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Dedicated to the memory
of Professor Milan Kolibiar

ISOMETRIES IN NON-ABELIAN MULTILATTICE GROUPS

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(*Communicated by Tibor Katriňák*)

ABSTRACT. In this paper, it is proved that for every stable isometry in a distributive multilattice group G there exists a direct decomposition $G = A \times B$ of G with B abelian such that $f(x) = x(A) - x(B)$ for each $x \in G$. Further, the actions of stable isometries on convex subsets are studied.

Swamy [15] introduced the concept of an isometry in an abelian lattice ordered group C as a bijection $f\colon C \to C$ such that

$$|x - y| = |f(x) - f(y)| \qquad \text{for each} \quad x, y \in C\,. \tag{1}$$

Jakubík [4], [5] has applied this definition also for non-abelian lattice ordered groups and proved the following assertion:

(A) *Let f be a stable isometry in a lattice ordered group C. Then there exists a direct decomposition $C = A \times B$ of C such that $f(x) = x(A) - x(B)$ for each $x \in C$.*

In [2], Holland gave a different proof of the assertion (A) and moreover, he showed that B is an abelian group.

Jakubík and Kolibiar [7] put $|x| = \{2t - x\,,\ t \in x \vee_m 0\}$ for any element x of a multilattice group C and defined an isometry in a multilattice group C to be a bijection $f\colon C \to C$ which satisfies the condition (1). They obtained an analogous result to assertion (A) for abelian distributive multilattice groups.

In [14], Rachůnek generalized the notion of the isometry for any partially ordered group and studied the isometries in a certain class of Riesz groups. He

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defined an isometry in a partially ordered group C as a bijection $f\colon C \to C$ satisfying (1) making use of the relation $|z| = U(z, -z)$ for any $z \in C$.

Weak isometries in lattice ordered groups were introduced by J a k u b í k [6]. He defined a weak isometry f in an lattice ordered group C to be a mapping f which satisfies the condition (1).

Isometries and weak isometries in some types of partially ordered groups have been investigated by the author in [8]–[13]. In [11], it was proved that every weak isometry in a directed group is a bijection. Hence the notions of weak isometry and isometry are equivalent in multilattice groups.

In this paper, R a c h ů n e k's definition of the isometry is used in the investigation of isometries in multilattice groups.

First we recall some notions and notations used in the paper.

Let C be a partially ordered group (po-group). The group operation will be written additively. We denote $C^+ = \{x \in C;\ x \geq 0\}$. If $A \subseteq C$, then we denote by $U(A)$ and $L(A)$ the set of all upper bounds and the set of all lower bounds of the set A in C, respectively. For $A = \{a, b\}$ we shall write $U(a, b)$ ($L(a, b)$) instead of $U(\{a, b\})$ ($L(\{a, b\})$). For each a $\in C$, $|a| = U(a, -a)$. If a and b are elements of C, then we denote by a $\vee_m b$ the set of all minimal elements of the set $U(a, b)$, and analogously, $a \wedge_m b$ is defined to be the set of all maximal elements of the set $L(a, b)$. If for $a, b \in C$ there exists the least upper bound (greatest lower bound) of the set $\{a, b\}$ in C, then it will be denoted by $a \vee b$ ($a \wedge b$). If $C = P \times Q$ is a direct decomposition of C, then for $x \in C$ we denote by $x(P)$ and $x(Q)$ the components of x in the direct factors P and Q, respectively. An isometry f in C is called a stable isometry if $f(0) = 0$.

The partially ordered set P is said to be a multilattice (B e n a d o [1]) if it fulfils the following conditions for each pair $a, b \in P$:

(m_1) If $x \in U(a, b)$, then there is $x_1 \in a \vee_m b$ such that $x_1 \leq x$.
(m_2) If $y \in L(a, b)$, then there is $y_1 \in a \wedge_m b$ such that $y_1 \geq y$.

A multilattice P is called distributive if, whenever a, b, c are elements of P such that $(a \wedge_m b) \cap (a \wedge_m c) \neq \emptyset$ and $(a \vee_m b) \cap (a \vee_m c) \neq \emptyset$, then $b = c$.

Let G be a partially ordered group such that

(i) G is directed,
(ii) the partially ordered set $(G, \leq)$ is a multilattice.

Then G is called a multilattice group. (See [1].)

A quadruple (a, b, u, v) of elements of a multilattice group G is said to be regular if $u \in a \wedge_m b$, $v \in a \vee_m b$ and $v - a = b - u$.

Throughout the paper, we assume that H is a multilattice group.

1. Lemma. *Let $a, b \in H$.*

(i) *If $v \in a \vee_m b$, $u = a - v + b$, then (a, b, u, v) is a regular quadruple in H.*
(ii) *If $u \in a \wedge_m b$, $v = b - u + a$, then (a, b, u, v) is a regular quadruple in H.*

Proof.

(i) It suffices to verify that $u \in a \wedge_m b$. From the relation $0 \leq v - a = b - u$ we obtain $b \geq u$. Since $0 \leq -b + v = -u + a$, we get $u \leq a$. Thus $u \in L(a, b)$. Then there exists $u_1 \in a \wedge_m b$ such that $u_1 \geq u$. Let $v_1 = b - u_1 + a$. Clearly, $v_1 \in U(a, b)$. Since $u_1 \geq u$, we have $v - a = b - u \geq b - u_1$. From this we get $v \geq v_1$. Because of $v \in a \vee_m b$, we obtain $v = v_1$. Therefore $u_1 = u$.

Assertion (ii) can be verified analogously. □

2. Lemma. *Let (a, b, u, v) be a regular quadruple in H.*

(i) *If $a_1 \in H$, $a_1 \in [u, a]$, $b_1 = b - u + a_1$, then (a_1, b, u, b_1), (a, b_1, a_1, v) are regular quadruples in H.*
(ii) *If $b_1 \in H$, $b_1 \in [b, v]$, $a_1 = u - b + b_1$, then (a_1, b, u, b_1), (a, b_1, a_1, v) are regular quadruples in H.*
(iii) *If $b_2 \in H$, $b_2 \in [u, b]$, $a_2 = b_2 - u + a$, then (a, b_2, u, a_2), (a_2, b, b_2, v) are regular quadruples in H.*
(iv) *If $a_2 \in H$, $a_2 \in [a, v]$, $b_2 = a_2 - a + u$, then (a, b_2, u, a_2), (a_2, b, b_2, v) are regular quadruples in H.*

Proof.

(i) Clearly, $u \in a_1 \wedge_m b$. Then from 1 (ii) we obtain that (a_1, b, u, b_1) is a regular quadruple. Obviously $v \in a \vee_m b_1$. Since $a_1 = a - v + b_1 = u - b + b_1$, from 1 (i) we get that (a, b_1, a_1, v) is a regular quadruple.

(ii) This is a consequence of (i).

The proof of (iii) is analogous to the proof of (i).

(iv) This is a consequence of (iii). □

The following construction concerning non-abelian multilattice groups is essentially a modification of a construction given by Jakubík and Kolibiar [7] for abelian multilattice groups.

Let (a, b, u, v) be a regular quadruple in H. Let $x \in [u, v]$, $a_1 \in a \wedge_m x$, $a_1 \geq u$. Let $\bar{b}_2 = b - u + a_1$. By 2 (i), $(a_1, b, u, \bar{b}_2)$ and $(a, \bar{b}_2, a_1, v)$ are regular quadruples in H. Further, there exists $u_1 \in \bar{b}_2 \wedge_m x$, $u_1 \geq a_1$. Let $\bar{a}_2 = u_1 - a_1 + a$, $b_1 = u_1 - a_1 + u$. From 2 (iii) and (iv) it follows that (a_1, b_1, u, u_1), $(u_1, b, b_1, \bar{b}_2)$, $(a, u_1, a_1, \bar{a}_2)$, $(\bar{a}_2, \bar{b}_2, u_1, v)$ are regular quadruples.

Now, we shall prove that $u_1 \in \bar{a}_2 \wedge_m x$. Since $u_1 \in L(\bar{a}_2, x)$, then there exists $z \in \bar{a}_2 \wedge_m x$ such that $z \geq u_1$. Let $\bar{z} = a_1 - u_1 + z$. By 2 (ii), $(a, z, \bar{z}, \bar{a}_2)$, (z, u_1, a_1, z) are regular quadruples. Thus $\bar{z} \in L(a, x)$, $\bar{z} \geq a_1$. Since $a_1 \in a \wedge_m x$, then $\bar{z} = a_1$. Thus $z = u_1$. Therefore $u_1 \in \bar{a}_2 \wedge_m x$.

Further there exists $a_2 \in \bar{a}_2 \vee_m x$, $a_2 \le v$. Let $\bar{b} = a_2 - \bar{a}_2 + u_1$. Then from 2 (iv), we infer that $(a_2, \bar{b}_2, \bar{b}, v)$ and $(\bar{a}_2, \bar{b}, u_1, a_2)$ are regular quadruples. Let $\bar{b}_1 = \bar{b} - u_1 + b_1$. By 2 (iv), $(u_1, \bar{b}_1, b_1, \bar{b})$ and $(\bar{b}, b, \bar{b}_1, \bar{b}_2)$ are regular quadruples. Clearly, $u_1 \in \bar{b} \wedge_m x$. Further, there exists $v_1 \in \bar{b} \vee_m x$, $v_1 \le a_2$. Let $b_2 = \bar{b}_2 - \bar{b} + v_1$, $\bar{a} = u_1 - \bar{b} + v_1$, $\bar{a}_1 = a_1 - u_1 + \bar{a}$. Then $\bar{a}_1 = a_1 - \bar{b} + v_1$. From 2 (i) and (ii), we obtain that (a_2, b_2, v_1, v), $(v_1, \bar{b}_2, \bar{b}, b_2)$, $(\bar{a}_2, v_1, \bar{a}, a_2)$, $(\bar{a}, \bar{b}, u_1, v_1)$, $(a, \bar{a}, \bar{a}_1, \bar{a}_2)$, $(\bar{a}_1, u_1, a_1, \bar{a})$ are regular quadruples in H. Clearly, $u_1 \in \bar{a} \wedge_m x$. The proof that $v_1 \in \bar{a} \vee_m x$ is analogous to the above proof that $u_1 \in \bar{a}_2 \wedge_m x$.

Under these denotations, we have the following two lemmas.

3. LEMMA. *If $a_1 = \bar{a}_1$ (i.e., $b_2 = \bar{b}_2$) or $a_2 = \bar{a}_2$ (i.e., $b_1 = \bar{b}_1$), then (a, x, a_1, a_2), (x, b, b_1, b_2), (a, b_1, u, x), (a_2, b_2, x, v) are regular quadruples in H.*

Proof. This is obvious. □

4. LEMMA. *If $\bar{a}_2 < a_2$, then H fails to be distributive.*

Proof. If H is distributive, then from the definition of distributivity it follows that $\bar{a} = \bar{b}$. Since $(\bar{a}, \bar{b}, u_1, v_1)$, $(\bar{a}_2, v_1, \bar{a}, a_2)$ are regular quadruples in H, we obtain $\bar{a}_2 = a_2$, a contradiction. This ends the proof. □

From 3 and 4, we obtain:

5. THEOREM. *Let H be distributive. Let (a, b, u, v) be a regular quadruple in H, and let $x \in [u, v]$, $a_1 \in a \wedge_m x$, $a_1 \ge u$. Then there are elements $b_1 \in [u, b]$, $a_2 \in [a, v]$, $b_2 \in [b, v]$ such that (a, x, a_1, a_2), (x, b, b_1, b_2), (a_1, b_1, u, x), (a_2, b_2, x, v) are regular quadruples in H.*

For the remainder of this paper, let G be a distributive multilattice group, and let f be a stable isometry in G.

6. LEMMA. *For each $x \in G^+$ there exists the least upper bound of $\{0, f(x)\}$ in G^+.*

Proof. Let $x \in G^+$. Then $U(x) = |x| = |f(x)| = U(-f(x), f(x))$. Therefore $-f(x) \vee f(x) = x$. By 1 (i), $(-f(x), f(x), -f(x)-x+f(x), x)$ is a regular quadruple. Clearly, $-f(x) - x + f(x) \le 0$. Let $a_1 \in -f(x) \wedge_m 0$, $a_1 \ge -f(x) - x + f(x)$. According to Theorem 5, there exist elements $b_1 \in [-f(x)-x+f(x), f(x)]$, $b_2 \in [f(x), x]$, $a_2 \in [-f(x), x]$ such that $(-f(x), 0, a_1, a_2)$, $(0, f(x), b_1, b_2)$, $(a_1, b_1, -f(x) - x + f(x), 0)$, $(a_2, b_2, 0, x)$ are regular quadruples. Let $z \in U(0, f(x))$. Since $a_2 \in U(-f(x), 0)$, we have $z + a_2 \in U(-f(x), f(x)) = U(x)$. Then, from $z + a_2 \ge x = b_2 + a_2$, we obtain $z \ge b_2$. Therefore $b_2 = 0 \vee f(x)$. □

7. THEOREM. *Let G be a distributive multilattice group, and let f be a stable isometry in G. Let $A_1 = \{x \in G^+ ,\ f(x) = x\}$, $B_1 = \{x \in G^+ ,\ f(x) = -x\}$, $A = A_1 - A_1$, $B = B_1 - B_1$. Then G is the direct product of the po-group A and the abelian po-group B, and $f(z) = z(A) - z(B)$ for each $z \in G$.*

Proof. It follows from 6 and [12; Theorem 2]. □

Remark. In [13; Theorem 2.6], it was shown that if $C = P \times Q$ is a direct decomposition of a po-group C with Q abelian, and if we put $g(x) = x(A) - x(B)$ for each $x \in C$, then g is a stable isometry in C.

Theorem 7 generalizes Theorem 2.5 of Jakubík [4] and with Theorem 2.6 ([13]) generalize Theorem 4 of Holland [2].

Theorem 7 also shows that the result of Jakubík and Kolibiar concerning isometries and direct decompositions of distributive multilattice groups can be extended to non-abelian distributive multilattice groups using the usual definition of the absolute of an element in a po-group. The notation from Theorem 7 will be also adopted in the following three Theorems.

8. THEOREM.

(i) *If $x, y \in G$, $y \le x$, then $f([y,x]) = [y(A) - x(B), x(A) - y(B)]$.*

(ii) *If $x, y \in G$, $f(y) \le f(x)$, then $[f(y), f(x)] = f([y(A)+x(B), x(A)+y(B)])$.*

(iii) *A non-void subset M of G is a directed convex subset of G if and only if $f(M)$ is a directed convex subset of G.*

The proof is the same as the proof of [13; Theorem 2.2].

9. THEOREM. *Let g be an isometry in G. Then $g\big(U(L(x,y)) \cap L(U(x,y))\big) = U\big(L(g(x), g(y))\big) \cap L\big(U(g(x), g(y))\big)$ for each $x, y \in G$.*

The proof is analogous to the proof of [13; Theorem 2.3], only instead of Theorem 2.2 ([13]), it is needed to use Theorem 8 (i) above.

10. THEOREM. *Let C be a directed convex subgroup of G. Then $f(C) = C$.*

Proof. Let $x \in C$. Then there exist $u, v \in C$ such that $u \in L(x, 0)$, $v \in U(x, 0)$. In view of Theorem 7, we have $v \ge x(A) \ge u$, $v \ge x(B) \ge u$. Then by the convexity of C, $x(A), x(B) \in C$. Thus $x(A) - x(B) \in C$. From Theorem 7, it follows that $f(x(A) - x(B)) = x(A) + x(B) = x$. Therefore $C \subseteq f(C)$.

If $y' \in f(C)$, then $y' = f(y)$ for some $y \in C$. By using similar considerations as above for x, we get $y(A), y(B) \in C$. Thus $y(A) - y(B) = f(y) = y' \in C$. Hence $f(C) \subseteq C$. □

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