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ON POSITIVELY EXPANSIVE DIFFERENTIABLE MAPS

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ABSTRACT. We obtain a necessary and sufficient condition for a positively expansive differentiable map to be expanding.

Let M be a C^∞ closed manifold, and let $C^1(M)$ be the space of C^1 maps of M endowed with C^1 topology. We denote by d a Riemannian distance of M, and let $f \in C^1(M)$. We say that f is *positively expansive* if there exists a constant $c > 0$ such that for $x, y \in M$, $d\big(f^n(x), f^n(y)\big) \le c$ $(n \ge 0)$ implies $x = y$. It is well known that the set of all periodic points of the positively expansive map f, $P(f)$, is dense in M (see [5; Lemma 2]). If there is a Riemannian metric $\|\cdot\|$ on TM and $\lambda > 1$ such that $\|\mathrm{D} f^n(v)\| \ge \lambda^n \|v\|$ for $v \in TM$ and $n \ge 0$, then f is called *expanding*.

Every expanding C^1 map is positively expansive (see [3; Theorem 2]), but the converse is not true. Indeed, there is an example of a positively expansive C^1 map f on the unit circle which has a fixed point p such that $\mathrm{D}_p f = \mathrm{id}$ (see [1]). In this note, by using Mañé's technique stated in [2], we prove the following theorem.

THEOREM. *Let $f \in C^1(M)$ be positively expansive. Then the following two conditions are equivalent:*

(i) *f is expanding,*

(ii) *there is a Riemannian metric $\|\cdot\|$ on TM and $\gamma > 1$ such that*

$$\inf_{p \in P(f)} \left[\prod_{i=0}^{\pi(p)-1} |\mathrm{D}_{f^i(p)} f| \right]^{\frac{1}{\pi(p)}} \ge \gamma ,$$

where $|\mathrm{D}_x f| = \min_{\|v\|=1} \|\mathrm{D}_x f(v)\|$, and $\pi(p)$ is the minimum period of $p \in P(f)$.

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Let M and d be as before, and let $f\colon M \to M$ be a continuous map. A sequence $\{x_k\}_{k=0}^{\infty}$ of points is called a *δ-pseudo-orbit* of f if $d\big(f(x_k), x_{k+1}\big) < \delta$ for $k \geq 0$. Given $\varepsilon > 0$, $\{x_k\}_{k=0}^{\infty}$ is said to be *ε-shadowed* by $x \in M$ if $d\big(f^k(x), x_k\big) < \varepsilon$ for $k \geq 0$. We say that f has the *shadowing property* if for $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-shadowed by some point of M. If f is positively expansive, then it is an open map (and so $f(M) = M$) since f is locally one-to-one. Thus f has the shadowing property (see [4]).

Denote by $\mathcal{M}(M)$ the set of all probabilities on the Borel σ-algebra of M endowed with its usual topology such that

$$\mu_n \to \mu \iff \int \xi \,\mathrm{d}\mu_n \to \int \xi \,\mathrm{d}\mu$$

for every continuous function $\xi\colon M \to \mathbb{R}$. For a continuous map $f\colon M \to M$, we denote by $\mathcal{M}(f)$ the set of all f-invariant elements of $\mathcal{M}(M)$. Take $x \in M$ and define a probability $\mu_n(x) \in \mathcal{M}(M)$ $(n > 0)$ by

$$\mu_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}\,.$$

Then it is easy to see that if μ is an accumulation point of $\big\{\mu_n(x)\big\}_{n=1}^{\infty}$, then $\mu \in \mathcal{M}(f)$.

To prove our theorem, we need the following two lemmas. The first one is well known.

Lemma 1. *Let f be a continuous map of M onto itself. Then there is a Borel set $c(f) \subset \big\{x \in M :\ x \in \omega(x)\big\}$ such that $\mu\big(c(f)\big) = 1$ for every $\mu \in \mathcal{M}(f)$.*

Lemma 2. *If f is positively expansive and $\mu \in \mathcal{M}(f)$ is ergodic, then for every neighbourhood $\mathcal{U}(\mu) \subset \mathcal{M}(M)$ of μ, there is $p \in P(f)$ such that $\mu_{\pi(p)}(p) \in \mathcal{U}(\mu)$.*

Proof. For any neighbourhood $\mathcal{U}(\mu)$ of an ergodic measure μ, there are an $\alpha > 0$ and a finite sequence of continuous functions $\{\xi_j\}_{j=1}^{\ell}$ from M to $\mathbb{R}$ such that if

$$\max_{1\leq j\leq \ell}\left|\int \xi_j \,\mathrm{d}\mu - \int \xi_j \,\mathrm{d}\nu\right| \leq \alpha \qquad (\nu \in \mathcal{M}(M)),$$

then $\nu \in \mathcal{U}(\mu)$. By Birkhoff's ergodic theorem, there is a Borel set A $(\mu(A) = 1)$ such that for all $x \in A$, there is $N(x) > 0$ satisfying

$$\max_{1\leq j\leq \ell}\left|\frac{1}{n}\sum_{i=0}^{n-1} \xi_j\big(f^i(x)\big) - \int \xi_j \,\mathrm{d}\mu\right| \leq \frac{\alpha}{2} \qquad \text{for} \quad n \geq N(x).$$

Fix $x \in c(f) \cap A$, and let $c > 0$ be an expansive constant for f. Then there is $0 < \varepsilon \leq c/2$ such that $d(x,y) < \varepsilon$ $(x, y \in M)$ implies $\max_{1\leq j\leq \ell} |\xi_j(x) - \xi_j(y)| \leq \alpha/2$. Let $0 < \delta = \delta(\varepsilon) \leq \varepsilon$ be as in the definition of the shadowing property of f. Then, since $x \in \omega(x)$, there is $n \geq N(x)$ such that $\{x, f(x), f^2(x), \ldots \ldots, f^{n-1}(x), x, f(x), \ldots\}$ is a cyclic δ-pseudo-orbit of f. Thus we can find $p \in P(f)$ $(f^n(p) = p)$ such that $d(f^i(x), f^i(p)) \leq \varepsilon$ for $0 \leq i \leq n-1$. Therefore we have

$$\max_{1\leq j\leq \ell} \left| \frac{1}{n} \sum_{i=0}^{n-1} \xi_j(f^i(x)) - \frac{1}{n} \sum_{i=0}^{n-1} \xi_j(f^i(p)) \right| \leq \frac{\alpha}{2},$$

and so

$$\max_{1\leq j\leq \ell} \left| \int \xi_j \, \mathrm{d}\mu - \int \xi_j \, \mathrm{d}\mu_n(p) \right| \leq \alpha.$$

□

Proof of Theorem. Let $f \in C^1(M)$ be positively expansive. Then, to get the conclusion, it is enough to show that if there exist a Riemannian metric $\|\cdot\|$ on TM and $\gamma > 1$ such that

$$\prod_{i=0}^{\pi(p)-1} |\mathrm{D}_{f^i(p)} f| \geq \gamma^{\pi(p)} \qquad (p \in P(f)),$$

then f is expanding. Put $\varphi(x) = \log |\mathrm{D}_x f| = \log \inf_{\|v\|=1} \|\mathrm{D}_x f(v)\|$ $(x \in M)$. Then $\varphi \colon M \to \mathbb{R}$ is continuous. Thus we have $\int \varphi \, \mathrm{d}\mu > 0$ for every $\mu \in \mathcal{M}(f)$. Indeed, if there is $\mu \in \mathcal{M}(f)$ such that $\int \varphi \, \mathrm{d}\mu \leq 0$, then by using the ergodic decomposition theorem we can find an ergodic measure $\bar{\mu} \in \mathcal{M}(f)$ such that $\int \varphi \, \mathrm{d}\bar{\mu} \leq 0$. Fix $0 < \varepsilon < \log \gamma$. Then, by Lemma 2, there are $p \in P(f)$ $(n = \pi(p))$ and $\mu_n(p) \in \mathcal{M}(f)$ such that $\int \varphi \, \mathrm{d}\mu_n(p) < \varepsilon$. Thus

$$\gamma^n \leq \prod_{i=0}^{n-1} |\mathrm{D}_{f^i(p)} f| < e^{n\varepsilon}.$$

Therefore we have $\log \gamma < \varepsilon$, which is a contradiction.

We claim that for every $x \in M$ there is $m(x) > 0$ such that $|\mathrm{D}_x f^{m(x)}| > 1$. If this is established, then it can be checked that there are constants $m > 0$ and $\lambda > 1$ such that $\|\mathrm{D} f^m(v)\| \geq \lambda \|v\|$ $(v \in TM)$. Thus, if we put $\lambda' = \lambda^{1/m} > 1$ and

$$\|v\|' = \sum_{i=0}^{m-1} \frac{1}{\lambda'^i} \|\mathrm{D} f^i(v)\| \qquad (v \in TM),$$

then f is expanding with respect to $\|\cdot\|'$ and λ'.

To prove the claim, we suppose that there exists $x \in M$ such that $|\mathrm{D}_x f^n| \leq 1$ for all $n > 0$. Then

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) = \frac{1}{n}\sum_{i=0}^{n-1} \log|\mathrm{D}_{f^i(x)} f| \leq 0 \qquad \text{for} \quad n > 0 .$$

Put $\mu_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)}$, and fix an accumulation point $\mu \in \mathcal{M}(f)$ of $\{\mu_n(x)\}_{n=1}^{\infty}$. Then we have $\int \varphi \, \mathrm{d}\mu \leq 0$ and this is a contradiction. □

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