{f'(x)}M-\frac{e(\bar{x})}{f'(\bar{x})}M(\bar{x})\right|\leq\operatorname{const}\tilde{e}(\varrho,\kappa)A^{\hat{\omega}_2}\bar{x}(\bar{x}-x)^h$$

with $\hat{\omega}_2=\alpha_3-(s_*+h+1)$, whence and from (19) we obtain

$$|\hat{\mathcal{G}}_\Phi(x)-\hat{\mathcal{G}}_\Phi(\bar{x})|\ \leq\operatorname{const}\tilde{e}(\varrho,\kappa)A^{\hat{\omega}}\bar{x}(\bar{x}-x)^h \tag{47}$$

where $\hat{\omega}=\min(\hat{\omega}_1,\hat{\omega}_2)$.

Inequalities (45) and (47) yield (cp. (17))

$$|\mathcal{G}_\Phi(x)-\mathcal{G}_\Phi(\bar{x})| \le \{ma^h\kappa_2+\mathrm{const}[\varrho_2+\tilde{e}(\varrho,\kappa)(A^{\tilde{\omega}}+K_3A^{1-s_*})]\}\bar{x}(\bar{x}-x)^h \quad (48)$$

where $\tilde{\omega}=\min(\check{\omega},\,\hat{\omega})$.

It is easily proved by using Assumptions I, II, Corollary 1 and formulae (31), (32), (43), (44) that

$$\left|\int_{f(x)}^{y}\vartheta_\Phi(x,\eta)\,\mathrm{d}\eta-\int_{f(\bar{x})}^{\bar{y}}\vartheta_\Phi(\bar{x},\eta)\,\mathrm{d}\eta\right| \le \tilde{e}(\varrho,\kappa)A^{\tilde{\omega}_1}(\bar{x}+\bar{y})\left[(\bar{x}-x)^h+(\bar{y}-y)^h\right] \quad (49)$$

with $\tilde{\omega}_1=\min(1,\,\alpha_1-h)$.

On joining (23), (48) and (49), we get

$$|T_\Phi(x)-T_\Phi(\bar{x})| \le \{ma^h\kappa_2+\mathrm{const}[\varrho_2+\tilde{e}(\varrho,\kappa)(\mathcal{A}^{\omega_*}+K_3A^{1-s_*})]\}(\bar{x}+\bar{y})\left[(\bar{x}-x)^h+(\bar{y}-y)^h\right] \quad (50)$$

where $\omega_*=\min(\tilde{\omega},\,\tilde{\omega}_1)$.

In a similar way we obtain

$$\begin{aligned}&|\hat{T}_\Phi(x)-\hat{T}_\Phi(\bar{x})|\\ &\le \{\tilde{m}\tilde{a}^h\kappa_1+\mathrm{const}[\varrho_1+\tilde{e}(\varrho,\kappa)(\mathcal{A}^{\omega_*}+K_3B^{1-s_*})]\}\cdot\\ &\qquad\cdot(\bar{x}+\bar{y})\left[(\bar{x}-x)^h+(\bar{y}-y)^h\right].\end{aligned} \quad (51)$$

As a consequence of (50), (51) we can assert that the functions $\tilde{v}$ and $\tilde{w}$ (cp. (30)) satisfy conditions (28) if the system of inequalities

$$\begin{aligned}&ma^h\kappa_2+C_*[\varrho_2+\tilde{e}(\varrho,\kappa)(\mathcal{A}^{\omega_*}+K_3\mathcal{A}^{1-s_*})]\le\kappa_1,\\ &\tilde{m}\tilde{a}^h\kappa_1+C_*[\varrho_1+\tilde{e}(\varrho,\kappa)(\mathcal{A}^{\omega_*}+K_3\mathcal{A}^{1-s_*})]\le\kappa_2\end{aligned} \quad (52)$$

is valid, where C_* is a constant of the same type as C in (39).

The discussion of (52) is analogous to that of (39).

Basing on (8) (cf. the footnote [1]), we can assert that $m\tilde{m}(a\tilde{a})^h=\theta_1^2\in(0,1)$, and choose ϱ_ν, κ_ν $(\nu=1,2)$ so that (40) and

$$\frac{\kappa_2}{\kappa_1}=\frac{\theta_1}{ma^h}=\frac{\tilde{m}a^h\tilde{a}^h}{\theta_1};\qquad C_*\varrho_2\le\frac{1-\theta_1}{2}\kappa_1;\qquad C_*\varrho_1\le\frac{1-\theta}{2}\kappa_2 \quad (53)$$

are satisfied.

As a result of (53), inequalities (52) reduce to

$$C_*\tilde{e}(\varrho,\kappa)(\mathcal{A}^{\omega_*}+K_3\mathcal{A}^{1-s_*})\le\frac{1-\theta_1}{2}\min(\kappa_1,\kappa_2). \quad (54)$$

It is easily observed that inequality (54) holds good if

$$\mathcal{A} \le \left\{ \frac{(1-\theta_1)\min(\kappa_1,\kappa_2)}{2\tilde{C}_*\left[1+4\max\limits_{1\le\nu\le 2}\max(\varrho_\nu,\kappa_\nu)\right]} \right\}^{1/\omega'_*}, \tag{55}$$

where $\omega'_* = \min(\omega_*,\, 1-s_*)$ and $\tilde{C}_*$ is a positive constant independent of ρ, κ.

The above-obtained results make it possible to assert that if $\mathcal{A}$ is sufficiently small, so that inequalities (42) and (55) are satisfied, then $T(\mathcal{Z}) \subset \mathcal{Z}$.

One can also prove the following lemma:

LEMMA 2. *The transformation T (cp. (30)) is continuous.*

Thus, all assumptions of Schauder's fixed point theorem are satisfied and hence we can assert that there is a fixed point of transformation T, that is a system $\Phi_0 = (v_0, w_0) \in \mathcal{Z}$ satisfying the system of integro-functional equations (22) in $\mathcal{D}_*$. Setting

$$v_*(x,y) = \begin{cases} 0 & \text{for } x = y = 0\,, \\ v_0(x,y) & \text{for } (x,y) \in \mathcal{D}_*\,, \end{cases}$$

$$w_*(x,y) = \begin{cases} 0 & \text{for } x = y = 0\,, \\ w_0(x,y) & \text{for } (x,y) \in \mathcal{D}_* \end{cases}$$

we get the system $\bar{\Phi}_* = (v_*, w_*)$ of continuous functions satisfying (22) in $\mathcal{D}$. As a result (cp. Lemma 1), problem (Σ) has a solution $u_* = \Lambda^1_{\Phi_*} \in \mathcal{K}_1$ which, by the equivalence of problems (P) and (Σ), is also a solution to problem (P).

As a result, we can formulate the following final theorem:

THEOREM. *If Assumptions* I–V *are satisfied and $\mathcal{A} = \max(A,B)$ is sufficiently small, so that inequalities* (42) *and* (55) *hold good, then problem* (P) *has a solution.*

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