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EDGE-TRANSITIVE MAPS AND NON-ORIENTABLE SURFACES

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ABSTRACT. This paper presents two classifications concerning group actions on surfaces. First, we classify maps whose symmetry groups are transitive on edges. Second, we classify all actions on a surface by a group whose order is larger than a certain minimum. Then we show the connections between these classifications: all but one of the larger-than minimal actions is the symmetry group of an edge-transitive map on the surface. Even the exceptional case (which is not large) can be interpreted and understood in terms of edge-transitive maps.

Definitions and notation

A *map* is an embedding of a graph into a surface which divides the surface into simply-connected regions. We will call these regions *faces* and consider the map as a kind of topological polyhedron. We subdivide the map by drawing a line from each face-center to each surrounding edge-center and vertex. This cuts the map into triangular regions called *flags*, as in Figure 1.

We can describe the map combinatorially by using the connections between flags. If f is any one flag, let fr_0, fr_1, fr_2 be the flags adjacent to f as in Figure 1. We think of r_i as a connection, the link between the flags f and fr_i; at the same time, we can think of each r_i as a permutation on the set of flags. The r_i's generate a group $C = C(M)$, the *connection* group of M. If Ω is the set of all flags, then C is a subgroup of S_Ω.

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Key words: symmetry group, non-orientable surface, map, edge-transitive map, face, flag.

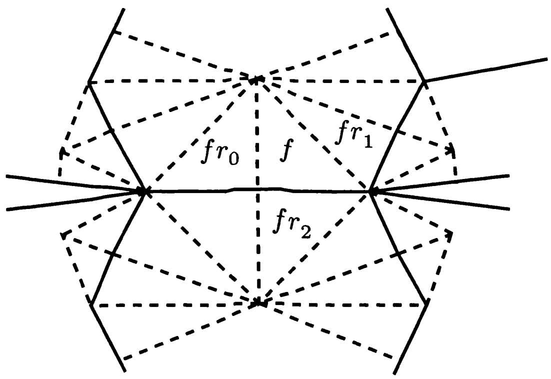


FIGURE 1. Subdividing the map into flags.

We then have two viewpoints on maps, one essentially topological, the other combinatorial. We can use these two viewpoints each to illuminate the other. For example, we can define a symmetry of a map M in two distinct but equivalent ways:

DEFINITION 1. A *symmetry* (or *automorphism*) of a map M is a homeomorphism of the surface onto itself which preserves the map structure.

DEFINITION 2. A *symmetry* (or *automorphism*) of a map M is a permutation of the flags which commutes with every element of C.

Let $G = G(M)$ be the group of all symmetries of M under composition. We wish to examine maps whose symmetry groups are relatively large. How symmetric shall we require a map to be in order for it to be of interest to us? One of the strongest requirements is that the map be *reflexible*; this means that G is transitive on flags. The group G, then, is generated by the set of reflections in the three sides of any one flag. Slightly weaker is the requirement that M be *rotary*; here G must contain symmetries which act as rotations about some face and adjacent vertex. The word *regular* has been applied to both of these cases in different settings. Here, we use words that mean what they say and reserve "regular" for use as a vague term in general discussions. A map which is rotary but not reflexible is called *chiral*. In this paper, we will also be interested in a still weaker form of regularity with an even more straightforward name: M is said to be *edge-transitive* provided that $G(M)$ is transitive on edges.

We mention that C acts on the orbits (under G) of flags. For example, suppose that flags f and g are joined by r_0, that f is in orbit A, g in orbit B. Let h be any other flag of orbit A. Then for some s in G, $fs = h$; then $gs = fr_0s = fsr_0 = hr_0$, and so r_0 joins h to gs which must be in orbit B.

So every flag in orbit A is joined by r_0 to some flag in B; in short, $Ar_0 = B$. Thus C acts on the orbits, as claimed.

Varieties of edge-transitive maps

Our first task is the classification of edge-transitive maps according to the number and arrangement of orbits of flags. Because we want to link this classification with actions of a group on a surface, and because a map and its dual induce the same action on a surface, we will consider a map M and its dual $D(M)$ to be essentially the same.

Because there are four flags at each edge, and because there is just one orbit on edges, there are three possibilities for the number of orbits of flags in M: one orbit, two orbits or 4 orbits.

Case I: one orbit on flags. These are precisely the reflexible maps, the most symmetric maps. They are at the center of the study of regularity. The fundamental regions for the action of the group on the surface are the flags themselves, and the group is generated by reflections in the sides of any one flag.

Case II: 2 orbits. Then each r_i must either interchange the two orbits or preserve them, and at least one of r_0, r_2 must interchange orbits in order for the group to be transitive on edges. We can begin classifying the possibilities by asking which of the r_i's switch the two orbits and which send each orbit to itself. This dichotomy is a partition of $\{0, 1, 2\}$. There are eight such partitions, and duals correspond to switching 0 and 2. If we let $\bar{i}$ indicate that r_i interchanges the orbits, while an unmarked i indicates that r_i preserves each orbit, then we can list these with duals paired thus:

1. $\bar{0}\bar{1}\bar{2}$,
2. $\bar{0}\bar{1}2 = D(0\bar{1}\bar{2})$,
3. $\bar{0}12 = D(01\bar{2})$,
4. $\bar{0}1\bar{2}$,
5. $0\bar{1}2$,
6. 012.

For example, item 3 corresponds to maps in which r_0 joins flags of different orbits, but r_1 joins flags in the same orbit, as does r_2. This possibility is shown in Figure 3c below.

Of these, items 5 and 6 are impossible by our comments on transitivity above. The remaining four are all possible, and we display them in Figure 3 below. In this diagram, we have shown the flags of one orbit as black, those of the other orbit as white. The diagrams show how the orbits are arranged in the map in each case.

We have arranged these in pairs, matching a map type with its "opposite". If M is a map, the map opp(M), the *opposite* of M ([W]), is formed from M by cutting the map apart along the edges and then re-attaching the two sides of each edge with the opposite orientation, as in Figure 2:

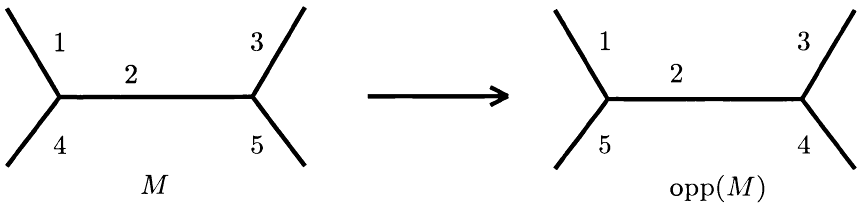


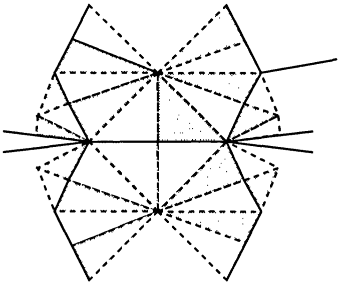
FIGURE 2. The neighbourhood of an edge in a map and its opposite.

The symmetry groups of a map and its opposite are identical as permutations of flags, but the different arrangement of flags gives the same symmetry different surface actions in the two maps.

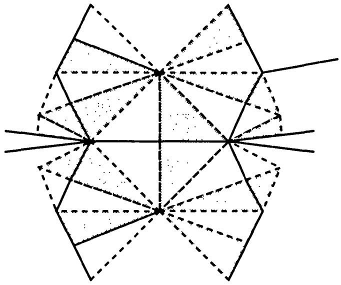
The first of the four possibilities above correspond to the situation in a chiral map, where only orientation-preserving symmetries exist. Its group is generated by one-step rotations about a face and a vertex. The second is the opposite of the first and is of special interest because of its underlying graph. The underlying graph of such a map is "semi-transitive", which means that it has a group of symmetries (in this case the group of the map) which acts transitively on edges and on vertices but not on the "darts" (half-edges) of the graph.

The third and fourth possibilities relate to hypermaps. "Hypermap" is a generalization of "map". One way to view the generalization is combinatorially: in a map, $(r_0r_2)^2$ must be the identity. If we remove this restriction, the resulting structure is a hypermap. Or consider an edge of a map, together with its midpoint, as a 2-star. If we now allow edges to be 3-stars or 4-stars, etc., we are thinking about hypermaps. This second notion, this drawing of a hypermap on a surface, can be viewed as a map in its own right, a map with two kinds of vertices, one called "vertices", one called "edges". This map is the *Walsh map* of the hypermap. See [B], [CS].

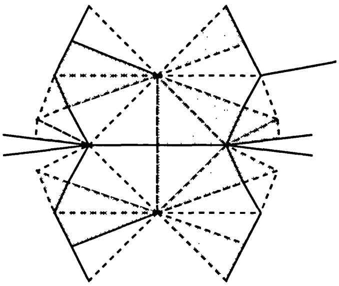
The third diagram, Figure 3c, then, has two kinds of vertices, one surrounded by flags of the white orbit, one by those of the black. Thus it is the Walsh map of some hypermap. The hypermap is reflexible because its group is generated by three reflections: one about the edge in the center of the diagram, and one each about the hypotenuses of the flags just above it. Figure 3d is the opposite of this situation; its group is generated by two reflections (as in 3c) and a 180° rotation (a "tweak") about the midpoint of an edge.



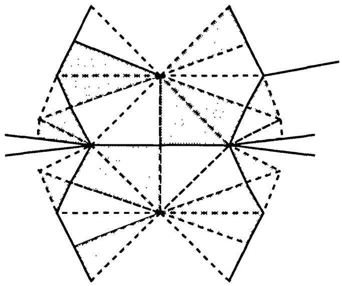
a: $\bar{0}\bar{1}\bar{2}$ = Chiral (Orientable only)



b: $\bar{0}\bar{1}2 = D(0\bar{1}\bar{2})$ = opp(chiral)

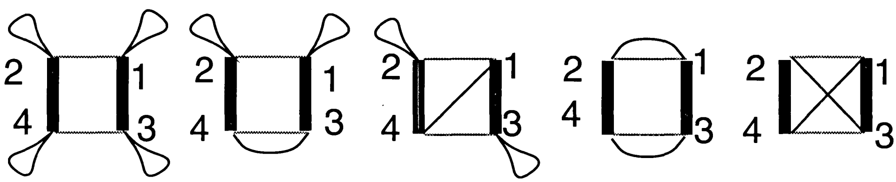


c: $\bar{0}12 = D(01\bar{2})$
= Walsh map
of a reflexible hypermap



d: $\bar{0}1\bar{2} = \mathrm{opp}(3c)$

FIGURE 3. Possibilities for an edge-transitive map in which each edge meets two orbits of flags.



a b c d e

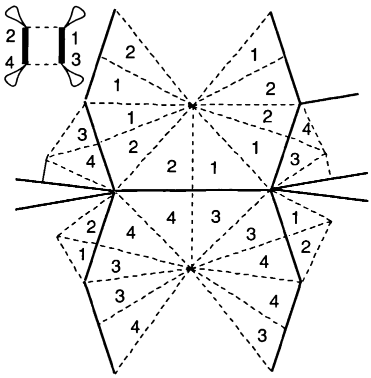
FIGURE 4. The five diagrams.

Case III: 4 orbits. Label the orbits 1, 2, 3, 4; then r_0 and r_2 must exchange these in pairs. Assume the numbering is such that r_0 switches $(1,2)(3,4)$ and r_2 switches $(1,3)(2,4)$. We can diagram the action by using a dotted line for r_0, a single line for r_1 and a double-width line for r_2. Again considering a map and its dual as essentially the same, we have 5 possibilities for the diagram, shown in Figure 4.

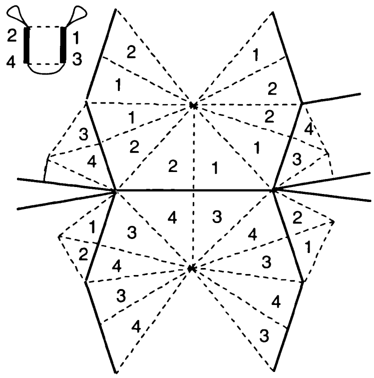
For instance, in the third of these five, flags of orbits 1 and 4 meet along their hypotenuses (that's the r_1 connection), while flags of orbit 2 meet flags of orbit 2, 3 meets 3.

In order to visualize the action of the group on the surface, we show the relative locations of the four orbits in each possibility. This is Figure 5, below.

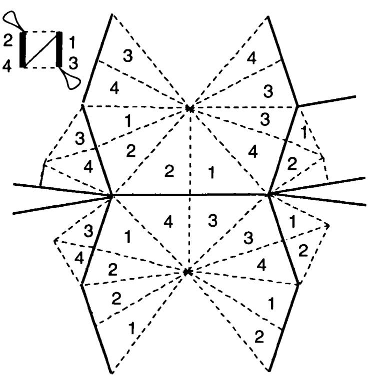
We examine these five actions carefully. Look at the central edge e in each diagram in Figure 5. It meets two faces, A above and B below, as well as two vertices, u on the left, v on the right. In each case, the four flags around e form a fundamental region for the action. The first group is generated by the four reflections about the sides of this fundamental region. The second is generated by the upper two reflections, together with a rotation one step about face B.



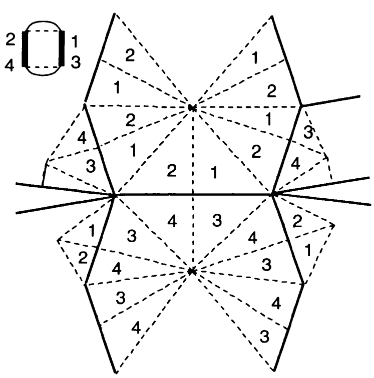
a: (4 reflections)



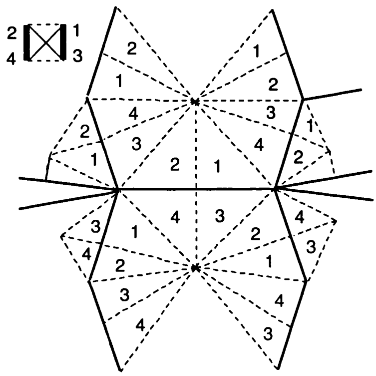
b: (2 reflections, 1 rotation)



c: (2 reflections, one glide)



d: (2 rotations) orientable only



e: (2 glides or one glide and one rotation)

FIGURE 5. Possibilities for an edge-transitive map in which each edge meets four distinct orbits of flags.

The third is generated by two reflections in the axes A-u and B-v, and by an action like a glide reflection with an axis oriented southwest-to-northeast through the midpoint of e; it sends B-u to A-v. The fourth possibility is generated by rotations of one step about each of A and B together with a 2-step rotation around u (or v). This corresponds to the dual of the Walsh map of a chiral hypermap and can occur on orientable surfaces only. The last case has a group generated by the two glide reflections on axes through the midpoint of e, or by one of those glides and a 2-step rotation about one face. In every case, maps do exist whose complete symmetry group has the given action.

Non-orientable surface groups

Let S be a topological surface. If M is a map on S having F faces, E edges and V vertices, then the number $F - E + V$ is $\chi(M)$, the Euler characteristic of M. This number is constant for all maps M on S, and so we can set $\chi(S) = \chi(M)$. Since this number is usually negative, we will refer to the negative characteristic $N = N(S) = -\chi(S)$ in this paper. If S is orientable of genus g, then $N = 2g - 2$; if S is non-orientable of genus g, then $N = g - 2$.

We focus on N because in many results dealing with orders of groups acting on S, it is N which is often most directly involved. For one example, consider a non-orientable surface S and its orientable smooth two-fold cover, which we can call $2S$. Then we have the very pretty formula $N(2S) = 2N(S)$.

As another example, we state the famous Riemann-Hurwitz formula. This theorem applies to a finite group G acting on an orientable surface S, preserving orientation. Let $S_0 = S/G$; i.e., S_0 is the surface resulting from S by identifying two points if they lie in the same orbit under G. Let φ be this identification, the natural projection of S onto S_0. For almost every point x on S, there is some neighbourhood of x such that the restriction of φ is one-to-one in that neighbourhood. If x is not such a point, then in some neighbourhood of x, φ is one-to-one at x but m-to-one in the rest of the neighbourhood. This happens if some element of G of order m fixes x. We say then that $y = \varphi(x)$ is a *branch point of multiplicity* m.

THEOREM. (**RIEMANN-HURWITZ FORMULA**) *Suppose that* G *is a finite group of orientation-preserving symmetries acting on an orientable surface* S, *let* $S_0 = S/G$, *and suppose the action has branch points* $y_1, y_2, \ldots, y_r$ *of multiplicities* $m_1, m_2, \ldots, m_r$. *Let* $N = N(S)$, $N_0 = N(S_0)$. *Then*

$$|G| = \frac{N}{N_0 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)} .$$

In [S], Singerman shows that under these hypotheses, if $|G| > 6N$, then the group is the group of some rotary hypermap, and that if $|G| > 12N$, the hypermap is a map. Our object in this paper is to produce similar results about non-orientable surfaces.

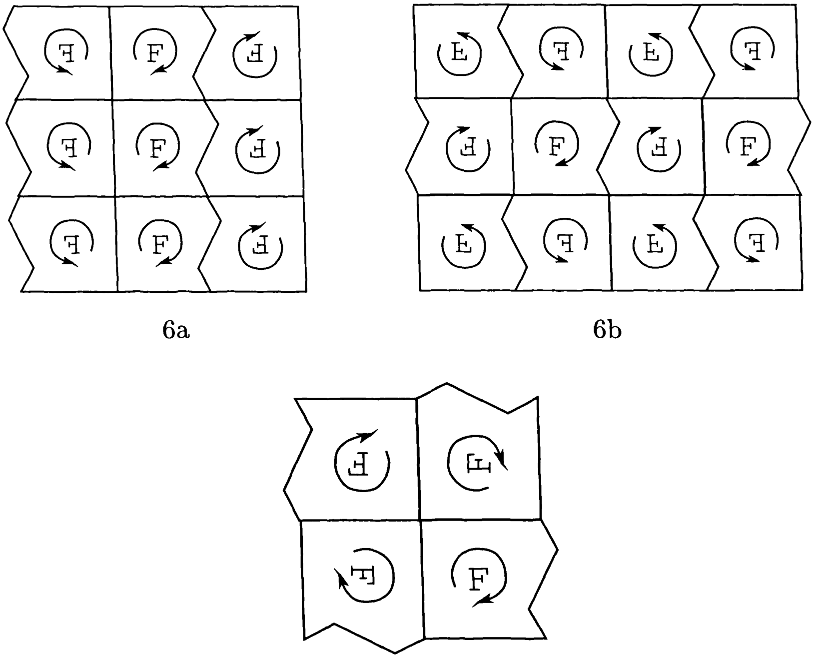
In order to apply these results to groups on non-orientable surfaces, we need to establish the relationship between a group G acting on a non-orientable surface S and related groups acting on $2S$, its smooth orientable twofold covering.

A *fundamental region* for G is an open, simply connected subset F of S such that:

(1) if $F \cap Fg$ is non-empty for some g in G, then g is trivial,
(2) the union of the images under G of the closure of F is all of S.

Such an F always exists, and we can assume without any loss of generality that its boundary is as "nice" as we wish. The copies of the closure of F, then, tessellate S, covering all of S and meeting only along their boundaries. This tessellation may or may not be unique, depending on the group G, as we shall see below.

We decorate F with an ornament indicating orientation, and we let the elements of G carry this ornament onto all of the other copies of F. Some examples of fundamental regions are shown in Figure 6:



6a 6b

6c

FIGURE 6. Neighboring fundamental regions of a group action on a surface.

Figure 6 shows some of the varieties of ways that fundamental regions can border one another. We label these types of attachments according to the kind of group element which carries the region to its neighbour. Thus regions which border each other as the vertically adjacent pairs in Figure 6a are said to be adjacent "by translation", and we will call the top and bottom edges of F "translation edges". Similarly, the third straight edge of F in Figure 6a is a "reflection edge", and the top and bottom edges of F in Figure 6b are "glide edges" because the vertical motion is, locally, a glide reflection. "Rotation edges" (such as the jagged fourth edge of F in all three diagrams and the top and left sides of F in Figure 6c) will be lumped together in the same category with translation edges.

Another way to look at this distinction is to orient an edge of F with an arrow pointing along the edge, and carry that arrow to all copies of F by means of the group. Then there are three possibilities:

(1) Each region will meet only one arrow. Then the edge is a reflection edge, and the symmetry which takes the region to the corresponding neighbour is (locally) a reflection.

(2) Each region will meet two copies of the arrow, pointing in the same direction (i.e., locally both clockwise or both counterclockwise). Then the edge is a glide edge and the symmetry which takes the region to the corresponding neighbour is (locally) a glide reflection.

(3) Each region will meet two copies of the arrow, pointing in opposite directions (that is, locally one clockwise and one counterclockwise). Then the edge is a translation edge, and the symmetry which takes the region to the corresponding neighbour is (locally) a translation (or a rotation).

In cases (1) and (2), adjacent regions will bear ornaments for opposite orientations, while in case (3), the orientations will be the same.

Now let $2S$ be the smooth two-fold orientable covering of the non-orientable surface S, and let φ be the natural projection of $2S$ onto S. If F is a fundamental region of G acting on S, then $\varphi^{-1}(F)$ consists of two copies of F, one in which the ornament points clockwise, and one in which it points counterclockwise. The function β which switches each such pair, i.e., which sends a point x of $2S$ to the other point in $\varphi^{-1}(\varphi(x))$ is a symmetry of $2S$. It reverses orientation.

On the other hand, if σ is any element of G, then consider σ acting on $2S$ via the function which, for each fundamental region F, sends each point in a pre-image of F to the corresponding point in the pre-image of $F\sigma$ which has the same orientation. This is an orientation-preserving symmetry of $2S$, and so the group G acts on $2S$ as orientation-preserving symmetries. Note that β commutes with everything in G, and so the group G' generated by G and β acts on $2S$ and is isomorphic to $G \oplus \mathbb{Z}_2$. Any pre-image of F is a fundamental region for G'. A fundamental region for G acting on $2S$ consists of a pre-image of two adjacent copies of F in S which are joined by reflection or glide edges; in other words, two adjacent copies of F having opposite orientation.

Our object now is to suppose that a group of larger-than-minimal order acts on S and to classify the possibilities for that action by appealing to the above relationships between G acting on S, G acting on $2S$ and G' acting on $2S$.

Non-orientable groups of not-too-small order

We intend to suppose for the rest of the paper that G is a group of order strictly greater than $4N$ acting on a non-orientable surface S of negative characteristic $N(S) = N$. This is not too stringent a requirement, and if we are looking for the largest group acting on S, it turns out to be no restriction at all; Conder and MacLachlan, using the latter's similar results about orientable surfaces from [M], have shown that if N is odd, S admits groups of orders $4N$, $4(N+1)$ and $4(N+2)$, and if N is even, S admits groups of order

$8N$ ([CMW]). Thus, if we are looking for the largest group acting on S, its size must be at least $4N+8$ and could be as large as $84N$.

So, for all that follows, suppose that S is non-orientable, $N(S)=N>0$, G is a finite group of automorphisms of S with $|G|>4N$. Then G acts on $2S$ in the manner specified above. Let φ be the natural projection of $2S$ onto $2S/G=S_0$.

Lemma 1. *Under these hypotheses, S_0 is the sphere, and the number of branch points is 3 or 4.*

P r o o f . From the hypothesis, the Riemann-Hurwitz theorem applies to the action of G on $2S$, and so $4N<|G|=\dfrac{N(2S)}{N_0+\sum_{i=1}^{r}(1-\frac{1}{m_i})}=\dfrac{2N}{N_0+\sum_{i=1}^{r}(1-\frac{1}{m_i})}$.

It follows that the denominator $N_0+\sum_{i=1}^{r}(1-\frac{1}{m_i})$ must be a positive number strictly less than $\frac{1}{2}$. Since N_0 must be even and no less than -2, we see immediately that N_0 must be either -2 or 0. Can N_0 be 0? If that were the case, then $\sum_{i=1}^{r}(1-\frac{1}{m_i})$ would be strictly less than $\frac{1}{2}$, and since each summand is at least $\frac{1}{2}$, the sum must be empty, making the entire denominator 0, which is forbidden. Thus N_0 is -2 and S_0 is the sphere.

Because $N_0=-2$, then $2<\sum_{i=1}^{r}(1-\frac{1}{m_i})<\frac{5}{2}$. Again, each summand is between $\frac{1}{2}$ and 1, and so there must be at least 3 summands but no more than 4, as required. □

Corollary A. *If $r=3$, then $\frac{1}{2}<\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}<1$, and if $r=4$, then $\frac{3}{2}<\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_4}<2$. In the latter case, assuming $m_1\le m_2\le m_3\le m_4$, we have $m_1=m_2=2$, and either $m_3=2$ or $m_3=3$ and $m_4=3,4$ or 5.*

Corollary B. *Let M stand for $\sum_{i=1}^{r}\frac{1}{m_i}$, and $m=\{m_1,m_2,\dots\}$. Then, if $r=3$, then $|G|=\frac{2N}{1-M}$; this is largest when $m=2,3,7$, and so $|G|=84N$. If $r=4$, then $|G|=\frac{2N}{2-M}$; this is largest when $m=\{2,2,2,3\}$ and $|G|=12N$.*

Fundamental regions

We wish to classify these larger-than-minimal groups, and our classification will be geometric. That is, we will determine the possibilities for the fundamental regions for such actions and the way in which the regions fit together on the surface. This geometric classification will translate to a classification on the algebraic structure of the group.

In what follows, suppose that F is a fundamental region for an action of G on S, where $|G| > 4N(S)$.

LEMMA 2. *F cannot contain both glide and reflection edges.*

Proof. Suppose that F does contain a reflection edge a and a glide edge b. Then F is as in Figure 7, in which the dotted lines represent edges, if any, other than the ones mentioned.

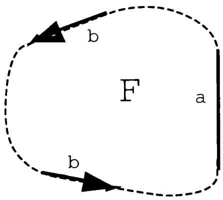


FIGURE 7. A fundamental region in S.

Then a fundamental region for G acting on $2S$ consists of two of these which are adjacent and oppositely oriented, as in Figure 8, where upper and lower case indicates which copy of the glide edge b is to be identified with which.

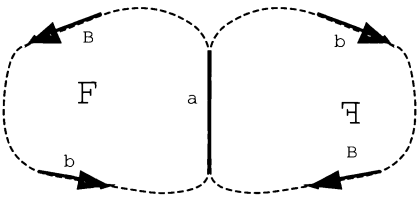


FIGURE 8. A fundamental region in $2S$.

The projection φ of $2S$ onto S_0, the sphere, may be regarded simply as identification of corresponding edges in this figure. But this cannot be the sphere because it has non-separating cycles such as the one shown in Figure 9:

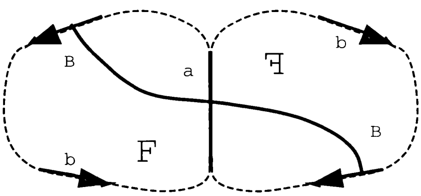


FIGURE 9. A non-separating cycle in S_0. □

This contradiction shows that F cannot have both reflections and glides. The following lemmas about the arrangement of edges on the boundary of F are proved by similar arguments:

Lemma 3. *A translation cannot separate two reflections.*

Note. If b is a translation edge, its two copies in F separate the boundary of F into two connected components. The lemma, then, says that it is impossible for both of these components to contain reflection edges.

Lemma 4. *A translation cannot separate the two copies of a glide.*

Lemma 5. *A translation cannot separate the two copies of another translation.*

Lemma 6. *Any two glides must separate each other.*

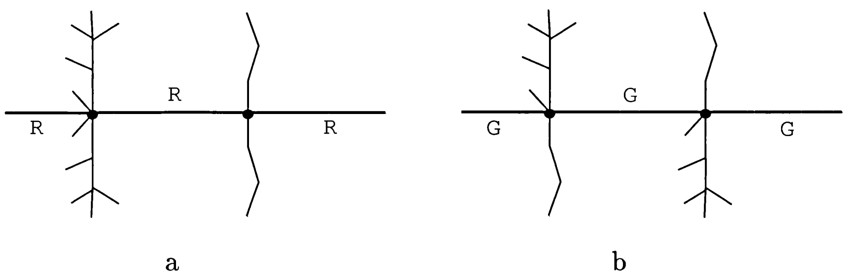


Figure 10. Part of the equator in S_0.

Lemmas 3 – 5 imply that each segment of the boundary of F which is composed entirely of translations consists of matching pairs (the two copies of each translation edge), and that these pairs are nested. Thus, when these identifications are made, the segment becomes a tree of edges; when all translation identifications are made, F becomes a disk whose outer edges are all the reflections or all the glides, together with a tree of translations at some vertices. When the two copies of F are attached, each tree appears once in each copy. See Figure 10.

In this diagram, "R" or "G" stands for a reflection or a glide edge, and the other edges represent the identified translation edges. For instance, if the translation edges in one segment are nested in the order $a\,a\,b\,c\,c\,d\,e\,e\,f\,f\,g\,g\,d\,b$, the resulting tree would be the one at the left in Figure 11a.

Since F must have some orientation-reversing edges, there must be reflections or glides; by Lemma 2, there cannot be both.

First, let us suppose that F has a glide edge, such as a in Figure 11a. The dotted lines x and y contain other edges which might be glides or translations.

Notice that by Lemma 4, if one copy of a translation edge is in x, so is the other copy; similarly, Lemma 6 says that if one copy of a glide edge is in x, the other half must be in y and vice versa. Now draw a line across F from the tip of one a to the tip of the other.

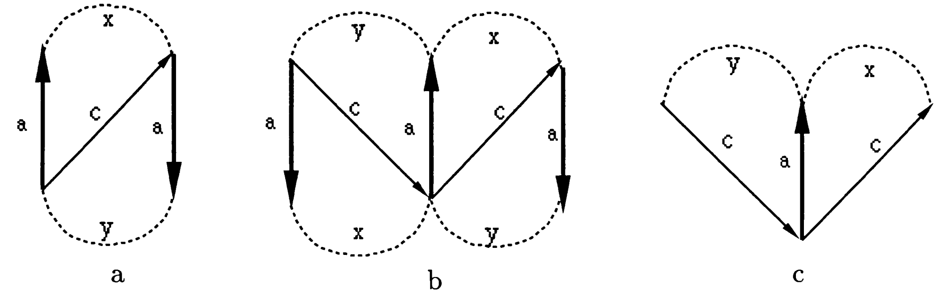


FIGURE 11. Surgery on a fundamental region containing a glide.

Looking at two copies of F adjacent along a, we see that the top and bottom halves of F, considered as connected along a instead of c, form a different fundamental region F' for the group. F' has the property that it can have no other glide edges, for every glide edge of F has had one of its two parts reversed and so is now a translation edge, while every translation edge has had both or neither half reversed, and so is now still a translation. Thus we can assume that if F has a glide edge, then it has only one, and that all the rest of its edges are translations. A similar surgery will assure us if we wish that the glide edge has endpoints which are branch points.

Now assume that F has reflections but no glides. If there are any branch points which lie on reflection axes, a surgery similar to the one above will assure us that every edge of F, reflection or translation, has endpoints which are branch points. It may happen, however, that no branch points lie on any of the reflection axes.

Classification

Suppose that F has k translations (i.e., $2k$ matching pairs of translation edges). If there are t reflections, then the projection φ has $t+2k$ or $2k$ branch points depending on whether or not there are some branch points lying on the reflection axes. If there is a glide, then the number of branch points is $2+2k$ (here we assume the glide is arranged to join branch points). Because this number is r, which must be 3 or 4, there are only six possibilities for the boundary of F:

(1) four reflections,
(2) three reflections,

(3) two reflections and one translation,
(4) one reflection joining branch points and one translation,
(5) one reflection missing branch points and two translations,
(6) one glide and one translation.

We examine each of the six possibilities in turn. In each case, we examine the fundamental region F for G acting on S and a pair of them joined up to form a fundamental region for G acting on $2S$. From this, we can sometimes deduce restrictions on the branching orders. We also consider how to express these actions as symmetry groups of edge-transitive maps.

(1) Four reflections.

Here, $r = 4$ and there are no restrictions on m other than the bounds given in Corollary a. F is a quadrilateral, and if we draw a line from corner to opposite corner, copies of this edge (call it the central edge) will divide S into a map; the vertices will be the images of the endpoints of the central edge, and the face-centers will be the copies of the other two corners of F. This corresponds exactly to the arrangement of Figures 4a & 5a.

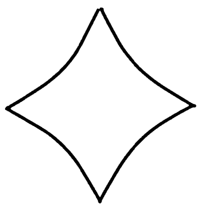


FIGURE 12. Four reflections.

(2) Three reflections.

This case is the best-studied of all our possibilities because it includes the highest possible orders for G. Here, $r = 3$, and the resulting structure is either a reflexible map (our un-diagrammed Case I of edge-transitive maps) in the case where $m_1 = 2$ or a reflexible hypermap if $m_1 > 2$. This second possibility is Figure 3c. The largest order for G occurs when $m = \{2, 3, 7\}$, and then $|G| = 84N$. If the structure is a hypermap but not a map, then $|G|$ has its greatest value of $24N$ at $m = \{3, 3, 4\}$.

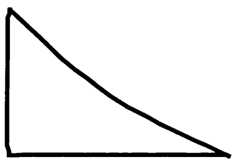


FIGURE 13. Three reflections.

(3) Two reflections and one translation.

There are four branch points (a, b, c, d in Figure 14) in the projection of $2S$ onto S_0. But since a and d are equivalent under G', they must have the same order. By Corollary a, this must be 2 or 3. If that common order is 2, $|G|$ is at most $12N$, and if that common order is 3, then $m = \{2,2,3,3\}$, $|G| = 6N$. To make the corresponding map, draw a line across each fundamental region joining the corners where a translation edge meets a reflection (b-b in Figure 14). Then copies of the center of rotation (a) form one class of face-centers, and copies of the opposite corner form another. There is only one class of vertex. This is the structure of Figures 4b & 5b.

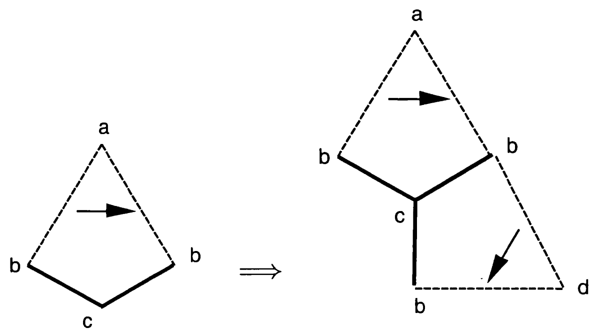


FIGURE 14. Two reflections and one rotation.

(4) One reflection, one translation.

Make the axis of reflection the central edge. The copies of this edge divide S into a map, and the flags are the left and right halves of the triangle in Figure 15 below. This is exactly the structure of the opposite of a chiral map, as in Figure 3b.

The projection of $2S$ onto S_0 has three branch points. One corresponds to vertices; the other two correspond to face-centers and so must have equal order. From Corollary a, this joint order must be 3 or more, and so the group has its largest order, $24N$, at $m = \{3,3,4\}$.

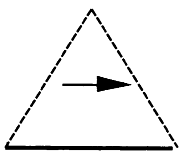


FIGURE 15. One reflection, one rotation.

(5) One reflection, two translations.

Here $r = 4$, and m must be $\{2,2,3,3\}$, $|G| = 6N$. Figure 16 shows one possibility for the shape of the fundamental region and the way copies are arranged on S:

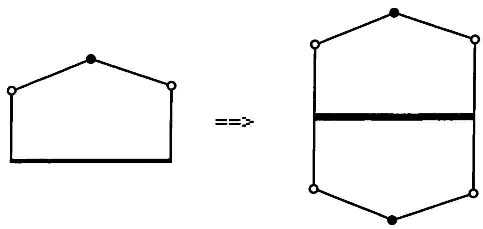


FIGURE 16. One reflection, two rotations.

While the resulting surface action has no corresponding map or hypermap, we note that if the surface is separated along the reflection axes, each component may be described as a rotary map M of type $\{3,k\}$ with a disk removed from each face. The original surface then can be described as several copies of M adjoined along the boundaries of these removed disks. The smallest N for which such an action is known to exist is $N = 4$. This action is a subgroup of index 2 of another action, the larger of type (1) in this classification. The smallest surface for which this author knows an action of type (5) which does not extend to a larger action is that with $N = 84$. This is formed from 4 copies of the map $M = \{6,3\}_{4,1}$ attached in an appropriate way as described above.

(6) One glide and one translation.

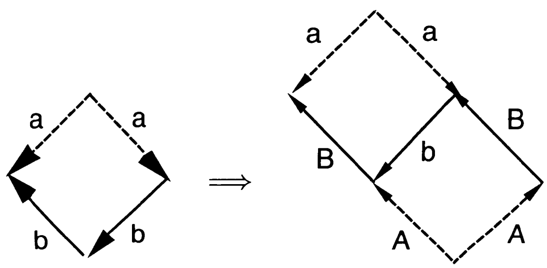
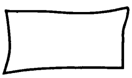
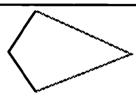
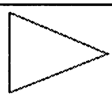
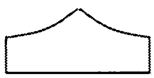
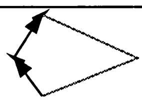


FIGURE 17. One glide and one rotation.

The fundamental region for G acting on S and $2S$ is as shown. This gives $r = 4$, and the four branch points come in two pairs, the points in each pair having equal order. Then $m = \{2,2,3,3\}$, and $|G| = 6N$. To make a map, draw an edge along each of the glide edges. This forms a map of type $\{4,6\}$ or $\{6,4\}$. This map is of the kind shown in Figures 4e & 5e. This group can be generated by two glides through the central edge, or by one of the glides plus rotation two steps about the top face. A fundamental region for this action is a set of flags labelled 1, 2, 3, 4 in that order clockwise in a face in Figure 5e.

Summary.

In the table below, we summarize the classification of group actions and their relations to edge-transitive maps. In this table, "type" abbreviates the kinds of edges around a fundamental region. Here "R" stands for a reflection edge, "T" stands for a matched pair of translation edges, and "G" stands for a matched pair of glide edges. Under "Shape", a dotted line stands for a translation (=rotation) edge, a solid line for a reflection, an arrowed line for a glide. Under m, "any" means any possibility not ruled out by Corollary a.

	type	Shape of F	r	m	|G|	E-T map type
1	R R R R		4	any	≤ 12N	III a
2	R R R		3	any	≤ 84N	I & II c
3	R R T		4	any	≤ 12N	III b
4	R T		3	a,a,b a ≥ 3	≤ 24N	II b
5	R T T		4	2,2,3,3	= 6N	none
6	G T		4	2,2,3,3	= 6N	III e

In short, if one is looking for large groups of automorphisms on a non-orientable surface, he should first examine rotary maps and hypermaps. These give, either directly or through Walsh maps or opposites, cases (2) and (4) above; these are the cases whose groups have the largest possible orders. A corollary of the above classification is that any group of order larger than $12N$ must be the group of such a map. However, the largest order might not be in this category. Antonio Breda d'Azevedo and the present author have shown that for the non-orientable surface with $N = 16$, no map in these categories exists [BW]; Marston Conder has shown that the largest group on this surface is of variety (1) above and has order $192 = 12N$.

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