Oleg Palumbíny

On the existence of monotone solutions to a certain class of n-th order nonlinear differential equations

Mathematica Slovaca, Vol. 47 (1997), No. 5, 527--537

Persistent URL: `http://dml.cz/dmlcz/136712`

Mathematica Slovaca
©1997
Mathematical Institute
Slovak Academy of Sciences

Math. Slovaca, **47** (1997), No. 5, 527–537

ON THE EXISTENCE OF MONOTONE SOLUTIONS OF A CERTAIN CLASS OF nTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

OLEG PALUMBÍNY

(*Communicated by Milan Medveď*)

ABSTRACT. This paper deals with existence of monotone solutions of nth order nonlinear differential equations with quasi-derivatives.

1. Introduction

The purpose of our paper is to study some conditions for the existence of monotone solutions of the differential equation

$$L(y) \equiv 0\,, \tag{L}$$

where

$$\begin{aligned}
L(y) &\equiv L_n y + \sum_{k=1}^{n-1} P_k(t) L_k y + f(t,y)\,,\\
L_0 y(t) &= y(t)\,,\\
L_1 y(t) &= p_1(t)\big(L_0 y(t)\big)' = p_1(t)\,\mathrm{d}y(t)/\,\mathrm{d}t\,,\\
L_k y(t) &= p_k(t)\big(L_{k-1} y(t)\big)' \qquad \text{for} \quad k = 2,3,\dots,n-1\,,\\
L_n y(t) &= \big(L_{n-1} y(t)\big)'\,,
\end{aligned}$$

n is an arbitrary positive integer, $n \ge 2$. It is assumed throughout that $P_k(t)$, $k = 1,\dots,n-1$, $p_i(t)$, $i = 1,2,\dots,n-1$, are real-valued continuous functions on an interval $I_a = [a,\infty)$, $-\infty < a < \infty$, and $f(t,y)$ is a real-valued function continuous on $I_a \times E_1$, where $E_1 = (-\infty,\infty)$, $a \in E_1$.

AMS Subject Classification (1991): Primary 34C10; Secondary 34D05.
Key words: nonlinear differential equation, quasi-derivative, monotone solution.

If $n = 1$, then $L(y) \equiv L_1 y + f(t,y) = y' + f(t,y)$, where $f(t,y)$ is a real-valued continuous function on $I_a \times E_1$, $a \in E_1$.

The following condition will play important role in our considerations:

(A) $P_k(t) \le 0$, $p_i(t) > 0$ for all $t \in I_a$, $k = 1,\dots,n-1$, $i = 1,2,\dots,n-1$; $f(t,y) \le 0$ for all $(t,y) \in I_a \times E_1$; n is an arbitrary positive integer, $n \ge 2$. If $n = 1$, then $f(t,y) \le 0$ for all $(t,y) \in I_a \times E_1$.

Similar problems for third order ordinary differential equations with quasi-derivatives have been studied in several papers ([4], [6], [9]). The equation (L), where $p_i(t) \equiv 1$, $i = 1,2,3$, $(n = 4)$ has been studied, for example, in [5], [8], [10], [11]. An equation of fourth order with quasi-derivatives has also been studied, for instance, in [1], [3], [12]. nth order equation with (ordinary) derivatives has been studied in [7]. Therefore some results achieved in the papers mentioned above are special cases of ours.

Theorem 1 of our paper gives sufficient conditions for a solution of (L) on I_a to be monotone on I_a. Theorem 2 gives sufficient conditions for the existence as well as monotony of a solution of (L) on I_a. Theorem 3 deals with the existence of a monotone solution for the nth order linear differential equation on I_a.

DEFINITION 1. A nontrivial solution $y(t)$ of a differential equation of the nth order is called *monotone on the interval* $[t_0,\infty)$ if and only if $L_k y(t) \ge 0$ for all $t \ge t_0$, $k = 1,\dots,n-1$, and $y(t) > 0$ on $[t_0,\infty)$.

DEFINITION 2. Let J be an arbitrary type of interval with bounds t_1, t_2, where $-\infty \le t_1 < t_2 \le \infty$. The interval J is called the *maximal interval of existence* of $u\colon J \to E_1^n$, where $u(t)$ is a solution of the differential system $u' = F(t,u)$ if and only if $u(t)$ can be continued neither to the right nor to the left of J.

DEFINITION 3. Let $y' = U(t,y)$ be a scalar differential equation. Then $y^0(t)$ is called the *maximal solution of the Cauchy problem*

$$y' = U(t,y), \qquad y(t_0) = y_0 \tag{$*$}$$

if and only if $y^0(t)$ is a solution of $(*)$ on the maximal interval of existence, and if $y(t)$ is another solution of $(*)$, then $y(t) \le y^0(t)$ for all t belonging to the common interval of existence of $y(t)$ and $y^0(t)$.

We introduce some preliminary results.

LEMMA 1. *Let $A(t,s)$ be a nonnegative and continuous function for $t_0 \le s \le t$. If $g(t)$, $\varphi(t)$ are continuous functions in the interval $[t_0,\infty)$ and*

$$\varphi(t) \le g(t) + \int_{t_0}^{t} A(t,s)\varphi(s)\,\mathrm{d}s, \qquad \text{for} \quad t \in [t_0,\infty),$$

then every solution $y(t)$ *of the integral equation*

$$y(t) = g(t) + \int_{t_0}^{t} A(t,s)y(s)\,\mathrm{d}s$$

satisfies the inequality $y(t) \geq \varphi(t)$ *in* $[t_0, \infty)$.

P r o o f. See [5; p. 331]. □

LEMMA 2. (Wintner) *Let* $U(t,u)$ *be a continuous function on a domain* $t_0 \leq t \leq t_0 + \alpha$, $\alpha > 0$, $u \geq 0$, *and let* $u(t)$ *be a maximal solution of the Cauchy problem* $u' = U(t,u)$, $u(t_0) = u_0 \geq 0$, ($u' = U(t,u)$ *is a scalar differential equation) existing on* $[t_0, t_0 + \alpha]$; *for example, let* $U(t,u) = \psi(u)$, *where* $\psi(u)$ *is a continuous and positive function for* $u \geq 0$ *such that*

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} = \infty .$$

Let us assume $f(t,y)$ *is continuous on* $t_0 \leq t \leq t_0 + \alpha$, $y \in E_1^n$, *where* y *is arbitrary and satisfies a condition*

$$|f(t,y)| \leq U(t, |y|) .$$

Then the maximal interval of existence of the solution of the Cauchy problem

$$y' = f(t,y), y(t_0) = y_0 ,$$

where $|y_0| \leq u_0$, *is* $[t_0, t_0 + \alpha]$.

P r o o f. See [2; Theorem III.5.1] □

2. Results

LEMMA 3. *Let* (A) *hold, and let there exist real nonnegative functions* $a_1(t)$, $a_2(t)$ *such that* $|f(t,y)| \leq a_1(t)|y| + a_2(t)$ *for all* $(t,y) \in I_a \times E_1$. *Let initial values* $L_k y(a) = b_k$ *be given for* $k = 0, 1, \dots, n-1$. *Then there exists a solution* $y(t)$ *of* (L) *on* $[a, \infty)$ *which fulfils these initial conditions.*

P r o o f. Let $n \geq 2$. The equation (L) is equivalent to the following system

$$u_k'(t) = u_{k+1}(t)/p_k(t) \qquad \text{for} \quad k = 1, 2, \dots, n-1 , \tag{S}$$

$$u_n'(t) = -\sum_{k=1}^{n-1} P_k(t) u_{k+1}(t) - f\big(t, u_1(t)\big) ,$$

where $u_k(t) = L_{k-1}y(t)$ for $k = 1,2,\dots,n$. Let us denote $f_k = f_k(t,u_1,u_2,\dots,u_n)$ for $k = 1,2,\dots,n$, where

$$\begin{aligned} f_k &= u_{k+1}/p_k\,, \qquad k = 1,2,\dots,n-1\,,\\ f_n &= -\sum_{k=1}^{n-1} P_k u_{k+1} - f(t,u_1)\,,\\ F(t,u) &= (f_1,f_2,\dots,f_n)\,,\\ u = u(t) &= \big(u_1(t),u_2(t),\dots,u_n(t)\big)\,,\\ u' = u'(t) &= \big(u_1'(t),u_2'(t),\dots,u_n'(t)\big)\,. \end{aligned}$$

It is obvious that the f_k are continuous on a set M_b, where $M_b = [a,b] \times E_1^n$, $a < b < \infty$. Let

$$(t,u) = (t,u_1,u_2,\dots,u_n)\,,$$

where (t,u) is an arbitrary pair from M_b, and let

$$|u| = \sum_{k=1}^{n} |u_k|\,, \qquad |F(t,u)| = \sum_{k=1}^{n} |f_k|\,.$$

Then

$$\begin{aligned} |F(t,u)| &= \sum_{k=1}^{n-1} |u_{k+1}/p_k| + \Big|-\sum_{k=1}^{n-1} P_k u_{k+1} - f(t,u_1)\Big|\\ &\le \sum_{k=1}^{n-1} |u_{k+1}/p_k| - \sum_{k=1}^{n-1} P_k |u_{k+1}| - f(t,u_1)\\ &= \sum_{k=2}^{n} (-P_{k-1} + 1/p_{k-1})|u_k| - f(t,u_1)\\ &\le K_1 \sum_{k=2}^{n} |u_k| + a_1|u_1| + a_2\\ &\le K_2 \sum_{k=1}^{n} |u_k| + a_2 \le K(1+|u|)\,, \end{aligned}$$

where K_1, K_2, K are the following constants:

$$\begin{aligned} K_1 &= \max\big\{-P_{k-1}(t) + 1/p_{k-1}(t)\,,\ t \in [a,b]\,,\ k = 2,3,\dots,n\big\}\,,\\ K_2 &= \max\big\{K_1, a_1(t)\,,\ t \in [a,b]\big\}\,,\\ K &= \max\big\{1, K_2, a_2(t)\,,\ t \in [a,b]\big\}\,. \end{aligned}$$

Let $\psi(y) = K(1+y)$. Then Cauchy problem $y' = \psi(y)$, $y(a) = y_0 \geq 0$ admits the unique solution $y(t) = (1+y_0)\exp\bigl(K(t-a)\bigr)-1$ on $[a,b]$, $\int^{\infty}\bigl(1/\psi(s)\bigr)\,\mathrm{d}s = \infty$, $|F(t,u)| \leq U(t,|u|) = \psi(|u|)$. Then, according to Lemma 2, the system (S) admits a solution $u(t)$ on $[a,b]$, which satisfies the initial conditions $u_{k+1}(a) = b_k$, $k = 0,1,\dots,n-1$. Because $b > a$, b is an arbitrary real number, we obtain the assertion of the lemma for $n \geq 2$ by going from (S) to (L). If $n = 1$, the system (S) is generated by the unique equation $y' = -f(t,y)$, and in this case, the proof is analogous to that one for $n \geq 2$. So it is omitted. □

Lemma 4. *Let $y(t)$ be a solution of (L) on I_a, and let (A) hold. Let $t_0 \in I_a$ and $L_k y(t_0) \geq 0$ for $k = 0,1,\dots,n-1$. Then $L_k y(t) \geq 0$ on the interval $[t_0,\infty)$ for $k = 0,1,\dots,n-1$.*

Proof. Let $n \geq 2$. Integration of the relationship $L_n y = (L_{n-1}y)'$ over $[t_0,t]$, $t_0 < t$, yields

$$\begin{aligned} L_{n-1}y(t) &= L_{n-1}y(t_0) - \sum_{k=1}^{n-1}\int_{t_0}^{t} P_k(s)L_k y(s)\,\mathrm{d}s - \int_{t_0}^{t} f\bigl(s,y(s)\bigr)\,\mathrm{d}s \\ &= L_{n-1}y(t_0) + \int_{t_0}^{t}\bigl(-f\bigl(s,y(s)\bigr)\bigr)\,\mathrm{d}s + \int_{t_0}^{t}\sum_{k=1}^{n-1}\bigl(-P_{n-k}(s)L_{n-k}y(s)\bigr)\,\mathrm{d}s. \end{aligned}$$

Let us denote $L_{n-1}y(t_0) + \int_{t_0}^{t}\bigl(-f\bigl(s,y(s)\bigr)\bigr)\,\mathrm{d}s$ by $K(t)$. This notation is correct because the function $y(t)$ is fixed. It is obvious that $K(t) \geq 0$. We have

$$L_{n-1}y(t) = K(t) + \int_{t_0}^{t}\sum_{k=1}^{n-1}\bigl(-P_{n-k}(s)L_{n-k}y(s)\bigr)\,\mathrm{d}s. \tag{1}$$

It can be proved that $(s_0 = s)$

$$
\begin{aligned}
&L_{n-k}y(s) = \\
&= L_{n-k}y(t_0) + L_{n-k+1}y(t_0)\int_{t_0}^{s}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)} \\
&\quad + L_{n-k+2}y(t_0)\int_{t_0}^{s}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\int_{t_0}^{s_1}\frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)} + \dots \\
&\quad + L_{n-2}y(t_0)\int_{t_0}^{s}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\int_{t_0}^{s_1}\frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)}\cdots\int_{t_0}^{s_{k-3}}\frac{\mathrm{d}s_{k-2}}{p_{n-2}(s_{k-2})} \\
&\quad + \int_{t_0}^{s}\frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)}\int_{t_0}^{s_1}\frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)}\int_{t_0}^{s_2}\frac{\mathrm{d}s_3}{p_{n-k+3}(s_3)}\cdots\int_{t_0}^{s_{k-2}}\frac{L_{n-1}y(s_{k-1})}{p_{n-1}(s_{k-1})}\,\mathrm{d}s_{k-1}
\end{aligned}
$$

for $k = 2,3,\dots,n-1$. Denoting the last $(k-1)$-dimensional integral by $I_k(s)$, and the previous sum by $G_k(s)$, $G_1(s) = 0$, $I_1(s) = L_{n-1}y(s)$ for $k = 2,3,\dots,n-1$ we have $(s_0 = s)$

$$L_{n-k}y(s) = G_k(s) + I_k(s)$$

for $k = 1,2,\dots,n-1$. Hence

$$
\begin{aligned}
L_{n-1}y(t) &= K(t) + \int_{t_0}^{t}\sum_{k=1}^{n-1}\Bigl(-P_{n-k}(s)\bigl[G_k(s) + I_k(s)\bigr]\Bigr)\,\mathrm{d}s \\
&= K(t) + \int_{t_0}^{t}\sum_{k=1}^{n-1}\bigl(-P_{n-k}(s)G_k(s)\bigr)\,\mathrm{d}s + \int_{t_0}^{t}\sum_{k=1}^{n-1}\bigl(-P_{n-k}(s)I_k(s)\bigr)\,\mathrm{d}s\,.
\end{aligned}
$$

Denoting $K(t) + \int_{t_0}^{t}\sum_{k=1}^{n-1}\bigl(-P_{n-k}(s)G_k(s)\bigr)\,\mathrm{d}s$ by $g(t)$ and $\int_{t_0}^{t}\bigl(-P_{n-k}(s)I_k(s)\bigr)\,\mathrm{d}s$ by $J_k(t)$, we have

$$L_{n-1}y(t) = g(t) + \sum_{k=1}^{n-1}J_k(t)\,.$$

It is obvious that

$$J_k(t) = \int_{t_0}^{t} (-P_{n-k}(s))\, \mathrm{d}s \int_{t_0}^{s} \frac{\mathrm{d}s_1}{p_{n-k+1}(s_1)} \int_{t_0}^{s_1} \frac{\mathrm{d}s_2}{p_{n-k+2}(s_2)} \cdots$$

$$\cdots \int_{t_0}^{s_{k-2}} \frac{L_{n-1}y(s_{k-1})}{p_{n-1}(s_{k-1})}\, \mathrm{d}s_{k-1}\,,$$

$$J_1(t) = \int_{t_0}^{t} (-P_{n-1}(s) L_{n-1}y(s))\, \mathrm{d}s$$

for $k = 2, 3, \dots, n-1$. By a change of notation, we get

$$J_k(t) = \int_{t_0}^{t} (-P_{n-k}(s_{k-1}))\, \mathrm{d}s_{k-1} \int_{t_0}^{s_{k-1}} \frac{\mathrm{d}s_{k-2}}{p_{n-k+1}(s_{k-2})} \int_{t_0}^{s_{k-2}} \frac{\mathrm{d}s_{k-3}}{p_{n-k+2}(s_{k-3})} \cdots$$

$$\cdots \int_{t_0}^{s_1} \frac{L_{n-1}y(s)}{p_{n-1}(s)}\, \mathrm{d}s$$

for $k = 2, 3, \dots, n-1$. Changing the order of the variables $s, s_1, s_2, \dots, s_{k-1}$ yields:

$$J_k(t) = \int_{t_0}^{t} L_{n-1}y(s)\, \mathrm{d}s \int_{s}^{t} \frac{\mathrm{d}s_1}{p_{n-2}(s_1)} \int_{s_1}^{t} \frac{\mathrm{d}s_2}{p_{n-3}(s_2)} \cdots \int_{s_{k-2}}^{t} \left(-\frac{P_{n-k}(s_{k-1})}{p_{n-1}(s)}\right) \mathrm{d}s_{k-1}\,.$$

The last integral can be rewritten in the form

$$J_k(t) = \int_{t_0}^{t} M_k(t,s) L_{n-1}y(s)\, \mathrm{d}s\,, \qquad k = 1, 2, \dots, n-1\,,$$

where

$$M_k(t,s) = \int_{s}^{t} \frac{\mathrm{d}s_1}{p_{n-2}(s_1)} \int_{s_1}^{t} \frac{\mathrm{d}s_2}{p_{n-3}(s_2)} \cdots \int_{s_{k-2}}^{t} \left(-\frac{P_{n-k}(s_{k-1})}{p_{n-1}(s)}\right) \mathrm{d}s_{k-1}\,,$$

$$M_1(t,s) = -P_{n-1}(s)$$

for $k = 2, 3, \dots, n-1$. Hence

$$L_{n-1}y(t) = g(t) + \sum_{k=1}^{n-1} \int_{t_0}^{t} M_k(t,s) L_{n-1}y(s)\, \mathrm{d}s$$

$$= g(t) + \int_{t_0}^{t} \sum_{k=1}^{n-1} M_k(t,s) L_{n-1}y(s)\, \mathrm{d}s$$

and

$$L_{n-1}y(t) = g(t) + \int_{t_0}^{t} A(t,s)L_{n-1}y(s)\,\mathrm{d}s\,, \tag{2}$$

where

$$A(t,s) = \sum_{k=1}^{n-1} M_k(t,s)\,.$$

Because $t_0 \le s$, $t_0 \le s_k$, $s \le t$, $s_k \le t$, we have $g(t) \ge 0$, $A(t,s) \ge 0$. It is obvious that

$$0 \le \int_{t_0}^{t} A(t,s)g(s)\,\mathrm{d}s\,,$$

$$g(t) \le g(t) + \int_{t_0}^{t} A(t,s)g(s)\,\mathrm{d}s\,.$$

Because

$$L_{n-1}y(t) = g(t) + \int_{t_0}^{t} A(t,s)L_{n-1}y(s)\,\mathrm{d}s\,,$$

according to Lemma 1, we have

$$L_{n-1}y(t) \ge g(t) = \varphi(t) \ge 0 \qquad \text{on} \quad [t_0,\infty)\,.$$

Because

$$L_{n-2}y(t) = L_{n-2}y(t_0) + \int_{t_0}^{t} \frac{L_{n-1}y(s)}{p_{n-1}(s)}\,\mathrm{d}s \ge L_{n-2}y(t_0)\,,$$

we have $L_{n-2}y(t) \ge 0$ on $[t_0,\infty)$. By using a similar procedure, we will get $L_k y(t) \ge L_k y(t_0) \ge 0$ on $[t_0,\infty)$ for $k = 0,1,\dots,n-3$.

We note that if $n = 2$, then the expressions (1) and (2) are the same. If $n = 1$, then the assertion of the lemma follows from the fact that

$$y'(t) = -f\big(t,y(t)\big) \ge 0 \qquad \text{for} \quad t \ge t_0\,.$$

The lemma is proved. □

Now let us consider the linear differential equation (L') and the condition (A'), where

(L') $L_n y + \sum_{k=0}^{n-1} P_k(t)L_k y \equiv 0$,

(A') $P_k(t) \le 0$, $p_i(t) > 0$ for all $t \in I_a$, P_k, p_i are continuous functions on I_a for $k = 0,1\dots,n-1$, $i = 1,2,\dots,n-1$; n is an arbitrary positive integer.

LEMMA 5. *Let* (A') *hold, and let the initial values* $L_k y(a) = b_k$ *be given for* $k = 0, 1, \dots, n-1$. *Then there exists a solution* $y(t)$ *of* (L') *on* I_a *which fulfils these initial conditions.*

Proof. The proof of this lemma is similar to the proof of Lemma 3, and so it is omitted. □

LEMMA 6. *Let* (A') *hold, and let* $y(t)$ *be a solution of the linear differential equation* (L') *on the interval* I_a *which satisfies the following initial conditions* $L_k y(t_0) \geq 0$, $t_0 \in I_a$ *for* $k = 0, 1, \dots, n-1$. *Then* $L_k y(t) \geq 0$ *on* $[t_0, \infty)$ *for* $k = 0, 1, \dots, n-1$.

Proof. The proof of this lemma is similar to the proof of Lemma 4, and so it is omitted. □

THEOREM 1. *Let* (A) *hold. If the equation* (L) *has a solution* $y(t)$ *on* $[a, \infty)$, *and if* $L_k y(a) \geq 0$ *for* $k = 1, 2, \dots, n-1$, $y(a) > 0$, *then* $y(t)$ *is monotone on* $[a, \infty)$.

Proof. This is an immediate corollary of Lemma 4 for $t_0 = a$, and the fact that $y(t) \geq y(a)$ for all $t \in I_a$. □

Remark. If $L_k y(a) > 0$ for $k = 1, 2, \dots, n-1$ in Theorem 1, then $L_k y(t) > 0$ for $t \geq a$, $k = 0, 1, \dots, n-1$. This follows from the proof of Lemma 4 for $t_0 = a$ because $L_k y(a) > 0$, and $L_k y(t) \geq L_k y(a)$ on $[a, \infty)$ for $k = 0, 1, \dots, n-1$.

THEOREM 2. *Let* (A) *hold, and let there exist nonnegative real functions* $a_1(t)$, $a_2(t)$, *such that* $|f(t,y)| \leq a_1(t)|y| + a_2(t)$ *for all* $(t,y) \in I_a \times E_1$. *Let the initial values* $L_0 y(a) = y(a) > 0$, $L_k y(a) \geq 0$ *be given for* $k = 1, 2, \dots, n-1$. *Then there exists a solution* $y(t)$ *of* (L) *on* $[a, \infty)$ *which fulfils these initial conditions, and this solution is monotone on* $[a, \infty)$.

Proof. The existence of this solution follows from Lemma 3. The monotony of this solution follows from Lemma 4 and the fact $y(t) \geq y(a)$ for all $t \geq a$. □

THEOREM 3. *Let* (A') *hold, and let the initial conditions* $y(a) > 0$, $L_k y(a) \geq 0$, $k = 1, 2, \dots, n-1$, *be given. Then there exists a solution* $y(t)$ *of* (L') *on* $[a, \infty)$ *which satisfies these initial conditions, and this solution* $y(t)$ *is monotone on* $[a, \infty)$.

Proof. The existence of the solution satisfying the above initial conditions follows from Lemma 5. This solution is monotone according to Lemma 6 and the fact $y(t) \geq y(a)$ for all $t \geq a$. □

3. Examples

EXAMPLE 1. The equation (L), where $n = 5$, $p_i(t) = t^i$, $i = 1,2,3,4$, $P_1(t) = -5t^5$, $P_2(t) = -10t^4$, $P_3(t) = -3t^2$, $P_4(t) = -1/t$, $f(t,y) = -334t^3y^2$ admits a solution $y(t) = t^2$ on $[1,\infty)$ such that $L_k y(1) > 0$ for $k = 0,1,2,3,4$. According to Theorem 1, this solution $y(t)$ is monotone on $[1,\infty)$. We note that Theorem 2 cannot be used because of the form of $f(t,y)$.

EXAMPLE 2. Let $n = 5$ in (L), $p_k(t) = \mathrm{e}^{kt}$ for $k = 1,2,3,4$, $P_1(t) = -2\,\mathrm{e}^{9t}$, $P_2(t) = -2\,\mathrm{e}^{7t}$, $P_3(t) = -6\,\mathrm{e}^{4t}$, $P_4(t) \equiv -10$, $f(t,y) = -\mathrm{e}^{10t}\sqrt{3\,\mathrm{e}^{2t}+y^2}$, $L_0y(1) = \mathrm{e}$, $L_1y(1) = \mathrm{e}^2$, $L_2y(1) = 2\,\mathrm{e}^4$, $L_3y(1) = 8\,\mathrm{e}^7$, $L_4y(1) = 56\,\mathrm{e}^{11}$. It is obvious that $|f(t,y)| \le \mathrm{e}^{10t}\big(\sqrt{3}\,\mathrm{e}^t + |y|\big) = \mathrm{e}^{10t}|y| + \sqrt{3}\,\mathrm{e}^{11t}$ for all $(t,y) \in I_1 \times E_1$. According to Theorem 2, the equation (L) admits a monotone solution $y(t)$ on $[1,\infty)$, where $L_k y(t) > 0$ for $t \ge 1$, $k = 0,1,2,3,4$. This solution $y(t)$ is the function e^t.

EXAMPLE 3. Let n be an arbitrary number from $\{1,2,\dots\}$, let $p_k(t) = t^k$, $k = 1,2,\dots,n-1$, $P_k(t) = -\mathrm{e}^{-kt}$, $k = 0,1,\dots,n-1$, $f(t,y) = -\mathrm{e}^{-t}\sqrt{1+y^2}$. Then $|f(t,y)| \le \mathrm{e}^{-t}(1+|y|) = \mathrm{e}^{-t}|y| + \mathrm{e}^{-t}$ for all $(t,y) \in I_1 \times E_1$, $L_k y(1) = 1$ for $k = 0,1,\dots,n-1$ in the equation (L). According to Theorem 2, then there exists a solution $y(t)$ of (L) which is monotone on $[1,\infty)$.

EXAMPLE 4. Every solution of the linear differential equation (L') on $[a,\infty)$, where $p_i(t) = 1 + t^{2i}$, $P_k(t) = -\mathrm{e}^{kt}$, $i = 1,2,\dots,n-1$, $k = 0,1,\dots,n-1$, n is an arbitrary fixed positive integer which fulfils the initial conditions $y(a) > 0$, $L_k y(a) \ge 0$ for $k = 1,2,\dots,n-1$, $a \in E_1$, is monotone on $[a,\infty)$ according to Theorem 3.

Acknowledgement

I would like to thank Professor Jozef Rovder for his stimulative suggestions during the period in which this paper was beeing prepared — wherever that great heart may now be since he left us in April 1997.

REFERENCES

[1] BARRETT, J. H.: *Two-point boundary problems for linear self-adjoint differential equations of the fourth-order with middle term*, Duke Math. J. **29** (1962), 543–554.

[2] HARTMAN, P.: *Ordinary Differential Equations*, John Wiley & Sons, New York, 1964.

[3] LEIGHTON, W.—NEHARI, Z.: *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc. **89** (1958), 325–377.

[4] PHILOS, CH. G.: *Oscillation and asymptotic behaviour of third order linear differential equations*, Bull. Inst. Math. Acad. Sinica **11** (1983), 141-160.

[5] REGENDA, J.: *Oscillatory and nonoscillatory properties of solutions of the differential equation* $y^{(4)} + P(t)y'' + Q(t)y = 0$, Math. Slovaca **28** (1978), 329–342.

[6] ROVDER, J.: *Comparison theorems for third-order linear differential equations*, Bull. Inst. Math. Acad. Sinica **19** (1991), 43–52.

[7] ROVDEROVÁ, E.: *Existence of a monotone solution of a nonlinear differential equation*, J. Math. Anal. Appl. **192** (1995), 1–15.

[8] SHAIR, A.: *On the oscillation of solutions of a class of linear fourth order differential equations*, Pacific J. Math. **34** (1970), 289–299.

[9] ŠKERLÍK, A.: *Criteria of property* **A** *for third order superlinear differential equations*, Math. Slovaca **43** (1993), 171–183.

[10] ŠVEC, M.: *On various properties of the solutions of third and fourth order linear differential equations.* In: Proceedings of the conference held in Prague in September 1962, pp. 187–198.

[11] ŠVEC, M.: *Über einige neue Eigenschaften der oszillatorischen Lösungen der linearen homogenen Differentialgleichung vierter Ordnung*, Czechoslovak Math. J. **4(79)** (1954), 75–94.

[12] TÓTHOVÁ, M.—PALUMBÍNY, O.: *On monotone solutions of the fourth order ordinary differential equations*, Czechoslovak Math. J. **45(120)** (1995), 737–746.

Received December 2, 1994
Revised April 4, 1995

Department of Mathematics
Faculty of Material Sciences
and Technology
Slovak University of Thechnology
in Bratislava
Paulínska 16
SK-917 24 Trnava
SLOVAKIA
E-mail: palum@sun.mtf.stuba.sk