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PSEUDO-PETRIE OPERATORS ON GRÜNBAUM POLYHEDRA

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(*Communicated by Martin Škoviera*)

ABSTRACT. In this article, we introduce and study the pseudo-Petrie operator, its properties, and the semi-regular polyhedra it produces. As a consequence, we obtain the 48th Grünbaum polyhedron as the pseudo-Petrie polyhedron of one of the 47 Grünbaum polyhedra.

1. Introduction

In 1977, B. Grünbaum [3] introduced a more general concept of regular polyhedron by allowing skew polygons as faces. He gave a systematic list of 47 such polyhedra and described the relations among these polyhedra in terms of the dual and the Petrie operator.

Later, A. Dress [2] gave a complete classification of the Grünbaum polyhedra. Grünbaum's list was found to be complete, except for one additional polyhedron, the 48th Grünbaum polyhedron.

In this article, we introduce a pseudo-Petrie operator, study its properties, and, as a consequence, we obtain the 48th Grünbaum polyhedron as the pseudo-Petrie polyhedron of one of the 47 Grünbaum polyhedra.

In Section 2, we summarize and regroup known results, such as regular polygons, regular polyhedra and Petrie operators. In Section 3, we introduce and study the pseudo-Petrie operator Ψ and the semi-regular polyhedra it produces.

We follow the notations used by Grünbaum [3].

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Key words: pseudo-Petrie operator, polygon, (semi-regular, Petrie-Coxeter, pseudo-Petrie, Grünbaum) polyhedron.

2. Operators on regular polyhedra

2.1. Regular polygons.

As the regular polygons are the building blocks of the Grünbaum polyhedra, we shortly recall the definition, notation and include an elementary property.

A *polygon* $P = [\dots V_1, V_2, V_3, \dots]$ in Euclidean space E^k is the figure formed by the distinct points (*vertices*) $\dots V_1, V_2, V_3, \dots$ together with the segments (*edges*) $E_i = [V_i, V_{i+1}]$.

In case P is *finite*, there is an additional edge $[V_{\text{first}}, V_{\text{last}}]$.

In case P is *infinite*, there is an additional condition: Each compact subset of E^k meets only finitely many edges.

A *flag* of P is a pair consisting of a vertex V of P incident with an edge E of P.

A polygon is said to be *regular* if its group of *symmetries* acts transitively on the family of all flags of P. This group of symmetries can be generated by two fundamental isometries ρ and σ, ρ fixing E_0 and σ fixing V_0.

A complete enumeration of regular polygons is given in [3].

From a slightly different point of view, the regular polygons $[\dots V_1, V_2, V_3, \dots]$ are the polygons for which $V_i = \phi^i(V_0)$, where ϕ is an isometry generating a finite or a discrete cyclic group [1; p. 45]. Thus we have:

Property 2.1.1. *A polygon $[\dots, V_0, V_1, V_2, \dots]$ is regular if and only if there exists an isometry ϕ such that for all V_i: $V_i = \phi^i V_0$.*

In all cases, except for the digon, the group generated by ϕ is not the whole symmetry group of the polygon, as it does not contain ρ.

2.2. Regular polyhedra.

A *polyhedron* P is a family of polygons (*faces*) with the following properties:

(i) each edge of a face is an edge of exactly one other face,
(ii) the family of polygons is connected,
(iii) each compact set meets only finitely many faces,
(iv) each vertex figure is a single polygon.

A *flag* of P is a triple consisting of a vertex V, an edge E and a face F of P, all mutually incident.

A polyhedron is said to be *regular* if its group of *symmetries* acts transitively on the family of all flags of P. This group of symmetries can be generated by three fundamental isometries ρ, σ and τ; ρ fixes E_0 and F_0, σ fixes V_0 and F_0, and τ fixes V_0 and E_0.

In Figures 1, 2, 3, we show three fundamental examples of Grünbaum polyhedra: the square tessellation $\{4,4\}$, the Petrie-Coxeter polyhedron $\{4, 6^{\frac{\pi}{3}}/1\}$, and the polyhedron $\{6^{\frac{\pi}{2}}/1, 4\}$. The notation $\{p, q\}$ means that the faces are p-gons, and the vertex figures are q-gons. We have enhanced the facets forming the vertex figure. More illustrations and a complete list can be found in [3], [2].

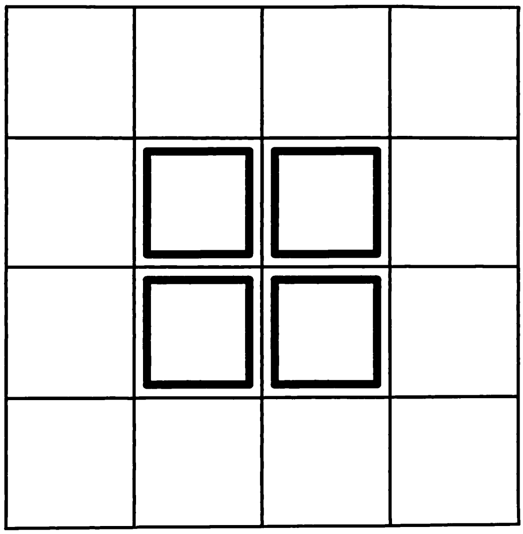

FIGURE 1.

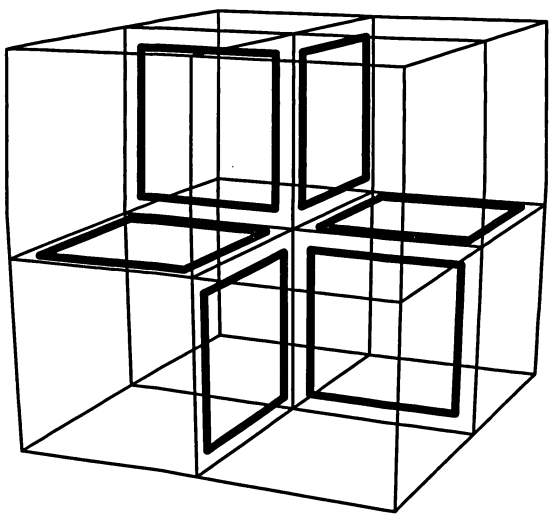

FIGURE 2.

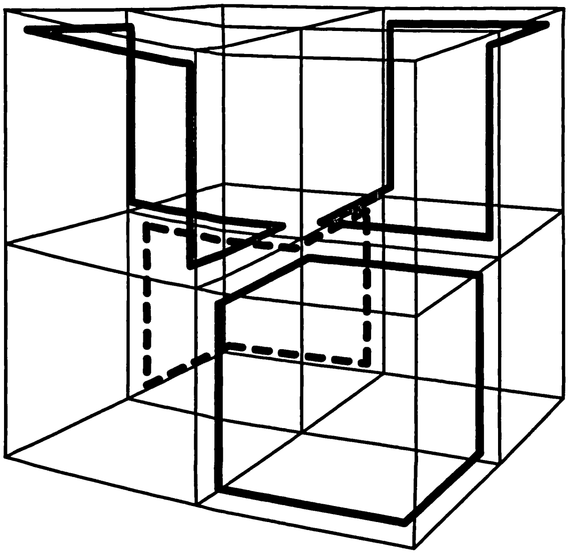

FIGURE 3.

2.3. The Petrie operator.

To get a clearer insight into the relationship between the different polyhedra, we can use operators on them. They transform a regular polyhedron into a regular polyhedron. The dual operator Δ and the Petrie operator Π were used in this way in [3], [2]. Further operations on regular polyhedra, such as facetting, halving, skewing, have been introduced in [4]. To clarify our construction, we are going to recall the definition of the Petrie operator.

The *Petrie* $\Pi(P)$ of a polyhedron P is the polyhedron defined by:

vertices: the vertices of P,
edges: the edges of P,
faces: polygons such that two successive edges are incident with one face of P, but no three successive edges are incident with the same face of P.

If the three fundamental generators of the group of a regular polyhedron P are in order (cf. Section 2) ρ, σ and τ, then those of $\Pi(P)$ are $\rho\tau$, σ and τ.

The effect of the Petrie operator on $\{4,4\}$, $\{4,6^{\frac{\pi}{3}}/1\}$ and $\{6^{\frac{\pi}{2}}/1,4\}$ is illustrated in Figures 4, 5 and 6 respectively. In the following list, we clarify the relationship between different Grünbaum polyhedra as given by the Petrie operator. For some of the notations refer to [3].

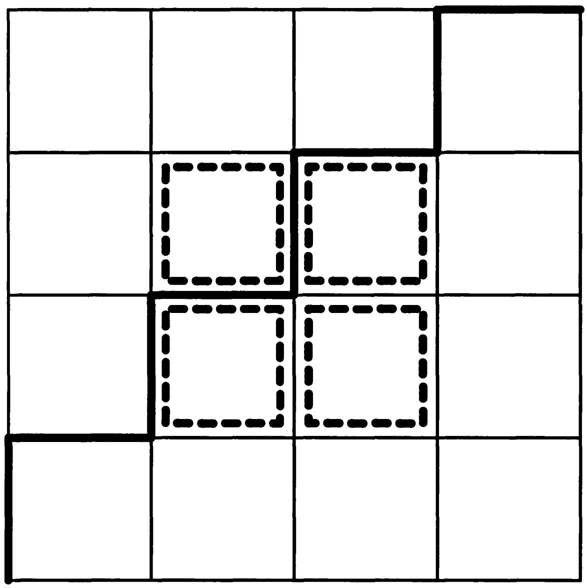

FIGURE 4.

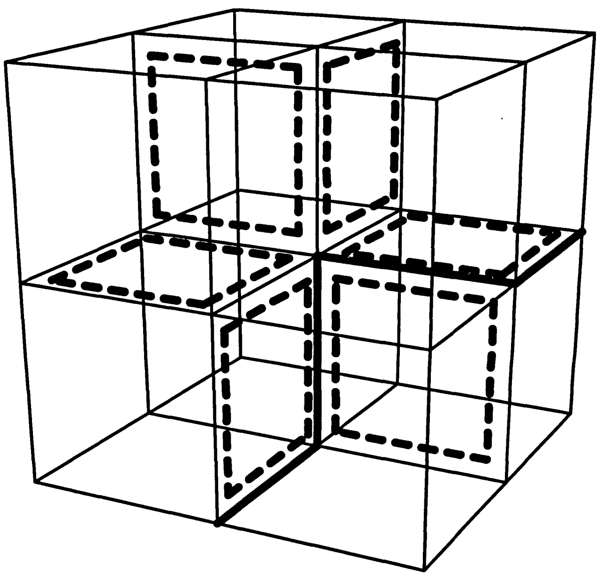

FIGURE 5.

polyhedron	Π(polyhedron)
platonic polyhedra	finite polyhedra with finite skew polygons
$\{3,3\}$ tetrahedron	$\{4^{\frac{\pi}{3}}/1,3\}$
$\{3,4\}$ octahedron	$\{6^{\frac{\pi}{3}}/1,4\}$
$\{4,3\}$ cube	$\{6^{\frac{\pi}{2}}/1,3\}$
$\{3,5\}$ icosahedron	$\{10^{\frac{\pi}{3}}/1,5\}$
$\{5,3\}$ dodecahedron	$\{10^{\frac{3\pi}{5}}/1,3\}$
Kepler-Poinsot polyhedra	finite polyhedra with finite skew polygons
$\{5,\frac{5}{2}\}$ great dodecahedron	$\{6^{\frac{3\pi}{5}}/1,\frac{5}{2}\}$
$\{3,\frac{5}{2}\}$ great icosahedron	$\{10^{\frac{\pi}{3}}/3,\frac{5}{2}\}$
$\{\frac{5}{2},5\}$ small stellated dodecahedron	$\{6^{\frac{\pi}{5}}/1,5\}$
$\{\frac{5}{2},3\}$ great stellated dodecahedron	$\{10^{\frac{\pi}{5}}/3,3\}$
planar tessellations	polyhedra with zig-zag polygons (with invariant plane)
$\{4,4\}$	$\{\infty^{\frac{\pi}{2}},4\}$
$\{3,6\}$	$\{\infty^{\frac{\pi}{3}},6\}$
$\{6,3\}$	$\{\infty^{\frac{2\pi}{3}},3\}$
infinite polyhedra (with finite skew polygons)	polyhedra with zig-zag polygons (with invariant plane)
$\{4^{\alpha}/1,4\}$ $(0<\alpha<\frac{\pi}{2})$	$\{\infty^{\alpha},4\}$
$\{2.3^{\alpha}/1,6\}$ $(0<\alpha<\frac{\pi}{3})$	$\{\infty^{\alpha},6\}$
$\{6^{\alpha}/1,3\}$ $(0<\alpha<\frac{2\pi}{3})$	$\{\infty^{\alpha},3\}$
Petrie-Coxeter polyhedra	polyhedra with helical polygons
$\{4,6^{\frac{\pi}{3}}/1\}$	$\{\infty^{\frac{\pi}{2},\frac{2\pi}{3}},6^{\frac{\pi}{3}}/1\}$
$\{6,4^{\alpha*}/1\}$	$\{\infty^{\frac{2\pi}{3},\frac{2\pi}{3}},4^{\frac{\pi}{3}}/1\}$
$\{6,6^{\alpha**}/1\}$	$\{\infty^{\frac{2\pi}{3},\frac{\pi}{2}},6^{\alpha**}/1\}$
polyhedra with helical polygons	polyhedra with zig-zag polygons
$\{\infty^{\alpha(b),\frac{\pi}{2}},4^{\alpha*(b)}/1\}$	$\{\infty^{\alpha(b)},4^{\alpha*(b)}/1\}$
$\{\infty^{\gamma(b),\frac{2\pi}{3}},6^{\gamma*(b)}/1\}$	$\{\infty^{\gamma(b)},6^{\gamma*(b)}/1\}$
$\{\infty^{\delta(b),\frac{\pi}{3}},2.3^{\delta*(b)}/1\}$	$\{\infty^{\delta(b)},2.3^{\delta*(b)}/1\}$
polyhedra with helical polygons	polyhedra with helical polygons
$\{\infty^{\frac{2\pi}{3},\frac{\pi}{2}},3\}$	$\{\infty^{\frac{2\pi}{3},\frac{2\pi}{3}},3\}$
infinite polyhedra (with finite skew polygons)	infinite polyhedra (with finite skew polygons)
$\{4^{\frac{\pi}{3}}/1,6\}$	$\{6^{\frac{\pi}{3}},6\}$
$\{6^{\frac{\pi}{2}}/1,4\}$	$\{6^{\frac{\pi}{2}}/1,4\}$

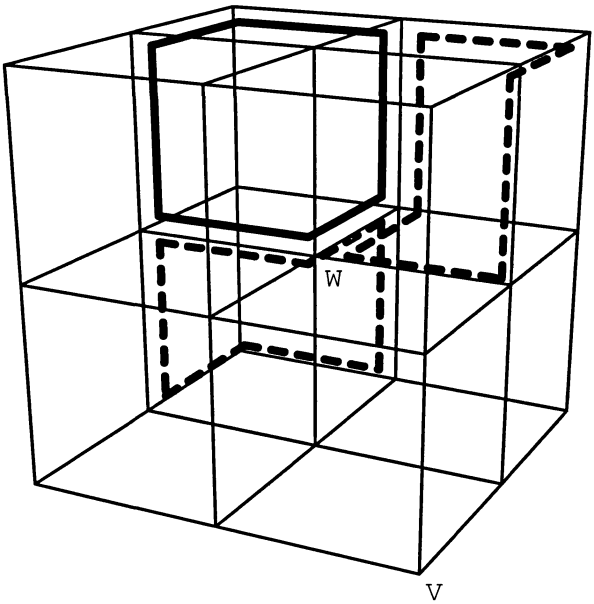

FIGURE 6.

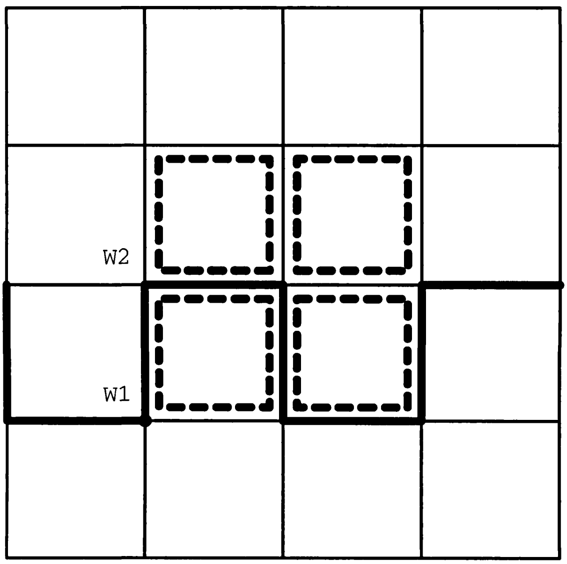

FIGURE 7.

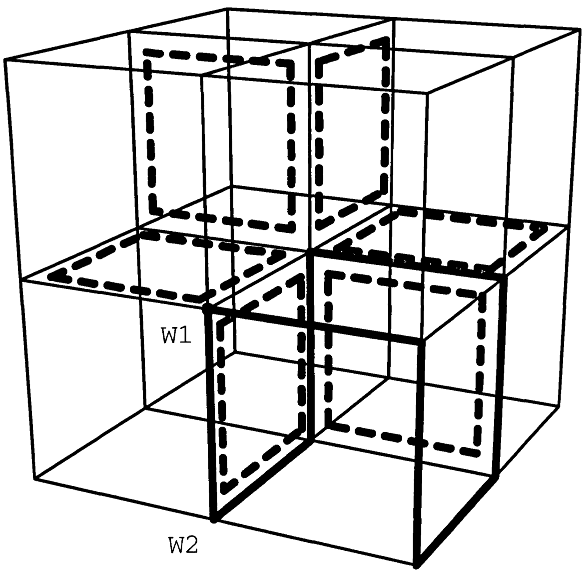

FIGURE 8.

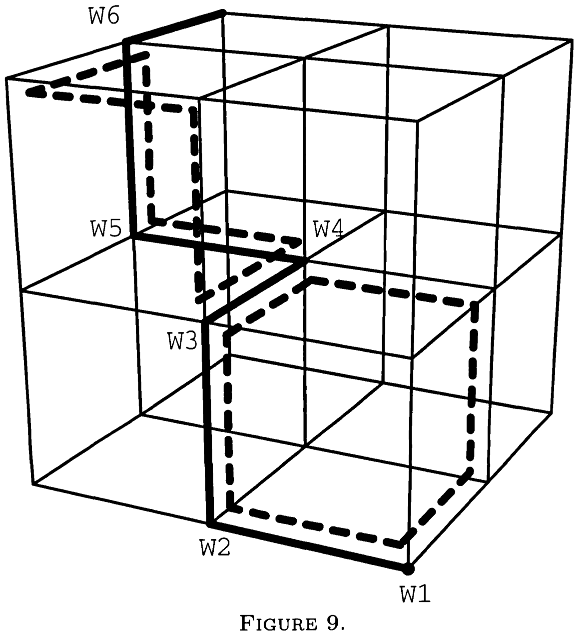

FIGURE 9.

3. Pseudo-Petrie operators on polyhedra

3.1. Definition of the pseudo-Petrie operator.

The preceeding table shows that the operator Π applied to a regular polyhedron P yields a polyhedron $\Pi(P)$ distinct from P, with one single exception, the last entry:

$\{6^{\frac{\pi}{2}}/1,4\}$ is *self-Petrie.*

Defining t to be the translation with vector $\overrightarrow{VW}$ and r to be the reflection in the horizontal plane through W (cf. Figure 6), we can write:

PROPERTY 3.1.1.

$$\Pi(\{6^{\frac{\pi}{2}}/1,4\}) = t(\{6^{\frac{\pi}{2}}/1,4\}),$$
$$\Pi(\{6^{\frac{\pi}{2}}/1,4\}) = r(\{6^{\frac{\pi}{2}}/1,4\}).$$

With the hope that a differently defined Petrie operator might yield a nontrivial action on $\{6^{\frac{\pi}{2}}/1,4\}$ and thus produce a new regular polyhedron, which could only be the 48th polyhedron discovered by Dress, we are going to generalize the definition of the Petrie operator.

To define the pseudo-Petrie $\Psi(P)$, we need to give a few restrictions on P. From now on, we only consider polyhedra P satisfying the following:

CONDITION 3.1.2.

(i) The *edge graph* is *bipartite*, and the vertices have been marked alternatively: call the marked vertices black vertices; call the unmarked vertices white.

(ii) The *edge graph of the dual* is *bipartite*, and thus all edges of the original graph can be marked coherently with arrows inducing an orientation on each of its faces: call this marking an orientation of the polyhedron.

DEFINITION 3.1.3. The *pseudo-Petrie* $\Psi(P)$ of a polyhedron P is defined as follows:

vertices: the vertices of P,

edges: the edges of P,

faces: faces are formed by a sequence of edges such that, following the arrows from a black vertex V of P, three successive edges are incident with a face of P, but no four successive edges are incident with the same face.

These conditions guarantee that an edge of a face is an edge of exactly another face, so we get:

PROPERTY 3.1.4. $\Psi(P)$ *is a polyhedron.*

The effect of Ψ on $\{4,4\}$, $\{4,6^{\frac{\pi}{3}}/1\}$, and $\{6^{\frac{\pi}{2}}/1,4\}$ is shown in Figures 7, 8, 9 respectively. (In these figures W_1 is a black vertex, and the orientation is given by $\overrightarrow{W_1W_2}$.)

3.2. Properties of Ψ.

Closer inspection of Figure 9 reveals that its faces are helical polygons $\{\infty^{\frac{\pi}{2},\frac{2\pi}{3}}\}$, four of each meeting at each vertex. We are going to prove it is the 48th Grünbaum polyhedron $\{\infty^{\frac{\pi}{2},\frac{2\pi}{3}},4\}$. To this effect we need a few lemmas.

Lemma 3.2.1.

a) *There exists a screw transformation s of a polyhedron P which transforms a pseudo-Petrie polygon denoted* $[\dots W_1,W_2,W_3,W_4,W_5,W_6,\dots]$ *into* $[\dots W_3,W_4,W_5,W_6,\dots]$ ($s\colon W_1\mapsto W_3$, $W_2\mapsto W_4$, $\dots$).

b) *s preserves the pseudo-Petrie polyhedra of P.*

Proof.

a) We can suppose without loss of generality that W_1 is a black vertex. Then W_1, W_2, W_3, W_4 are consecutive vertices of a polygon F of P. There exists a transformation ϕ of P which transforms W_1 to W_3, $[W_1,W_2]$ into $[W_3,W_4]$ and fixes F. Then $\phi(W_1)$, $\phi(W_2)$, $\phi(W_3)$, $\phi(W_4)$ are the consecutive vertices W_3, W_4, W_3', W_4' of F. There also exists a transformation ρ of P which fixes W_3, $[W_3,W_4]$ and transforms F into F', the other polygon of P incident with $[W_3,W_4]$.

Consider the transformation $s=\rho\circ\phi$.

It transforms W_1, W_2 to W_3, W_4; it transforms W_1, W_2, W_3, W_4 into four consecutive vertices of F', which must be the four consecutive vertices W_3, W_4, W_5, W_6 of the pseudo-Petrie polygon $[\dots W_1,W_2,W_3,W_4,W_5,W_6,W_7\dots]$.

Now s transforms W_1, W_2, W_3, W_4 (vertices of F) into W_3, W_4, W_5, W_6 (vertices of F'). Therefore it must transform W_3, W_4, W_5, W_6 into the vertices W_5, W_6, W_7, W_8 of a polygon F'' different from F', since otherwise, s^{-1} would transform F' into itself, which is impossible as $s^{-1}(F')=F$. So W_5, W_6, W_7, W_8 are part of the pseudo-Petrie polygon $[\dots W_1,W_2,W_3,W_4,W_5,W_6,\dots]$. Continuing in this way we get the desired result.

b) As ϕ and ρ preserve the marking of the vertices and the orientation, s transforms an arbitrary pseudo-Petrie polygon of P into a pseudo-Petrie polygon of P. □

Let $P_{b,\rightarrow}$ be the *polyhedron with marked black vertices and a fixed* (*right*) *orientation*; let $P_{b,\leftarrow}$ be the *polyhedron with marked black vertices and the opposite orientation.* A similar notation holds for white vertices.

If the orientation is clear from the context, we sometimes drop the arrow and write, e.g., P_b for the *polyhedron with marked black vertices.*

We immediately note the following:

LEMMA 3.2.2. $\Psi(P_{b,\rightarrow}) = \Psi(P_{w,\leftarrow})$.

Now we are able to describe the relationship between Π and Ψ.

LEMMA 3.2.3. $\Psi \circ \Pi(P_b) = \Psi(P_w)$.

Proof. Let W_1 be an arbitrary black vertex.

Consider the consecutive vertices W_1, W_2, W_3, W_4, W_5, W_6 of a face of $\Psi \circ \Pi(P_b)$.

We want to show that:

1) W_2, W_3, W_4, W_5 are four consecutive vertices of a face of P,
2) $[W_5, W_6]$ is not an edge of this face.

1) First note that three consecutive vertices of a face of $\Psi\circ\Pi(P_b)$ are necessarily consecutive vertices of a face of P. Therefore, W_2, W_3, W_4 are consecutive vertices of a face F of P. Also $[W_4, W_5]$ must be an edge of F. Otherwise, $[W_4, W_5]$ would have to be an edge of the only other polygon F' incident to $[W_3, W_4]$. But then $[W_4, W_5]$ would be an edge of the same face of $\Pi(P)$ as $[W_2, W_3]$ and $[W_3, W_4]$, and there would be five consecutive vertices W_1, W_2, W_3, W_4, W_5 of $\Pi(P)$ in $\Psi \circ \Pi(P)$.

2) As W_3, W_4, W_5, W_6 are four consecutive vertices of a face of $\Pi(P)$, $[W_5, W_6]$ cannot be adjacent to the same polygon F of which W_3, W_4, W_5 are vertices.

So every face of $\Psi\circ\Pi(P_b)$ is a face of $\Psi(P_w)$. Conversely, as a face of $\Psi(P_w)$ has obviously two edges in common with a face of $\Psi \circ \Pi(P_b)$, it has to coincide with a face of $\Psi \circ \Pi(P_b)$ as every face of $\Psi \circ \Pi(P_b)$ is a face of $\Psi(P_w)$. So each face of $\Psi(P_w)$ is a face of $\Psi \circ \Pi(P_b)$. □

LEMMA 3.2.4. $\Pi \circ \Psi(P_b) = \Psi(P_w)$.

Proof. Let W_1 be an arbitrary black vertex.

Consider the consecutive vertices W_1, W_2, W_3, W_4, W_5, W_6 of a polygon of $\Pi \circ \Psi(P_b)$, and proceed as before.

1) As W_1, W_2, W_3, W_4 are consecutive vertices of $\Pi \circ \Psi(P_b)$, W_2, W_3, W_4 are consecutive vertices of one face G of $\Psi(P_b)$ and therefore of one face F of P. Now, as W_3, W_4, W_5 are consecutive vertices of a different face G' of $\Psi(P_b)$, $[W_4, W_5]$ has to be an edge of F. In fact, otherwise, $[W_4, W_5]$ would be an edge of G by definition, as W_3 is a black vertex.

2) As W_3 is a black vertex, and W_3, W_4, W_5 are consecutive vertices of the polygon G' of $\Psi(P_b)$, $[W_5, W_6]$ cannot be incident to F, as otherwise, W_3, W_4,

W_5, W_6 would be part of one polygon of $\Psi(P_b)$, but three consecutive edges of a polygon do not belong to a Petrie polygon.

As in the proof of 3.2.3, each face of $\Psi(P_w)$ is a face of $\Pi \circ \Psi(P_b)$. $\square$

Combining Lemmas 3.2.3 and 3.2.4 we get:

COROLLARY 3.2.5. $\Pi \circ \Psi = \Psi \circ \Pi$.

As moreover $\Pi \circ \Pi = \mathrm{id}$, we can rewrite Corollary 3.2.5 as:

COROLLARY 3.2.6. $\Pi \circ \Psi \circ \Pi = \Psi$.

PROPOSITION 3.2.7. $\Psi(\{6^{\frac{\pi}{2}}/1, 4\})$ *is regular.*

Proof. We first prove that the faces of $\Psi(\{6^{\frac{\pi}{2}}/1, 4\})$ are regular and use this to show that there are three fundamental generators of the symmetry group.

a) By Property 3.1.1,

$$\Pi(\{6^{\frac{\pi}{2}}/1, 4\}) = t(\{6^{\frac{\pi}{2}}/1, 4\})$$

followed by a change of coloring. Therefore by Lemma 3.2.3,

$$\begin{aligned}\Psi(\{6^{\frac{\pi}{2}}/1, 4\}_b) &= \Psi \circ \Pi(\{6^{\frac{\pi}{2}}/1, 4\}_w)\\ &= \Psi \circ t(\{6^{\frac{\pi}{2}}/1, 4\}_b)\\ &= t \circ \Psi(\{6^{\frac{\pi}{2}}/1, 4\}_b)\end{aligned}$$

by commutativity of t and Ψ. Composing t with s^{-1} (where s is defined by Lemma 3.2.1) we obtain the screw transformation $\phi = s^{-1} \circ t$. Thus:

$$s^{-1} \circ \Psi(\{6^{\frac{\pi}{2}}/1, 4\}_b) = s^{-1} \circ t \circ \Psi(\{6^{\frac{\pi}{2}}/1, 4\}_b),$$

which, by 3.2.1, is equivalent to:

$$\Psi(\{6^{\frac{\pi}{2}}/1, 4\}) = \phi \circ \Psi(\{6^{\frac{\pi}{2}}/1, 4\}).$$

This proves in particular that the faces of $\Psi(\{6^{\frac{\pi}{2}}/1, 4\})$ are regular in view of Property 2.1.1.

b) By Property 3.1.1,

$$\Pi(\{6^{\frac{\pi}{2}}/1, 4\}) = r(\{6^{\frac{\pi}{2}}/1, 4\})$$

the colouring and orientation staying the same. Therefore, by Lemma 3.2.3,

$$\begin{aligned}\Psi(\{6^{\frac{\pi}{2}}/1, 4\}_b) &= \Psi \circ \Pi(\{6^{\frac{\pi}{2}}/1, 4\}_w)\\ &= \Psi \circ r(\{6^{\frac{\pi}{2}}/1, 4\}_w)\\ &= r \circ \Psi(\{6^{\frac{\pi}{2}}/1, 4\}_w)\end{aligned}$$

by commutativity. Composing r with σ (where σ is the transformation of P fixing W_4 and the face $[\dots W_3, W_4, W_5 \dots]$ of P) we get:

$$\begin{aligned}(\sigma \circ r) \circ \Psi\big(\{6^{\frac{\pi}{2}}/1, 4\}_{w,\leftarrow}\big) &= \sigma \circ (r \circ \Psi)\big(\{6^{\frac{\pi}{2}}/1, 4\}_{w,\leftarrow}\big) \\ &= \sigma \circ \Psi\big(\{6^{\frac{\pi}{2}}/1, 4\}_{b,\leftarrow}\big) \\ &= \Psi \circ \sigma\big(\{6^{\frac{\pi}{2}}/1, 4\}_{b,\leftarrow}\big) \\ &= \Psi\big(\{6^{\frac{\pi}{2}}/1, 4\}_{b,\rightarrow}\big) \\ &= \Psi\big(\{6^{\frac{\pi}{2}}/1, 4\}_{w,\leftarrow}\big)\,.\end{aligned}$$

Therefore, $\sigma \circ r$ transforms a pseudo-Petrie polygon of $\Psi\big(\{6^{\frac{\pi}{2}}/1, 4\}\big)$ into a pseudo-Petrie polygon. Also it fixes W_4 and exchanges $[W_3, W_4]$ and $[W_5, W_4]$. Therefore it transforms the regular polygon $[\dots W_1, W_2, W_3, W_4, W_5 \dots]$ into itself, as two distinct pseudo-polygons of $\Psi(P)$ have at most one edge in common. Thus $\sigma \circ r$ is a flag transformation denoted ρ which fixes a vertex and a face of $\Psi(P)$.

Together $\phi = s^{-1} \circ t$ and $\rho = \sigma \circ r$ generate the symmetries of $\Psi(P)$ which fix one of its faces.

Finally the transformation τ of P which fixes a vertex and an edge incident with it, preserves colouring and orientation, and therefore it is a flag transformation of $\Psi(P)$ with the same properties.

$\phi = s^{-1} \circ t$, $\rho = \sigma \circ r$ and τ generate the full symmetry group of $\Psi(P)$, and we can conclude that $\Psi(P)$ is regular. □

THEOREM 3.2.8. $\Psi\big(\{6^{\frac{\pi}{2}}/1, 4\}\big)$ *is* $\{\infty^{\frac{\pi}{2}, \frac{2\pi}{3}}, 4\}$ *and it is self-Petrie.*

P r o o f . The first statement results from the fact that $\Psi\big(\{6^{\frac{\pi}{2}}/1, 4\}\big)$ is regular and has the appropriate vertex figure $\{4\}$ and faces $\{\infty^{\frac{\pi}{2}, \frac{2\pi}{3}}\}$. (See also Figures 9 and 10.)

Also, by Corollary 3.2.5 and Property 3.1.1, we get

$$\begin{aligned}\Pi \circ \Psi\big(\{6^{\frac{\pi}{2}}/1, 4\}_b\big) &= \Psi \circ \Pi\big(\{6^{\frac{\pi}{2}}/1, 4\}_b\big) \\ &= \Psi \circ t\big(\{6^{\frac{\pi}{2}}/1, 4\}_w\big) \\ &= t \circ \Psi\big(\{6^{\frac{\pi}{2}}/1, 4\}_w\big)\,,\end{aligned}$$

implying the second part of the theorem. □

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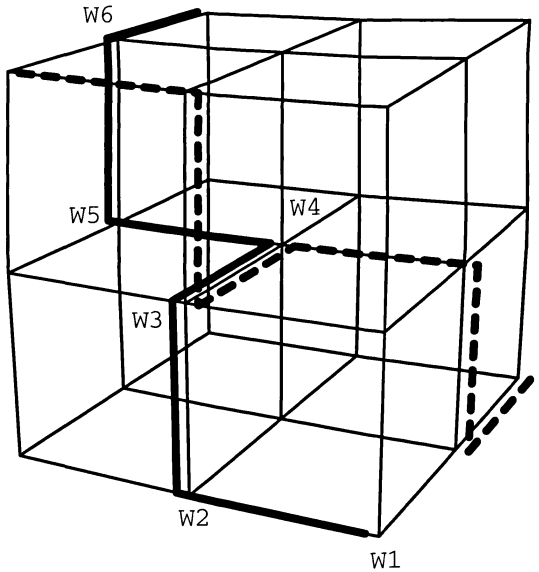

FIGURE 10.

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